Scott processes

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Abstract

The Scott process of a relational structure M is the transfinite sequence of sets of formulas given by the Scott analysis of M, as introduced in [18]. We present axioms for the class of Scott processes of infinite structures in a relational vocabulary τ, and use them to give a proof of an unpublished theorem of Leo Harrington from the 1970's, showing that a counterexample to Vaught's Conjecture has models of cofinally many Scott ranks below ω_2. Our approach also gives a theorem of Harnik and Makkai, showing that if there exists a counterexample to Vaught's Conjecture, then there is a counterexample whose uncountable models all have the same $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$-theory, and which has a model of Scott rank ω_1. Moreover, we show that if ϕ is a sentence of $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$ giving rise to a counterexample to Vaught's Conjecture, then for every limit ordinal α greater than the quantifier depth of ϕ and below ω_2, ϕ has a model of Scott rank α, and that for club many ordinals α below each of ω_1 and ω_2, ϕ has at least two nonisomorphic models of Scott rank α, generalizing a result of Sacks. We give a new proof using Scott processes of the fact that if there is a counterexample to Vaught's Conjecture in $\mathcal{L}_{\aleph_1,\aleph_0}$ then there is one of quantifier depth ω. We show that Scott processes give rise to a class of structures with the property if there is a counterexample to Vaught's Conjecture then there is one corresponding to a subclass of this class. We show that if countable structures M and N have the same Scott process though level δ, for δ a countable ordinal, and N has Scott rank δ, then M is isomorphic to a quantifier-depth-δ-elementary submodel of N.

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1 Introduction

The Scott analysis of a structure (introduced in [18]) is a procedure which assigns a transfinite sequence of infinitary formulas to the finite tuples of the structure. In the case of countable structures, this process culminates in a sentence in $\mathcal{L}_{\aleph_1,\aleph_0}$ which characterizes the structure up to isomorphism. This approach was used by Morley [15] to show that if a sentence in $\mathcal{L}_{\aleph_1,\aleph_0}$ has more than $\aleph_1$ many countable models, then it has continuum many. Vaught's Conjecture [20], which remains open, is the corresponding statement with $\aleph_1$ replaced by $\aleph_0$.

In this paper we give axioms for the class of sequences of sets of formulas which arise in the Scott analysis of infinite structures in a given relational vocabulary. For the case of countable structures, our axioms characterize this class exactly (see Section 6). For uncountable cardinals, while the sequences of formulas given by the Scott analysis satisfy our axioms, it can happen that a sequence satisfying our axioms does not have a model. Remark 10.8 shows this for a sequence of length ω_2; we do not know in general if this can happen for a sequence of cardinality $\aleph_1$ (the material in Section 7 shows that it cannot if we impose additional conditions on the sequence). We use this approach to give new proofs of several classical results on counterexamples to Vaught's Conjecture, including an unpublished theorem of Leo Harrington saying that the Scott ranks of the models of any counterexample to Vaught's Conjecture include a cofinal subset of ω_2 (in fact they include every limit ordinal of cardinality $\aleph_1$). Although our analysis concentrates on the scattered (Vaught's Conjecture counterexample) case, we expect that our approach will have applications to the general study of infinitary model theory.

We fix for this paper a relational vocabulary τ, and distinct variable symbols $\{x_n : n < \omega\}$. For notational convenience, we assume that τ contains a 0-ary relation symbol, as well as the binary symbol $=$, which is always interpreted as equality. We refer the reader to [9, 8, 13] for the definition of the language $\mathcal{L}_{\infty,\aleph_0}(\tau)$ and the languages $\mathcal{L}_{\kappa,\aleph_0}(\tau)$, for κ an infinite cardinal. In this paper, all formulas will have only finitely many free variables. Formally, we consider conjunctions and disjunctions of formulas as unordered, even when we write them as indexed by an ordered set (in this way, for instance, a formula in $\mathcal{L}_{\aleph_2,\aleph_0}(\tau)$ becomes a member of $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$ in a forcing extension in which the ω_1 of the ground model is countable). We begin by recalling the standard definition of the Scott process corresponding to an infinite τ-structure M, as introduced in [18] (see also [8, 13]), slightly modified to require the sequences $\bar{a}$ to consist of distinct elements.

{Scottformdef}

1.1 Definition. Given an infinite τ-structure M over a relational vocabulary τ, we define for each finite ordered tuple $\bar{a} = \langle a_0, \ldots, a_{|\bar{a}|-1}\rangle$ of distinct elements of M and each ordinal α the $|\bar{a}|$-ary $\mathcal{L}_{\infty,\aleph_0}(\tau)$-formula $\phi^M_{\bar{a},\alpha}$, recursively on α, as follows.

{Sdefone}

1. Each formula $\phi^M_{\bar{a},0}$ is the conjunction of all expressions of the two following forms:
   - $R(x_{f(0)}, \ldots, x_{f(k-1)})$, for R a k-ary relation symbol from τ and f a function from k to $|a|$, such that $M \models R(a_{f(0)}, \ldots, a_{f(k-1)})$,
   - $\neg R(x_{f(0)}, \ldots, x_{f(k-1)})$, for R a k-ary relation symbol from τ and f a function from k to $|a|$, such that $M \models \neg R(a_{f(0)}, \ldots, a_{f(k-1)})$.
2. Each formula $\phi^M_{\bar{a},\alpha+1}$ is the conjunction of the following three formulas:
   - $\phi^M_{\bar{a},\alpha}$,
   - $\bigwedge_{c \in M \setminus \{a_0,\ldots,a_{|\bar{a}|-1}\}} \exists x_{|\bar{a}|} \phi^M_{\bar{a}^\frown\langle c\rangle,\alpha}$,
   - $\forall x_{|\bar{a}|} \notin \{x_0, \ldots, x_{|a|-1}\} \bigvee_{c \in M \setminus \{a_0,\ldots,a_{|\bar{a}|-1}\}} \phi^M_{\bar{a}^\frown\langle c\rangle,\alpha}$.
3. For limit ordinals β, $\phi^M_{\bar{a},\beta} = \bigwedge_{\alpha<\beta} \phi^M_{\bar{a},\alpha}$.

We call $\phi^M_{\bar{a},\alpha}$ the *Scott formula* of $\bar{a}$ *in* M *at level* α.

For each infinite τ-structure M, each finite injective tuple $\bar{a}$ from M and each ordinal α, $\phi^M_{\bar{a},\alpha} \in \mathcal{L}_{|M\cup\tau\cup\alpha|^+,\aleph_0}(\tau)$ and $M \models \phi^M_{\bar{a},\alpha}(\bar{a})$. The following well-known fact can be proved by induction on α (see Theorem 3.5.2 of [8]). Again, we refer the reader to [9, 8, 13] for the definition of the *quantifier depth* of a formula, and note that each formula $\phi^M_{\bar{a},\alpha}$ as defined above has quantifier depth exactly α.

{wkh}

Theorem 1.2. *Given infinite τ-structures M and N, $n \in \omega$, an ordinal α and injective n-tuples $\bar{a}$ from M and $\bar{b}$ from N, $\phi^M_{\bar{a},\alpha} = \phi^N_{\bar{b},\alpha}$ if and only if, for each n-ary $\mathcal{L}_{\infty,\omega}(\tau)$ formula ψ of quantifier depth at most α, $\bar{a}$ satisfies ψ in M if and only if $\bar{b}$ satisfies ψ in N.*

1.3 Definition. Given an infinite τ-structure M and an ordinal β, we let $\Phi_\beta(M)$ denote the set of all formulas of the form $\phi^M_{\bar{a},\beta}$, for $\bar{a}$ a finite tuple of distinct elements of M. We call the class-length sequence $\langle \Phi_\alpha(M) : \alpha \in \mathrm{Ord}\rangle$ the *Scott process of* M.

This paper studies a proposed axiomatization of the class set-length initial segments of Scott processes of infinite τ-structures. Section 2 introduces an array of sets of formulas (properly) containing all the formulas appearing in the Scott process of any infinite τ-structure, and vertical and horizontal projection functions acting on this array. Section 3 introduces our general notion of a Scott process (i.e., without regard to a fixed τ-structure). Section 4 develops some of the basic consequences of this definition, showing the equivalence of our definition with a natural variation, and Section 5 defines the rank of a Scott process. The material in these two sections checks that Scott process in general, as defined here, satisfy various basic properties of Scott processes of τ-structures as defined by Scott. Section 6 shows that a Scott process of countable length whose last level is countable is an initial segment of the Scott process of some τ-structure. Theorem 6.11 shows that if countable structures M and N have the same Scott process though level δ, for δ a countable ordinal, and N has Scott rank δ, then M is isomorphic to a quantifier-depth-δ-elementary substructure of N. Section 7 shows (following Harrington) how to build models of cardinality $\aleph_1$ for certain Scott processes (roughly, those corresponding to Scott sentences). Section 8 develops more basic material on Scott processes, studying the way they reflect finite blocks of existential quantifiers. Section 9 looks at extending Scott processes of limit length. Section 10 is largely disjoint from the rest of the paper, and presents an argument showing that in some cases (for instance, counterexamples to Vaught's Conjecture in $\mathcal{L}_{\omega_1,\omega}(\tau)$) a Scott process which exists in a forcing extension can be shown to exist in the ground model. Put together, the material in Sections 7, 9 and 10 gives Harrington's theorem that a counterexample to Vaught's Conjecture has models of cofinally many Scott ranks below ω_2. Section 11 produces a second class of models of a counterexample to Vaught's Conjecture, among other things. In Section 12 we analyze the isomorphism relation on Scott subprocesses, and use it to give a new proof using Scott processes of the fact that if there is a counterexample to Vaught's Conjecture in $\mathcal{L}_{\aleph_1,\aleph_0}$ then there is one of quantifier depth ω. In Section 13 we define a class of structures corresponding to Scott processes (where the infinitary formulas become points) and observe that if there is a counterexample to Vaught's Conjecture then there is one given by a subclass of this class.

The material in this paper was inspired by the slides of a talk given by David Marker on Harrington's theorem [14]. Our proof is different in some respects from the proof outlined there. Marker's talk outlines a recursion-theoretic argument, assuming the existence of a counterexample ϕ to Vaught's Conjecture, for finding a sentence in $\mathcal{L}_{\omega_2,\omega}$ which will be the Scott sentence of a model of ϕ (of suitably high Scott rank) in a forcing extension collapsing ω_1. This part of the proof is replaced here by a forcing-absoluteness argument in Section 10 (essentially equivalent versions of these arguments appear in Section 1 of [6]).

The remainder of Harrington's proof builds a model of this Scott sentence. This we do in Section 7, guided by the argument in Marker's slides.

Other, different, proofs of Harrington's theorem appear in [1] and [10].

2 Formulas and projections

{fpsec}

For each $n \in \omega$, we let X_n denote the set $\{x_m : m < n\}$ and i_n denote the identity function on X_n. For all $m \leq n$ in ω, we let $\mathcal{I}_{m,n}$ denote the set of injections from X_m into X_n.

We start by defining a class of formulas which contains every formula appearing in the Scott process of any infinite τ-structure (see Remark 2.5). The sets Ψ_α defined below also contain formulas that do not appear in the Scott process of any τ-structure (i.e., which are not satisfiable). This extra degree of freedom is sometimes useful (for instance, in Definition 5.14); in any case strengthening the definition to rule out such formulas would raise issues that we would rather defer (see Remark 2.16, however). For the moment, the important point is that the sets Ψ_β ($\beta \in \mathrm{Ord}$) are small enough to carry the projection functions $V_{\alpha,\beta}$ and H^n_α defined below.

{psidef}

2.1 Definition. We define, for each ordinal α and each $n \in \omega$, the sets Ψ_α and Ψ^n_α, by recursion on α, as follows.

1. For each $n \in \omega$, Ψ^n_0 is the set of all n-ary formulas which are conjunctions consisting of, for each atomic τ-formula using variables from X_n, exactly one of the formula and its negation, including an instance of the formula $x_i = x_i$ for each $x_i \in X_n$, and an instance of $x_i \neq x_j$ for each pair of distinct x_i, x_j from X_n.

{nextstep}

2. For each ordinal α and each $n \in \omega$, $\Psi^n_{\alpha+1}$ is the set of formulas ϕ for which there exist a formula $\phi' \in \Psi^n_\alpha$ and a nonempty $E \subseteq \Psi^{n+1}_\alpha$ such that ϕ is the conjunction of ϕ' with the following two formulas.
   (a) $\bigwedge_{\psi \in E} \exists x_n \psi$;
   (b) $\forall x_n (x_n \notin \{x_0, \ldots, x_{n-1}\} \rightarrow \bigvee_{\psi \in E} \psi)$.

{pathstage}

3. For each limit ordinal α and each $n \in \omega$, Ψ^n_α is the set of conjunctions which consist of exactly one formula ψ_β from each Ψ^n_β, for $\beta < \alpha$, satisfying the following conditions.
   (a) For each $\beta < \alpha$, ψ_β is the formula ϕ' with respect to $\psi_{\beta+1}$, as in condition (2) (i.e., the unique conjunct of quantifier depth β).
   (b) For each limit ordinal $\beta < \alpha$, $\psi_\beta = \bigwedge\{\psi_\gamma : \gamma < \beta\}$.
4. For each ordinal α, $\Psi_\alpha = \bigcup_{n \in \omega} \Psi^n_\alpha$.

We can think of the sets Ψ^n_α as forming an array, with the rows indexed by α and the columns indexed by n. In the rest of this section we define the functions $V_{\alpha,\beta}$, which map between rows while preserving column rank, and the functions H^n_α which map between columns while preserving row rank.

{exactfree}

2.2 Remark. Each Ψ_α is a set of $\mathcal{L}_{\infty,\omega}(\tau)$ formulas of quantifier depth α, so the sets Ψ_α are disjoint for distinct α. Similarly, for each $n \in \omega$ and each ordinal α, X_n is the set of free variables for each formula in each Ψ^n_α.

2.3 Remark. As we require our vocabulary to contain a 0-ary relation as well as the binary relation $=$, Ψ^n_α is nonempty for each ordinal α and each $n \in \omega$.

{pdef}

2.4 Definition. For each ordinal α, and each formula ϕ in $\Psi_{\alpha+1}$, we let $E(\phi)$ denote the set E from condition (2) of Definition 2.1.

{mcont}

2.5 Remark. If M is a τ-structure, α is an ordinal and $\bar{a}$ is a finite tuple of distinct elements of M, then the Scott formula of $\bar{a}$ in M at level α as defined in Definition 1.1 (i.e., $\phi^M_{\bar{a},\alpha}$) is an element of $\Psi^{|\bar{a}|}_\alpha$. It follows that $\Phi_\alpha(M) \subseteq \Psi_\alpha$.

The functions $V_{\alpha,\beta}$, as defined below, are the *vertical projection functions.*

{vdef}

2.6 Definition. The functions $V_{\alpha,\beta}\colon \Psi_\beta \to \Psi_\alpha$, for all pairs of ordinals $\alpha \leq \beta$ are defined as follows.

1. Each function $V_{\alpha,\alpha}$ is the identity function on Ψ_α.

{vstep}

2. For each ordinal α, and each $\phi \in \Psi_{\alpha+1}$, $V_{\alpha,\alpha+1}(\phi)$ is the first conjunct of ϕ, i.e., the formula ϕ' in condition (2) of Definition 2.1.

{pathcond}

3. For each limit ordinal β, each formula $\phi \in \Psi_\beta$, and each $\alpha < \beta$, $V_{\alpha,\beta}(\phi)$ is the unique conjunct of ϕ in Ψ_α.

{vdir0}

4. For all ordinals $\alpha < \beta$, $V_{\alpha,\beta+1} = V_{\alpha,\beta} \circ V_{\beta,\beta+1}$.

{leqvdef}

2.7 Definition. For formulas ϕ, ψ in $\bigcup_{\alpha\in\mathrm{Ord}} \Psi_\alpha$, we write $\phi \leq_V \psi$ to mean that $V_{\alpha,\beta}(\psi) = \phi$, where $\phi \in \Psi_\alpha$ and $\psi \in \Psi_\beta$.

{vdir}

2.8 Remark. Conditions (2) and (3) of Definition 2.1 imply the following stronger version of condition (4) of Definition 2.6 : for all ordinals $\alpha \leq \beta \leq \gamma$, $V_{\alpha,\gamma} = V_{\alpha,\beta} \circ V_{\beta,\gamma}$.

{vone}

2.9 Remark. For all ordinals $\alpha \leq \beta$, each $n \in \omega$, and each $\phi \in \Psi^n_\beta$, $V_{\alpha,\beta}(\phi)$ is in Ψ^n_α, so ϕ and $V_{\alpha,\beta}(\phi)$ have the same free variables.

2.10 Remark. Since the domains of the functions $V_{\alpha,\beta}$ are disjoint for distinct β, one could drop β and simply write V_α (which would then be a definable class-sized function from $\bigcup_{\beta\in(\mathrm{Ord}\setminus\alpha)} \Psi_\beta$ to Ψ_α). We retain both subscripts for clarity.

We define the *horizontal projection functions* as follows.

{hdef}

2.11 Definition. The functions H^n_α, for each ordinal α and each $n \in \omega$, are defined recursively on α, as follows.

1. The domain of each H^n_α consists of all pairs (ϕ, j), where $\phi \in \Psi^n_\alpha$ and, for some $m \leq n$, $j \in \mathcal{I}_{m,n}$.

2. For all $m \leq n$ in ω, all formulas $\phi \in \Psi_0^n$, and all $j \in \mathcal{I}_{m,n}$, $H_0^n(\phi, j)$ is the conjunction of all conjuncts from ϕ whose variables are all in the range of j, with these variables replaced by their j-preimages.

{projfact}

3. For each ordinal α, all $m \leq n$ in ω, each $\phi \in \Psi_{\alpha+1}^n$, and each $j \in \mathcal{I}_{m,n}$, $H_{\alpha+1}^n(\phi, j)$ is the formula $\psi \in \Psi_{\alpha+1}^m$ such that

$$V_{\alpha,\alpha+1}(\psi) = H_\alpha^n(V_{\alpha,\alpha+1}(\phi), j)$$

and $E(\psi) = H_\alpha^{n+1}[E(\phi) \times \{j \cup \{(x_m, y)\} \mid y \in (X_{n+1} \setminus \mathrm{range}(j))\}]$.

{hsix}

4. For each limit ordinal α, each $m \leq n$ in ω, each $j \in \mathcal{I}_{m,n}$ and each $\phi \in \Psi_\alpha^n$,

$$H_\alpha^n(\phi, j) = \bigwedge \{H_\beta^n(V_{\beta,\alpha}(\phi), j) : \beta < \alpha\}.$$

2.12 Remark. Since the domains of the functions H_α^n are disjoint for distinct pairs (α, n), one could drop α and n and simply write H. We retain them for clarity. One could further streamline the notation used here by writing $\phi^{(j)}$ for $H_\beta^n(\phi, j)$ and $\phi_{(\alpha)}$ for $V_{\alpha,\beta}(\phi)$, for appropriate α, β, j and n. Again, we will stick to the more explicit notation in this paper.

{hidrem}

2.13 Remark. For all ordinals α, all $m \leq n$ in ω, all $j \in \mathcal{I}_{m,n}$ and all $\phi \in \Psi_\alpha^n$, $H_\alpha^n(\phi, j)$ is an element of Ψ_α^m, and $H_\alpha^n(\phi, i_n) = \phi$.

We leave it to the reader to verify (by induction on α) that if

- M is a τ-structure,
- α is an ordinal,
- $m \leq n$ are elements of ω,
- $\bar{b} = \langle b_0, \ldots, b_{n-1} \rangle$ is a sequence of distinct elements of M,
- $j^* : m \to n$ is an injection,
- $\bar{a}$ is the sequence $\langle b_{j^*(0)}, \ldots, b_{j^*(m-1)} \rangle$ and
- $j \in \mathcal{I}_{m,n}$ is such that $j(x_p) = x_{j^*(p)}$ for each $p < m$,

then $H_\alpha^n(\phi_{\bar{b},\alpha}^M, j) = \phi_{\bar{a},\alpha}^M$.

{tworems}
{permutevar}

2.14 Remark. The following facts can be easily verified by induction on α.

1. For each ordinal α, each $n \in \omega$, each $\phi \in \Psi_\alpha^n$ and each $j \in \mathcal{I}_{n,n}$, $H_\alpha^n(\phi, j)$ is the result of replacing each free variable in ϕ (i.e., each member of X_n) with its j-preimage.

{hcompose}

2. For each ordinal α, all $m \leq n \leq p$ in ω, all $\phi \in \Psi_\alpha^p$, all $j \in \mathcal{I}_{n,p}$ and all $k \in \mathcal{I}_{m,n}$, $H_\alpha^n(H_\alpha^p(\phi, j), k) = H_\alpha^p(\phi, j \circ k)$.

The following proposition shows that the vertical and horizontal projection functions commute appropriately.

{proppartone}

Proposition 2.15. *For all ordinal $\alpha \leq \beta$, all $m \leq n \in \omega$, all $j \in \mathcal{I}_{m,n}$, and all $\phi \in \Psi^n_\beta$,*

$$V_{\alpha,\beta}(H^n_\beta(\phi, j)) = H^n_\alpha(V_{\alpha,\beta}(\phi), j).$$

Proof. When $\alpha = \beta$, both sides are equal to $H^n_\alpha(\phi, j)$. When $\beta = \alpha + 1$, the proposition is part of condition (3) of Definition 2.11. When β is a limit ordinal, it follows from condition (3) of Definition 2.6 and condition (4) of Definition 2.11. The remaining cases can be proved by induction on β, fixing α, using the induction hypotheses for the pairs α, β and $\beta, \beta + 1$ at successor stages of the form $\beta + 1$.

□

{nocut}

2.16 Remark. For all ordinals α, all $n \in \omega$, all $\phi \in \Psi^n_{\alpha+1}$ appearing in the Scott process of a τ-structure, and all $\psi \in E(\phi)$, $H^{n+1}_\alpha(\psi, i_n) = V_{\alpha,\alpha+1}(\phi)$. Having defined the horizontal and vertical projection functions, we could now thin the sets Ψ^n_α by adding this as an additional requirement, but choose not to.

2.17 Example. Suppose that τ contains a single binary relation symbol R, along with $=$ and the 0-ary relation symbol S. The set Ψ^0_0 then consists of the sentences S and $\neg S$. The set Ψ^1_0 contains four formulas, $S \wedge R(x_0, x_0) \wedge x_0 = x_0$, $S \wedge \neg R(x_0, x_0) \wedge x_0 = x_0$, $\neg S \wedge R(x_0, x_0) \wedge x_0 = x_0$ and $\neg S \wedge \neg R(x_0, x_0) \wedge x_0 = x_0$. Call the first two of these formulas ψ^1_0 and ϕ^1_0, respectively. Then

$$H^1_0(\psi^1_0) = H^1_0(\phi^1_0) = S.$$

The set Ψ^2_0 then contains 32 formulas, for instance,

$$S \wedge \neg R(x_0, x_1) \wedge \neg R(x_1, x_0) \wedge R(x_0, x_0) \wedge R(x_1, x_1) \wedge x_0 \neq x_1 \wedge x_0 = x_0 \wedge x_1 = x_1$$

and

$$S \wedge \neg R(x_0, x_1) \wedge \neg R(x_1, x_0) \wedge R(x_0, x_0) \wedge \neg R(x_1, x_1) \wedge x_0 \neq x_1 \wedge x_0 = x_0 \wedge x_1 = x_1.$$

Call these formulas ψ^2_0 and ϕ^2_0, respectively. Then

$$H^2_0(\psi^2_0, i_1) = \psi^1_0$$

and

$$H^2_0(\phi^2_0, \{(x_0, x_1)\}) = \phi^1_0,$$

as defined in Definition 2.11. The set Ψ^3_0 then contains 2^{10} formulas, including the conjunction of S with every instance of $R(y, z)$ for $y, z \in X_3$ (and the requisite forumulas involving $=$). In general, Ψ^n_0 contains $2^{(n^2+1)}$ formulas.

The set Ψ^0_1 contains the sentence

$$S \wedge (\exists x_0 S \wedge R(x_0, x_0) \wedge x_0 = x_0) \wedge (\forall x_0 S \wedge R(x_0, x_0) \wedge x_0 = x_0)$$

(omitting one instance each of $\wedge$ and $\vee$, corresponding to a conjunction and a disjunction of of size 1, and a subformula asserting that x_0 is not in the emptyset) and the conjunction of the three following formulas

- S
- $(\exists x_0 S \wedge R(x_0, x_0) \wedge x_0 = x_0) \wedge (\exists x_0 S \wedge \neg R(x_0, x_0) \wedge x_0 = x_0)$
- $\forall x_0(x_0 \notin \emptyset \rightarrow ((S \wedge R(x_0, x_0) \wedge x_0 = x_0) \vee (S \wedge \neg R(x_0, x_0) \wedge x_0 = x_0)))$.

Call these sentences ψ_1^0 and ϕ_1^0, respectively. Then $E(\psi_1^0) = \{\psi_0^1\}$ and $E(\phi_1^0) = \{\psi_0^1, \phi_0^1\}$, as defined in Definition 2.4. The set Ψ_1^1 contains the formulas

$$\psi_0^1 \wedge (\exists x_1 \psi_0^2) \wedge (\forall x_1\, x_1 \neq x_0 \rightarrow \psi_0^2)$$

and

$$\psi_0^1 \wedge (\exists x_1 \phi_0^2) \wedge (\forall x_1\, x_1 \neq x_0 \rightarrow \phi_0^2),$$

with the same omissions as ψ_1^0. Call these formulas ψ_1^1 and ϕ_1^1. Then $E(\psi_1^1) = \{\psi_0^2\}$, $E(\phi_1^1) = \{\phi_0^2\}$,

$$V_{0,1}(\psi_1^1) = V_{0,1}(\phi_1^1) = \psi_0^1,$$

$H_1^1(\psi_1^1, i_0) = \psi_1^0$ and $H_1^1(\phi_1^1, i_0) = \phi_1^0$. Note that the function H_1^1 changes the bound variables (as well as the free variables).

3 Scott processes

{spsec}

This section introduces the central topic of the paper, the class of Scott processes (for a relational vocabulary τ).

{prodef}

3.1 Definition. A *Scott process* is a sequence $\langle \Phi_\alpha : \alpha < \delta \rangle$, for some nonzero ordinal δ (the *length* of the process), satisfying the following conditions, where for each ordinal α and each $n \in \omega$, Φ_α^n denotes the set $\Phi_\alpha \cap \Psi_\alpha^n$.

1. **The Formula Conditions** {formone}

   (a) Each Φ_α is a nonempty subset of the corresponding set Ψ_α. {econtain}

   (b) For each ordinal of the form $\alpha + 1 < \delta$, and each $\phi \in \Phi_{\alpha+1}$, $E(\phi)$ is a subset of Φ_α.

   (c) {vfive} For all $\alpha < \beta < \delta$, $\Phi_\alpha = V_{\alpha,\beta}[\Phi_\beta]$.

   (d) {htwozero} For all $\alpha < \delta$, all $n \in \omega$, all $j \in \mathcal{I}_{n,n}$ and all $\phi \in \Phi_\alpha^n$, $H_\alpha^n(\phi, j) \in \Phi_\alpha^n$.

   (e) {htwo} For all $\alpha < \delta$, and all $m < n$ in ω, $\Phi_\alpha^m = H_\alpha^n[\Phi_\alpha^n \times \{i_m\}]$.

2. **The Coherence Conditions** {hfour}

   (a) For each ordinal of the form $\alpha + 1$ below δ, each $n \in \omega$ and each $\phi \in \Phi_{\alpha+1}^n$,

   $$E(\phi) = V_{\alpha,\alpha+1}[\{\psi \in \Phi_{\alpha+1}^{n+1} \mid H_{\alpha+1}^{n+1}(\psi, i_n) = \phi\}].$$

   {ppath}

   (b) For all $\alpha < \beta < \delta$, all $n \in \omega$ and all $\phi \in \Phi_\beta^n$,

   $$E(V_{\alpha+1,\beta}(\phi)) \subseteq V_{\alpha,\beta}[\{\psi \in \Phi_\beta^{n+1} \mid H_\beta^{n+1}(\psi, i_n) = \phi\}].$$

{combine}

(c) For all $\alpha < \delta$, n, m in ω, $\phi \in \Phi^n_\alpha$ and $\psi \in \Phi^m_\alpha$, there exist $\theta \in \Phi^{n+m}_\alpha$ and $j \in \mathcal{I}_{m,n+m}$ such that $\phi = H^{n+m}_\alpha(\theta, i_n)$ and $\psi = H^{n+m}_\alpha(\theta, j)$.

The sets Φ_α are called the *levels* of the Scott process.

3.2 Remark. Condition (2b) of Definition 3.1 includes the left to right inclusion in condition (2a). We prefer the given formulation of condition (2a), as it gives a better sense of the meaning of $E(\phi)$.

{succrem}

3.3 Remark. Proposition 4.4 shows that equality holds in condition (2b) of Definition 3.1, for any Scott process, so that conditions (2a) and (2b) could equivalently be replaced by condition (2b) alone with $=$ in place of $\subseteq$.

{jandi}

3.4 Remark. Conditions (1d) and (1e) of Definition 3.1 combine to give the following: for all $\alpha < \delta$, all $m \leq n$ in ω and all $j \in \mathcal{I}_{m,n}$, $\Phi^m_\alpha = H^n_\alpha[\Phi^n_\alpha \times \{j\}]$.

Proposition 3.5 shows that each level of a Scott process contains a unique sentence.

{usent}

Proposition 3.5. *Whenever $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, Φ^0_α has a unique element, for each $\alpha < \delta$.*

Proof. That each Φ^0_α is nonempty follows from conditions (1a) and (1e) of Definition 3.1. Now, suppose that ϕ and ψ are elements of Φ^0_α. By condition (2c) of Definition 3.1, there exist $\theta \in \Phi^0_\alpha$ and $j \in \mathcal{I}_{0,0}$ such that $\phi = H^0_\alpha(\theta, i_0)$ and $\psi = H^0_\alpha(\theta, j)$. However, i_0 is the unique element of $\mathcal{I}_{0,0}$, so $\phi = \psi$. $\square$

It follows from Proposition 3.5 and conditions (1c), (1e) and (2a) of Definition 3.1 that whenever $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process and α is an ordinal with $\alpha + 1 < \delta$, if ϕ is the unique element of $\Phi^0_{\alpha+1}$, then $E(\phi) = \Phi^1_\alpha$.

4 Consequences of coherence

{ccsec}

In this section we prove some basic facts about Scott processes, primarily about sets of the form $E(\phi)$ and the projection functions. The main result of the section is Proposition 4.4, which was referred to in Remark 3.3. We fix for this section a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$.

Proposition 4.1 follows from Proposition 2.15 (i.e., the commutativity of the horizontal and vertical projections). The failure of the reverse inclusion is witnessed whenever a set of the form $V^{-1}_{\alpha,\beta}[\{\rho\}]$ has distinct members ϕ_1, ϕ_2.

{proppparttwo}

Proposition 4.1. *For all $\alpha \leq \beta < \delta$, all $m \leq n \in \omega$, all $j \in \mathcal{I}_{m,n}$, and all $\phi \in \Phi^m_\beta$,*

$$V_{\alpha,\beta}[\{\psi \in \Phi^n_\beta \mid H^n_\beta(\psi, j) = \phi\}] \subseteq \{\theta \in \Phi^n_\alpha \mid H^n_\alpha(\theta, j) = V_{\alpha,\beta}(\phi)\}.$$

The right-to-left inclusion in Proposition 4.2 says that every one-point extension of a formula ϕ at level α is a member of $E(\psi)$, for some $\psi \in V^{-1}_{\alpha,\alpha+1}[\phi]$. This proposition is used in Remark 5.15.

{proppartfour}

Proposition 4.2. *For each ordinal of the form $\alpha+1$ below δ, each $n \in \omega$ and each $\phi \in \Phi^n_\alpha$,*

$$\bigcup\{E(\psi) \mid \psi \in V^{-1}_{\alpha,\alpha+1}[\{\phi\}]\} = \{\theta \in \Phi^{n+1}_\alpha \mid H^{n+1}_\alpha(\theta, i_n) = \phi\}.$$

Proof. The left-to-right inclusion follows from Proposition 4.1 and condition (2a) of Definition 3.1. The reverse inclusion follows from conditions (1c) and (2a) of Definition 3.1, and Proposition 2.15. □

Proposition 4.3 is the successor case of Proposition 4.4, modulo condition (2a) of Definition 3.1.

{proppartthree}

Proposition 4.3. *For all $\alpha \leq \beta$ such that $\beta+1<\delta$, and all $\phi \in \Phi_{\beta+1}$,*

$$E(V_{\alpha+1,\beta+1}(\phi)) = V_{\alpha,\beta}[E(\phi)].$$

Proof. Fix $n \in \omega$ such that $\phi \in \Phi^n_{\beta+1}$. For the forward direction, condition (2b) of Definition 3.1 gives that

$$E(V_{\alpha+1,\beta+1}(\phi)) \subseteq V_{\alpha,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}],$$

which by condition (4) of Definition 2.6 is equal to

$$V_{\alpha,\beta}[V_{\beta,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}]],$$

which by condition (2a) of Definition 3.1 is equal to $V_{\alpha,\beta}[E(\phi)]$.

For the reverse direction we have from condition (2a) of Definition 3.1 that $V_{\alpha,\beta}[E(\phi)]$ is equal to

$$V_{\alpha,\beta}[V_{\beta,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}]],$$

which by Remark 2.8 is equal to

$$V_{\alpha,\alpha+1}[V_{\alpha+1,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}]],$$

which by Proposition 4.1 is contained in

$$V_{\alpha,\alpha+1}[\{\theta \in \Phi^{n+1}_{\alpha+1} \mid H^{n+1}_{\alpha+1}(\theta, i_n) = V_{\alpha+1,\beta+1}(\phi)\}]],$$

which by condition (2a) of Definition 3.1 is equal to $E(V_{\alpha+1,\beta+1}(\phi))$. □

We now show that the reverse inclusion of condition (2b) of Definition 3.1 holds for any Scott process.

{succcase}

Proposition 4.4. *For all $\alpha < \beta < \delta$, for all $n \in \omega$ and all $\phi \in \Phi^n_\beta$,*

$$E(V_{\alpha+1,\beta}(\phi)) = V_{\alpha,\beta}[\{\psi \in \Phi^{n+1}_\beta \mid H^{n+1}_\beta(\psi, i_n) = \phi\}].$$

Proof. When β is a successor ordinal, this is Proposition 4.3, using condition (2a) of Definition 3.1. For any β, the left-to-right inclusion is condition (2b) of Definition 3.1. For the reverse inclusion,

$$V_{\alpha,\beta}[\{\psi \in \Phi^{n+1}_{\beta} \mid H^{n+1}_{\beta}(\psi, i_n) = \phi\}]$$

is equal to

$$V_{\alpha,\alpha+1}[V_{\alpha+1,\beta}[\{\psi \in \Phi^{n+1}_{\beta} \mid H^{n+1}_{\beta}(\psi, i_n) = \phi\}]]$$

by Remark 2.8, and this is contained in

$$V_{\alpha,\alpha+1}[\{\psi \in \Phi^{n+1}_{\alpha+1} \mid H^{n+1}_{\alpha+1}(\psi, i_n) = V_{\alpha+1,\beta}(\phi)\}],$$

by Proposition 4.1. Finally, this last term is equal to $E(V_{\alpha+1,\beta}(\phi))$ by condition (2a) of Definition 3.1. □

{rssec}

5 Ranks and Scott sentences

The *Scott rank* of a τ-structure M (see [8, 13], for instance) is the least ordinal α such that $V_{\alpha,\alpha+1}$ is injective on $\Phi_{\alpha+1}(M)$ (which we defined in the introduction). If α is the Scott rank of M, then $V_{\beta,\beta+1}$ injective on $\Phi_{\beta+1}(M)$ for all $\beta \geq \alpha$ as well. Proposition 5.5 below verifies that Scott processes have the same property. We isolate the successor step of the proof as a separate proposition. The second part of the proposition is used in Remark 9.11.

{leastfixedalt}

Proposition 5.1. *Let β be an ordinal, and let $\langle \Phi_\alpha : \alpha \leq \beta + 2\rangle$ be a Scott process. If ϕ is an element of $\Phi_{\beta+1}$, then each of the following conditions implies that $V^{-1}_{\beta+1,\beta+2}[\{\phi\}] \cap \Phi_{\beta+2}$ is a singleton.*

{lf1}

1. *$V^{-1}_{\beta,\beta+1}[\{\psi\}] \cap \Phi_{\beta+1}$ is a singleton for each $\psi \in E(\phi)$.*

{lf2}

2. *There exists a $\psi \in E(\phi)$ such that $V^{-1}_{\beta,\beta+2}[\{\psi\}] \cap \Phi_{\beta+2}$ is a singleton.*

Proof. Condition (1c) of Definition 3.1 implies that $V^{-1}_{\beta+1,\beta+2}[\{\phi\}] \cap \Phi_{\beta+2}$ is nonempty. Suppose that $\phi' \in \Phi_{\beta+2}$ is such that $V_{\beta+1,\beta+2}(\phi') = \phi$.

If (1) holds then, by Proposition 4.3, $V_{\beta,\beta+1}[E(\phi')] = E(\phi)$. Since $V^{-1}_{\beta,\beta+1}[\{\psi\}] \cap \Phi_{\beta+1}$ is a singleton for each $\psi \in E(\phi)$, this implies that $E(\phi') = V^{-1}_{\beta,\beta+1}[E(\phi)] \cap \Phi_{\beta+1}$, which uniquely determines ϕ'.

If (2) holds, let ψ' be the unique member of $V^{-1}_{\beta,\beta+2}[\{\psi\}] \cap \Phi_{\beta+2}$. Since $\psi \in E(\phi)$, $V_{\beta+1,\beta+2}(\psi')$ is a member of $E(\phi')$, by Proposition 4.3. Let $n \in \omega$ be such that $\phi \in \Phi^{n}_{\beta+1}$. Then $\phi' = H^{n+1}_{\beta+2}(\psi', i_n)$, by part (2a) of Definition 3.1. □

Corollary 5.2 is a consequence of part (1) of Proposition 5.1, and Corollary 5.3 is a consequence of part (2) (using Proposition 4.3).

{fixeddown}

Corollary 5.2. *Let β be an ordinal, and let $\langle \Phi_\alpha : \alpha \leq \beta+2\rangle$ be a Scott process. Suppose that $n \in \omega$ is such that $V_{\beta,\beta+1}$ is injective on $\Phi^{n+1}_{\beta+1}$. Then $V_{\beta+1,\beta+2}$ is injective on $\Phi^{n}_{\beta+2}$.*

{fixedup}

Corollary 5.3. *Let β and δ be an ordinals, with $\delta \geq \beta + 2$, and let*

$$\langle \Phi_\alpha : \alpha \leq \delta \rangle$$

be a Scott process. Suppose that $\phi \in \Phi_{\beta+1}$ and $\psi \in E(\phi)$ are such that $V^{-1}_{\beta,\delta}[\{\psi\}] \cap V_\delta$ is a singleton. Then $V^{-1}_{\beta+1,\delta}[\{\phi\}] \cap \Phi_\delta$ is a singleton.

5.4 Remark. It is natural to ask whether part (1) of Proposition 5.1 has a converse, in the sense that if $\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$ is a Scott process and $\phi \in \Phi_{\beta+1}$ and $\psi \in E(\phi)$ are such that $V^{-1}_{\beta,\beta+1}[\{\psi\}]$ has at least two members then there must exist a set $\Phi_{\beta+2}$ such that $\langle \Phi_\alpha : \alpha \leq \beta + 2 \rangle$ is a Scott process and $V^{-1}_{\beta+1,\beta+2}[\{\phi\}]$ is not a singleton. This is not the case in general, however, as by Proposition 3.5, each function of the form $V_{\alpha,\alpha+1} \restriction \Phi^0_{\alpha+1}$ is always injective.

{leastfixed}

Proposition 5.5. *If $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, $\beta < \gamma$ are ordinals with $\gamma + 1 < \delta$, and $V_{\beta,\beta+1} \restriction \Phi_{\beta+1}$ is injective, then $V_{\gamma,\gamma+1} \restriction \Phi_{\gamma+1}$ is injective.*

Proof. Letting η be such that $\gamma = \beta + \eta$, we prove the proposition by induction on η, for all β and δ simultaneously. Applying the induction hypotheses, the limit case follows from Remark 2.8, and the successor case follows from part (1) of Proposition 5.1 (and also from Corollary 5.2). □

5.6 Definition. The *rank* of a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$ is the least β such that $V_{\beta,\beta+1} \restriction \Phi_{\beta+1}$ is injective, if such a β exists, and undefined otherwise. We say that a Scott processes is *terminating* (or *terminates*) if its rank is defined, and *nonterminating* otherwise.

The rank of (any sufficiently long set-sized initial segment of) the Scott process of a τ-structure M is the same then as the Scott rank of M.

5.7 Remark. Suppose that β and γ are ordinals, and $n \in \omega$ is such that $\gamma > \beta + n + 1$. Suppose that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is a Scott process, and that $V_{\beta,\beta+1}$ is injective on $\Phi^m_{\beta+1}$, for all $m > n$ in ω. By Corollary 5.2, the rank of $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is at most $\beta + n$ (since each Φ^0_α is a singleton, $V_{\alpha,\alpha+1} \restriction \Phi^0_{\alpha+1}$ is injective for all α).

In the following definition, j can equivalently be replaced with i_n, by condition (1d) of Definition 3.1.

{beyonddef}

5.8 Definition. Let β and γ be ordinals such that $\gamma > \beta + 1$, and let

$$\langle \Phi_\alpha : \alpha < \gamma \rangle$$

be a Scott process. Let n be an element of ω, and let ϕ be an element of Φ^n_β. We say that the Scott process $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is *injective beyond* ϕ if for all $m \in \omega \setminus n$, all $j \in \mathcal{I}_{n,m}$ and all $\psi \in \Phi^m_\beta$ such that $\phi = H^m_\beta(\psi, j)$, $V^{-1}_{\beta,\beta+1}[\{\psi\}] \cap \Phi_{\beta+1}$ is a singleton.

{preoneextension}

5.9 Remark. Let $\beta < \gamma$ be ordinals, and let $\langle \Phi_\alpha : \alpha \leq \gamma \rangle$ be a Scott process. Let n be an element of ω, and let $\phi \in \Phi^n_\beta$ be such that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is injective beyond ϕ. Applying part (1) of Proposition 5.1, one can show by induction that for all $\delta \in [\beta, \gamma]$,

- $V^{-1}_{\beta,\delta}[\{\phi\}] \cap \Phi_\delta$ is a singleton;
- if $\delta < \gamma$ then $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is injective beyond the unique member of $V^{-1}_{\beta,\delta}[\{\phi\}]$;
- for all $m \in \omega \setminus n$, all $j \in \mathcal{I}_{n,m}$ and all $\psi \in \Phi^m_\beta$ such that

$$\phi = H^m_\beta(\psi, j),$$

$V^{-1}_{\beta,\delta}[\{\psi\}] \cap \Phi_\delta$ is a singleton.

{oneextension}

5.10 Remark. Let β be an ordinal, and n an element of ω. Suppose that

$$\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$$

is a Scott process, and that $\phi \in \Phi^n_\beta$ is such that $\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$ is injective beyond ϕ. The proof of Scott's Isomorphism Theorem (Theorem 2.4.15 of [13]; using $\bar{a}$ in place of $\emptyset$ at stage 0) shows that for any two countable τ-structures M and N whose Scott processes agree with $\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$ though level $\beta + 1$, if $\bar{a}$ is an n-tuple from M and $\bar{b}$ is an n-tuple from N, each satisfying ϕ in their respective models, then there is an isomorphism of M and N sending $\bar{a}$ to $\bar{b}$. Alternately, one can show that for each ordinal $\gamma > \beta + 1$, there is a unique Scott process of length γ extending β, using either Remark 5.9 or Proposition 9.3.

{oneextension2}

5.11 Remark. In the situation of Definition 5.8, $\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$ need not have rank β. To see this, consider the Scott process of a countably infinite undirected graph G consisting of an infinite set of nodes which are not connected to anything, and another infinite set of nodes which are all connected to each other, but not to themselves. The formula in $\Phi^2_0(G)$ corresponding to a connected pair has the property of ϕ in Remark 5.10, but the Scott rank of G is 1, not 0, since the unique member of $\Phi^1_0(G)$ has two successors in $\Phi^1_1(G)$.

The following definition is inspired by Remarks 5.10 and 5.11. There may be some connection between the notion of pre-rank and the subject of [12].

{prerankdef}

5.12 Definition. The *pre-rank* of a Scott process $\langle \Phi_\alpha : \alpha < \beta \rangle$ is the least $\gamma \leq \beta$ such that for all ordinals $\eta > \gamma$, there exists a unique Scott process of length η extending $\langle \Phi_\alpha : \alpha < \gamma \rangle$ (if such a γ exists). The Scott *pre-rank* of a τ-structure is the pre-rank of any terminating initial segment of its Scott process.

By Proposition 5.5, the pre-rank of a terminating Scott process is at most one more than its rank; Remark 5.11 shows that it can be smaller. By Proposition 9.24, if a Scott process has countable length, and all of its levels are countable, then its rank is at most ω more than its pre-rank. Proposition 5.13 gives a tighter bound in the situation of Definition 5.8.

{prerankbound}

Proposition 5.13. *Let β be an ordinal, and n an element of ω. Suppose that $\langle \Phi_\alpha : \alpha \leq \beta + n + 1 \rangle$ is a Scott process, and that $\phi \in \Phi^n_\beta$ is such that $\langle \Phi_\alpha : \alpha \leq \beta + n + 1 \rangle$ is injective beyond ϕ. Then $\langle \Phi_\alpha : \alpha \leq \beta + n + 1 \rangle$ has rank at most $\beta + n$.*

Proof. For each $p \leq n$, let Υ_p be the set of $\psi \in \Phi_{\beta+p}$ for which

$$V^{-1}_{\beta+p,\beta+n+1}[\{\psi\}] \cap \Phi_{\beta+n+1}$$

is a singleton. We want to see that $\Upsilon_n = \Phi_{\beta+n}$.

By Remark 5.9, we have the following, for each $p \leq n$:

- $V^{-1}_{\beta,\beta+p}[\{\phi\}]$ has a single element (which we call ϕ_p);
- for each $m \in \omega \setminus n$, Υ_p contains each $\psi \in \Phi^m_{\beta+p}$ for which $\phi_p = H^m_\beta(\psi, j)$ for some $j \in \mathcal{I}_{n,m}$.

We prove the following statement by induction on $p \leq n$: if $k \in \omega$ is such that $k + p \geq n$ and $\theta \in \Phi^k_{\beta+p}$ is such that $\theta = H^{k+p}_{\beta+p}(\rho, i_k)$ for some $\rho \in \Phi^{k+p}_{\beta+p}$ such that $V_{\beta,\beta+p}(\rho) \in \Upsilon_0$, then θ is in Υ_p. For $p = 0$ this is immediate.

For the induction step from p to $p + 1$, fix

- $k \in \omega$ such that $k + p + 1 \geq n$,
- $\theta \in \Phi^k_{\beta+p+1}$ and
- $\rho \in \Phi^{k+p+1}_{\beta+p+1}$

such that $V_{\beta,\beta+p+1}(\rho)$ is in Υ_0 and $\theta = H^{k+p+1}_{\beta+p+1}(\rho, i_k)$. The induction hypothesis gives that

$$V_{\beta+p,\beta+p+1}(H^{k+p+1}_{\beta+p+1}(\rho, i_{k+1})),$$

is in Υ_p, which (as this formula is in $E(\theta)$) by Corollary 5.3 shows that θ is in Υ_{p+1} as desired.

Finally, the statement for $n = p$ implies the proposition, applying condition (2c) of Definition 3.1 to an arbitrary $\theta \in \Phi_{\beta+n}$ and ϕ_n to obtain the desired formula ρ. □

{maxdef}

5.14 Definition. Given an ordinal δ and a set $\Phi \subseteq \Psi_\delta$, the *maximal completion* of Φ is the set of $\phi \in \Psi_{\delta+1}$ such that for some $n \in \omega$ and some $\phi' \in \Phi \cap \Psi^n_\delta$, $V_{\delta,\delta+1}(\phi) = \phi'$, and

$$E(\phi) = \{\psi \in \Phi \cap \Psi^{n+1}_\delta \mid H^{n+1}_\delta(\psi, i_n) = \phi'\}.$$

The extension of a Scott process of successor length by the maximal completion of its last level may not be a Scott process (see Proposition 5.19 below).

{fivepointsix}

5.15 Remark. By Proposition 4.2, if $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, and β is an ordinal such that $\beta + 1 < \delta$, then $V_{\beta,\beta+1} \restriction \Phi_{\beta+1}$ is injective if and only if $\Phi_{\beta+1}$ is the maximal completion of Φ_β.

The following definition describes the situation in which no formula ϕ has incompatible horizontal extensions.

{unidef}

5.16 Definition. Given an ordinal δ, a set $\Phi \subseteq \Psi_\delta$ satisfies the *amalgamation property* (or *amalgamates*) if for all $m < n \in \omega$, $\phi \in \Phi \cap \Psi_\delta^{m+1}$, and $\psi \in \Phi \cap \Psi_\delta^n$ such that $H_\delta^{m+1}(\phi, i_m) = H_\delta^n(\psi, i_m)$, there exist $\theta \in \Phi \cap \Psi_\delta^{n+1}$ and $y \in X_{n+1} \setminus X_m$ such that $H_\delta^{n+1}(\theta, i_m \cup \{(x_m, y)\}) = \phi$ and $H_\delta^{n+1}(\theta, i_n) = \psi$.

{unidef2}

5.17 Remark. Given an ordinal δ and a set $\Phi \subseteq \Phi_\delta$ satisfying condition (1d) of Definition 3.1 (i.e., closure under the functions H_δ^n $(n \in \omega)$), the amalgamation property for a set $\Phi \subseteq \Psi_\delta$ is equivalent to the statement that for all $m \leq n \in \omega$, $\phi \in \Phi \cap \Psi_\delta^m$, $j \in \mathcal{F}_{m,n}$ and $\psi \in \Phi \cap \Psi_\delta^n$ such that $\phi = H_\delta^n(\psi, j)$,

$$\{\theta \in \Phi \cap \Psi_\delta^{m+1} \mid H_\delta^{m+1}(\theta, i_m) = \phi\}$$

is the same as

$$H_\delta^{n+1}[\{\rho \in \Phi \cap \Psi_\delta^{n+1} \mid H_\delta^{n+1}(\rho, i_n) = \psi\} \times \{j \cup \{(x_m, y)\} \mid y \in (X_{n+1} \setminus \mathrm{range}(j))\}].$$

This follows immediately from the definitions (using part (3) of Definition 2.11).

5.18 Remark. The set in the second displayed formula in Remark 5.17 is always contained in the set in the first, by part (2) of Remark 2.14.

{canmax}

Proposition 5.19. *The extension of a nonempty Scott process of nonlimit length by the maximal completion of its last level induces a Scott process if and only if its last level amalgamates.*

Proof. Let $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ be a Scott process. Conditions (1a)-(1c) of Definition 3.1 are always satisfied by the extension by the maximal completion. The other conditions depend on whether the functions $H_{\delta+1}^n$ $(n \in \omega)$ lift the actions of the functions H_δ^n $(n \in \omega)$, i.e., whether whenever $n \in \omega$, $j \in \bigcup_{m \leq n} \mathcal{I}_m$, $\psi \in \Phi_\delta^n$ and ψ' is the unique member of $V_{\delta,\delta+1}^{-1}[\{\psi\}]$ in the maximal completion of Φ_δ, $H_{\delta+1}^n(\psi', j)$ is the unique member of $V_{\delta,\delta+1}^{-1}[\{H_\delta^n(\psi, j)\}]$ in the maximal completion of Φ_δ. Comparing the condition (3) of Definition 2.11 with Definition 5.14 shows that is exactly the statement that Φ_δ amalgamates as expressed in Remark 5.17. $\square$

We conclude this section by giving a restatement of the amalgamation property which will be useful in Section 7. A failure of amalgamation gives a counterexample to Proposition 5.20 with $n = m + 1$.

{amalgamation}

Proposition 5.20. *Suppose that $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is Scott process whose last level amalgamates, and that $m, n, p \in \omega$ are such that $m \leq \min\{n, p\}$. Suppose now that $j \in \mathcal{I}_{m,n}$, $k \in \mathcal{I}_{m,p}$, $\psi \in \Phi^n_\delta$ and $\theta \in \Phi^p_\delta$ are such that*

$$H^n_\delta(\psi, j) = H^p_\delta(\theta, k).$$

Then there exist $q \in \omega \setminus \max\{n, p\}$, a formula $\rho \in \Phi^q_\delta$ and functions $j' \in \mathcal{I}_{n,q}$ and $k' \in \mathcal{I}_{p,q}$ such that $X_q = \mathrm{range}(j') \cup \mathrm{range}(k')$, $j' \circ j = k' \circ k$, $\psi = H^q_\delta(\rho, j')$ and $\theta = H^q_\delta(\rho, k')$.

Proof. Fixing m and p, we prove the proposition by induction on n. If $n = m$, then we can let $q = p$, $\rho = \theta$, $k' = i_p$ and $j' = k \circ j^{-1}$, using the second half of Remark 2.14, which we do repeatedly throughout this proof.

Suppose now that the proposition holds for some $n \in \omega \setminus m$. Let $j \in \mathcal{I}_{m,n+1}$, $k \in \mathcal{I}_{m,p}$, $\psi \in \Phi^{n+1}_\delta$ and $\theta \in \Phi^p_\delta$ be such that $H^{n+1}_\delta(\psi, j) = H^p_\delta(\theta, k)$. Let $f \in \mathcal{I}_{n+1,n+1}$ be i_{n+1} if $x_n \notin \mathrm{range}(j)$; otherwise, fix n' such that $x_{n'} \notin \mathrm{range}(j)$ and let f map x_n and $x_{n'}$ to each other and fix the rest of X_{n+1}. Then $f^{-1} = f$ and $x_n \notin \mathrm{range}(f \circ j)$. Let $\psi_0 = H^{n+1}_\delta(\psi, f)$; then $\psi = H^{n+1}_\delta(\psi, f)$. Let $\psi_1 = H^{n+1}_\delta(\psi_0, i_n)$. By the second part of Remark 2.14,

$$\begin{aligned}
H^n_\delta(\psi_1, f \circ j) &= H^n_\delta(H^{n+1}_\delta(\psi_0, i_n), f \circ j) \\
&= H^{n+1}_\delta(\psi_0, i_n \circ (f \circ j)) \\
&= H^{n+1}_\delta(\psi_0, f \circ j) \\
&= H^{n+1}_\delta(H^{n+1}_\delta(\psi_0, f), j) \\
&= H^{n+1}_\delta(\psi, j) \\
&= H^p_\delta(\theta, k).
\end{aligned}$$

Applying the induction hypothesis to $f \circ j$, k, ψ_1 and θ, we get $q_0 \in \omega \setminus \max\{n, p\}$, a formula $\rho_0 \in \Phi^{q_0}_\delta$ and functions $j_0 \in \mathcal{I}_{n,q_0}$ and $k' \in \mathcal{I}_{p,q_0}$ such that

$$X_{q_0} = \mathrm{range}(j_0) \cup \mathrm{range}(k'),$$

$j_0 \circ (f \circ j) = k' \circ k$, $\psi_1 = H^{q_0}_\delta(\rho_0, j_0)$ and $\theta = H^{q_0}_\delta(\rho_0, k')$.

Suppose first that there exists a $y \in X_{q_0} \setminus \mathrm{range}(j_0)$ such that

$$\psi_0 = H^{q_0}_\delta(\rho_0, j_0 \cup \{(x_n, y)\}).$$

Then q_0, ρ_0 and k' are as desired. If $f = i_{n+1}$, then we can let $j' = j_0 \cup \{(x_n, y)\}$ and we are done. Otherwise, let j' send $x_{n'}$ to y, x_n to $j_0(x_{n'})$ and every other member of X_n to the same place that j_0 does (i.e., let $j' = (j_0 \cup \{(x_n, y)\}) \circ f$). Then $j' \circ j = k' \circ k$, and

$$\begin{aligned}
\psi &= H^{n+1}_\delta(\psi_0, f) \\
&= H^{n+1}_\delta(H^q_\delta(\rho, j_0 \cup \{(x_n, y)\}), f) \\
&= H^q_\delta(\rho, (j_0 \cup \{(x_n, y)\}) \circ f) \\
&= H^q_\delta(\rho, j'),
\end{aligned}$$

as desired.

Finally, suppose that there is no such $y \in X_{q_0} \setminus \mathrm{range}(j_0)$. Putting together the amalgamation property of Φ and the equation $\psi_1 = H^{n+1}_{\delta+1}(\psi_0, i_n)$, we get that there exist a formula $\rho \in \Phi^{q_0+1}_{\delta}$ such that $H^{q_0+1}_{\delta}(\rho, i_{q_0}) = \rho_0$ and a $y \in X_{q_0+1} \setminus \mathrm{range}(j_0)$ such that $H^{q_0+1}_{\delta}(\rho, j_0 \cup \{(x_n, y)\}) = \psi_0$. Then k', ρ, and $q = q_0 + 1$ are as desired. If $f = i_{n+1}$, then we can let $j' = j_0 \cup \{(x_n, y)\}$, and we are done. Otherwise, as above, let $j' = (j_0 \cup \{(x_n, y)\}) \circ f$. Then again $j' \circ j = k' \circ k$ and $\psi = H^q_\delta(\rho, j')$, as desired. □

6 Building countable models

{threadsec}

We say that a τ-structure M is a *model* of a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$ if $\Phi_\alpha = \Phi_\alpha(M)$ for all $\alpha < \delta$. In this section we show that any Scott process of successor length has a countable model if its last level is countable. This in turn implies that such a sequence can be extended to any given ordinal length (although the rank of the Scott process of length ω_1 corresponding to a countable model is countable).

{threaddef}

6.1 Definition. Given an ordinal β, and a countable set $\Phi \subseteq \Psi_\beta$, a *thread* through Φ is a set of formulas $\{\phi_n : n \in \omega\} \subseteq \Phi$ such that

1. for all $n \in \omega$, $\phi_n \in \Psi^n_\beta$; {allin}
2. for all $m < n$ in ω, $\phi_m = H^n_\beta(\phi_n, i_m)$; {goodlift}
3. for all $m \in \omega$, all $\alpha < \beta$, and all $\psi \in E(V_{\alpha+1,\beta}(\phi_m))$, there exist an $n \in \omega \setminus (m+1)$ and a $y \in X_n \setminus X_m$ such that $\psi = V_{\alpha,\beta}(H^n_\beta(\phi_n, i_m \cup \{(x_m, y)\}))$.

6.2 Remark. If β is a successor ordinal, condition (3) of Definition 6.1 is equivalent to the restriction of the condition to the case where $\alpha = \beta - 1$. This follows from Proposition 4.3. Similarly, condition (3) of Definition 6.1 is equivalent to the restriction of the condition to the set of α in any cofinal subset of β.

{cofrestr}

6.3 Remark. Suppose that $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process, and $\beta < \delta$ is such that $V_{\beta,\delta} \restriction \Phi_\delta$ is injective. Then the $V_{\beta+1,\delta}$-preimage of a thread through $\Phi_{\beta+1}$ is a thread through Φ_δ. This follows from Remark 2.9, Proposition 2.15 and Proposition 4.3. For conditions (1) and (2) of Definition 6.1 this is almost immediate; for condition (3) it requires tracing through the horizontal and vertical projections.

The proof of Proposition 6.4 shows how to build thread for a Scott process whose last level is countable. The proof of Theorem 6.5 then shows how use such a thread to build a model of the Scott process.

{threadcon}

Proposition 6.4. *If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process with Φ_δ countable, then there exists a thread through Φ_δ.*

Proof. The case $\delta = 0$ follows easily from condition (1e) of Definition 3.1, so suppose that δ is positive. By Remark 6.3, if suffices to consider the cases where δ is either a successor ordinal or an ordinal of cofinality ω. Let A be $\{\delta - 1\}$ in the case where δ is a successor ordinal, and a countable cofinal subset of δ otherwise. We choose the formulas ϕ_n recursively, meeting instances of condition (3) of Definition 6.1 for $\alpha \in A$ while satisfying condition (2). Note that ϕ_0 is the unique element of Φ^0_δ. To satisfy an instance of condition (3), we need to see that if $m \leq n$ are in ω, $\alpha \in A$, ϕ_n has been chosen, and $\psi \in E(V_{\alpha+1,\beta}(\phi_m))$ is not equal to $V_{\alpha,\beta}(H^n_\beta(\phi_n, i_m \cup \{(x_m, y)\}))$ for any $y \in X_n \setminus X_m$, then ϕ_{n+1} can be chosen so that

$$\psi = V_{\alpha,\beta}(H^{n+1}_\beta(\phi_{n+1}, i_m \cup \{(x_m, x_n)\}))$$

(since Φ_δ is countable, the set of such formulas ψ is also countable). The existence of such a ϕ_{n+1} follows from condition (3) of Definition 2.11 applied to $V_{\alpha+1,\beta}(\phi_n)$ and i_m, giving a $\theta \in E(V_{\alpha+1,\beta}(\phi_n))$ such that

$$H^{n+1}_\alpha(\theta, i_m \cup \{(x_m, x_n)\}) = \psi,$$

followed by condition (2b) of Definition 3.1 applied to ϕ_n, giving ϕ_{n+1} as desired. □

{existmodel}

Theorem 6.5. *Given a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ with Φ_δ countable, a thread $\langle \phi_n : n \in \omega \rangle$ through Φ_δ and an injective sequence $C = \{c_n : n \in \omega\}$, there is a τ-structure with domain C in which each tuple $\langle c_m : m < n \rangle$ satisfies ϕ_n.*

Proof. Let each tuple $\langle c_m : m < n \rangle$ satisfy all the atomic formulas indicated by $V_{0,\delta}(\phi_n)$. We show by induction on $\alpha \leq \delta$ that each tuple $\langle c_m : m < n \rangle$ satisfies the formula $V_{\alpha,\delta}(\phi_n)$. This follows immediately for limit stages. For the induction step from α to $\alpha+1$, $\langle c_m : m < n \rangle$ satisfies $V_{\alpha+1,\delta}(\phi_n)$ if and only if

$$E(V_{\alpha+1,\delta}(\phi_n)) = V_{\alpha,\delta}[\{H^p_\delta(\phi_p, i_n \cup \{(x_n, y)\}) : p \in \omega \setminus (n+1), y \in X_p \setminus X_n\}].$$

That is, checking that $\langle c_m : m < n \rangle$ satisfies $V_{\alpha+1,\delta}(\phi_n)$ means showing that the left side of the equality is the set of formulas from Φ^{n+1}_α satisfied by extensions of $\langle c_m : m < n \rangle$ by one point, which by the induction hypothesis is what the right side is. The left-to-right containment follows from condition (3) of Definition 6.1. For the other direction, note first that by Proposition 4.4,

$$E(V_{\alpha+1,\delta}(\phi_n)) = V_{\alpha,\delta}[\{\theta \in \Phi^{n+1}_\delta \mid H^{n+1}_\delta(\theta, i_n) = \phi_n\}].$$

That

$$\{H^p_\delta(\phi_p, i_n \cup \{(x_n, y)\}) : p \in \omega \setminus (n+1), y \in X_p \setminus X_n\}$$

is contained in $\{\theta \in \Phi^{n+1}_\delta \mid H^{n+1}_\delta(\theta, i_n) = \phi_n\}$ follows from the assumption that $\phi_n = H^p_\delta(\phi_p, i_n)$, by part (2) of Remark 2.14. □

{cthreaddef}

6.6 Definition. Given an ordinal β, and a countable set $\Phi \subseteq \Psi_\beta$, a thread $\{\phi_n : n \in \omega\}$ through Φ is *complete* if for all $m \in \omega$ and all $\psi \in \Phi \cap \Psi^m_\beta$, there exist $n \in \omega$ and $j \in \mathcal{I}_{m,n}$ such that $\psi = H^n_\beta(\phi_n, j)$.

{cthreadrem}

6.7 Remark. The thread through Φ_δ given by Proposition 6.4 induces (via Theorem 6.5) a model of $\langle \Phi_\alpha : \alpha < \delta \rangle$ for which the δ-th level of the corresponding Scott process is contained in the given Φ_δ. The δ-th level is equal to Φ_δ if and only if the thread is complete. Condition (2c) of Definition 3.1 implies that one can add stages to the construction in Proposition 6.4 to produce a complete thread.

Proposition 6.4, Theorem 6.5 and Remark 6.7 give the following.

{ctblehasmodel}

Theorem 6.8. *Every Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ with Φ_δ countable has a countable model.*

6.9 Remark. Theorem 9.9 gives a stronger version of Theorem 6.8, showing that every Scott process with all levels countable (and possibly of limit length) has a model.

6.10 Remark. If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process, $\gamma < \delta$ and $\{\phi_n : n \in \omega\}$ is a thread through Φ_δ, then $\{V_{\gamma,\delta}(\phi_n) : n \in \omega\}$ is a thread through Φ_γ (this follows from Proposition 2.15). This thread induces (as in the proof of Theorem 6.5) the same class-length Scott process as $\{\phi_n : n \in \omega\}$.

We insert here two arguments for constructing pairs of models. With respect to Theorem 6.11, note that (by Theorem 6.8 and Scott's Isomporphism Theorem) if $\mathcal{P} = \langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that Φ_δ is countable and amalgamates, there is exactly one model of $\mathcal{P}$ of Scott rank at most δ, up to isomorphism. Whether or not the Scott rank of this model is less than δ depends on whether or not $\mathcal{P}$ terminates.

{extendtofull}

Theorem 6.11. *Let $\mathcal{P} = \langle \Phi_\alpha : \alpha \leq \delta \rangle$ be a Scott process such that Φ_δ is countable and amalgamates, and let M be a countable τ-structure such that every finite tuple from M satisfies a member of Φ_δ. Then there is a countable τ-structure N, modeling $\mathcal{P}$, such that M is a quantifier-depth-δ-elementary substructure of N, and N has Scott rank at most δ.*

Proof. Let $\langle c_n : n \in \omega \rangle$ be an enumeration of the domain of M. By Theorem 6.5, it suffices to find a thread $\{\phi_n : n \in \omega\}$ through Φ_δ and an infinite set $Y \subseteq \omega$ such that, for each $n \in \omega$,

{extendone}

1. letting
    - j_n be the order preserving map from X_n to the first n elements of the set $\{x_m : m \in Y\}$, and
    - k_n be the least element of ω such that $|Y \cap k_n| = n$,

   $H^{k_n}_\delta(\phi_{k_n}, j_n) = \phi^M_{\langle c_0,\ldots,c_{n-1}\rangle,\delta}$;

{extendtwo}

2. for each $\psi \in \Phi^{n+1}_\delta$ such that $H^{n+1}_\delta(\psi, i_n) = \phi_n$, there exist $m > n$ in ω and $y \in X_m \setminus X_n$ such that $H^m_\delta(\phi_m, i_n \cup \{(x_n, y)\}) = \psi$.

Condition (2), and the assumption that Φ_δ amalgamates, will then give the following:

- $\{\phi_n : n \in \omega\}$ is complete (one can show by induction on m, for instance, that
$$\Phi_\delta \cap \Psi^m_\delta = \{H^n_\delta(\phi_n, j) : n \in \omega \setminus m, j \in \mathcal{I}_{m,n}\}$$
for all $m \in \omega$);
- the model given by Theorem 6.8 will have Scott rank δ, as the induced $(\delta + 1)$-st level of the Scott process of N will be the maximal completion of Φ_δ.

Then $\{\phi_n : n \in \omega\}$ induces the desired τ-structure N, and the set Y induces the desired copy of M inside N.

We start (as we must) with ϕ_0 as the unique element of Φ^0_δ. In our construction, we alternate stages for putting new elements in Y (while preserving condition (1)) and meeting condition (2). At each stage we will have chosen ϕ_p and decided $Y \cap p$ for some $p \in \omega$. As we construct, we maintain the following condition (*) : for each $p \in \omega$, if ϕ_p and $Y \cap p$ have been chosen, and $|Y \cap p| = n$, then there do not exist $y \in X_p \setminus \{x_m : m \in Y \cap p\}$ and $c \in M \setminus \{c_0, \ldots, c_{n-1}\}$ such that $H^p_\delta(\phi_n, j_n \cup \{(x_n, y)\}) = \phi^M_{\langle c_0,\ldots,c_{n-1},c\rangle,\delta}$. As long as well do this, our assumption that Φ_δ amalgamates implies that we can choose ϕ_{n+1} in such a way that we can put $n \in Y$ and maintain condition (1). To meet condition (2), suppose that $p \in \omega$ is maximal such that ϕ_p and $Y \cap p$ have been chosen, and let ψ be given as in condition (2). Again, since Φ_δ amalgamates, we may assume that $H^{p+1}_\delta(\psi, i_p) = \phi_p$ (that is, to meet the condition for some $n \leq p$ we can meet it for p). If possible (while maintaining condition (1)), we satisfy this instance of condition (2) with a formula $\phi_q \in \Phi^q_\delta$ (for some $q > p$) while putting all of $q \setminus p$ in Y. If this is not possible, then we can let ϕ_{p+1} be ψ, and condition (*) is preserved. $\square$

The proof of the following theorem is similar, but we assume a weaker amalgamation property. Given an ordinal γ and sets $\Upsilon \subseteq \Phi \subseteq \Psi_\gamma$, say that Υ *weakly amalgamates with respect to* Φ if whenever

- m, n and p are in ω, with $m \leq n$,
- ϕ is in $\Phi \cap \Psi^n_\gamma$,
- $k \in \mathcal{I}_{m,n}$ is such that $H^n_\gamma(\phi, k) \in \Upsilon$ and
- ψ is in $\Upsilon \cap \Psi^p_\gamma$,

there exist $\theta \in \Phi \cap \Psi^{n+p}_\gamma$, $q \leq m + p$, $j \in \mathcal{I}_{p,n+p}$ and $k' \in \mathcal{I}_{q,n+p}$ extending k such that

- $H^{n+p}_{\gamma}(\theta, i_n) = \phi$,
- $H^{n+p}_{\gamma}(\theta, k') \in \Upsilon$ and
- $H^{n+p}_{\gamma}(\theta, j) = \psi$.

If Φ amalgamates, then it weakly amalgamates with respect to itself. The issue of extending Theorem 6.12 (or the weak version of it where Φ_γ is assumed to amalgamate) to uncountable models is discussed in Remark 7.15.

{addthis}

Theorem 6.12. *Let γ be a countable ordinal, and suppose that $\langle \Phi_\beta : \beta \leq \gamma \rangle$ is a Scott process with Φ_γ countable. Let Φ^* be a subset of Φ_γ such that the extension of $\langle \Phi_\beta : \beta < \gamma \rangle$ by Φ^* is also a Scott process, and Φ^* weakly amalgamates with respect to Φ_γ. Then there exists τ-structures M and N such that M is a substructure of N, N is a model of $\langle \Phi_\beta : \beta \leq \gamma \rangle$ and M is a model of the extension of $\langle \Phi_\beta :< \gamma \rangle$ by Φ^*.*

Proof. By Theorem 6.5, it suffices to find a complete thread $\bar{\phi} = \langle \phi_n : n \in \omega \rangle$ through Φ_γ and an infinite set $Y \subseteq \omega$ such that, letting, for each $n \in \omega$,

- j_n be the order preserving map from X_n to the first n elements of the set $\{x_m : m \in Y\}$,
- k_n be the least element of ω such that $|Y \cap k_n| = n$,

$\langle H^{k_n}_{\gamma}(\phi_{k_n}, j_n) : n \in \omega \rangle$ is a complete thread through Φ^*.

A construction of such a pair $\langle \phi_n : n \in \omega \rangle$, Y can be carried out in essentially the same manner as the proof of Theorem 6.4, recursively putting n into Y whenever $H^n_\gamma(\phi_n, j_{|Y \cap n|} \cup \{(x_{|Y \cap n|}, x_n\})$ is a member of Φ^*. Again, we let ϕ_0 be the unique member of Φ_0. The only new issue is the completeness of the two threads being constructed. For $\bar{\phi}$, completeness can be achieved using condition (2c) of Definition 3.1. That is, having chosen ϕ_n (and thus $Y \cap (n+1)$), and given some $\psi \in \Phi^m_\gamma$, we let ϕ_{n+m} be an element of Φ_γ (as given by Condition 2c of Definition 3.1) such that $H^{n+m}_\gamma(\phi_{n+m}, i_n) = \phi_n$ and $H^{n+m}_\gamma(\phi_{n+m}, j) = \psi$, for some $j \in \mathcal{I}_{m,n+m}$. We then use our recursive rule to decide $Y \cap (n+1, n+m)$.

The notion of weak amalgamation of Φ^* with respect to Φ_γ was defined to make the same argument work for the sequence $\langle H^{k_n}_\gamma(\phi_{k_n}, j_n) : n \in \omega \rangle$. Here we again have ϕ_n and $Y \cap (n+1)$, we are given a $\psi \in \Phi^* \cap \Psi^p_\gamma$, for some $p \in \omega$, and we want to find a formula $\phi_{n+p} \in \Phi^{n+p}_\gamma$ and an extension of $Y \cap (n+1)$ to $Y \cap (n+p+1)$ of size at least p such that $\psi = H^{n+m}_\gamma(\phi_{n+m}, k_p)$. Our recursive condition on Y guarantees that for the extension k' (the desired k_p) given by weak amalgmatation, the range of $k' \setminus k$ (where k is $k_{|Y \cap (n+1)|}$) is contained in $X_{n+p} \setminus X_n$, so that we can extend Y as desired. □

7 Models of cardinality $\aleph_1$

{arbcardsec}

In this section we show how to build models for Scott processes of length a successor ordinal, under the assumption that the last level of the process amalgamates and has cardinality at most $\aleph_1$.

Given two finite sets of ordinals $a \subseteq b$ with $a = \{\alpha_0, \ldots, \alpha_{n-1}\}$ (listed in increasing order), let $j_{a,b}$ be the function j in $\mathcal{I}_{n,|b|}$ such that $j(x_m) = x_{|b \cap \alpha_m|}$ for all $m < n$.

In the case $\gamma = \omega$, the following definition is essentially the same as Definition 6.1, as the formulas $\{\phi_n : n \in \omega\}$ of the weaving then satisfy Definition 6.1.

{threaddef2}

7.1 Definition. Suppose that δ is an ordinal and Φ is a subset of Ψ_δ. A *weaving* through Φ is a set of formulas $\{\phi_a : a \in [\gamma]^{<\omega}\} \subseteq \Phi$, for some infinite ordinal γ, such that the following hold.

{basic}

1. Each $\phi_a \in \Psi^{|a|}_\delta$.

{allin2}

2. For all $a \subseteq b \in [\gamma]^{<\omega}$, $\phi_a = H^{|b|}_\delta(\phi_b, j_{a,b})$.

{goodlift2}

3. For all $a \in [\gamma]^{<\omega}$, all $\alpha < \delta$, and all $\psi \in E(V_{\alpha+1,\delta}(\phi_a))$, there exist a $b \in [\gamma]^{|a|+1}$ containing a and a $y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})$ such that
$$\psi = V_{\alpha,\delta}(H^{|a|+1}_\delta(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\})).$$

The proof of Theorem 7.2 is an adaptation of the proof of Theorem 6.5.

{existmodel2}

Theorem 7.2. *Given a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$, an infinite ordinal γ, a weaving $\langle \phi_a : a \in [\gamma]^{<\omega} \rangle$ through Φ_δ and an injective sequence $C = \langle c_\alpha : \alpha < \gamma \rangle$, there is a τ-structure with domain C in which, for each $a \in [\gamma]^{<\omega}$, the tuple $\langle c_\alpha : \alpha \in a \rangle$ satisfies ϕ_a.*

Proof. For each $a \in [\gamma]^{<\omega}$, let the tuple $\langle c_\alpha : \alpha \in a \rangle$ satisfy all the atomic formulas indicated by $V_{0,\delta}(\phi_a)$. We show by induction on $\beta < \delta$ that each tuple $\langle c_\alpha : \alpha \in a \rangle$ satisfies the formula $V_{\beta,\delta}(\phi_a)$. This follows immediately for limit stages. For the induction step from β to $\beta + 1$, $\langle c_\alpha : \alpha \in a \rangle$ satisfies $V_{\beta+1,\delta}(\phi_a)$ if and only if $E(V_{\beta+1,\delta}(\phi_a))$ is equal to
$$V_{\beta,\delta}[\{H^{|b|}_\delta(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\}) : a \subseteq b \in [\gamma]^{<\omega}, y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})\}].$$
The left-to-right containment follows from condition (3) of Definition 7.1. For the other direction, note first that by Proposition 4.4,
$$E(V_{\beta+1,\delta}(\phi_a)) = V_{\beta,\delta}[\{\theta \in \Phi^{|a|+1}_\delta \mid H^{|a|+1}_\delta(\theta, i_{|a|}) = \phi_a\}].$$
That
$$\{H^{|b|}_\delta(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\}) : a \subseteq b \in [\gamma]^{<\omega}, y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})\}$$
is contained in $\{\theta \in \Phi^{|a|+1}_\delta \mid H^{|a|+1}_\delta(\theta, i_{|a|}) = \phi_a\}$ follows from condition (2) of Definition 7.1 and part (2) of Remark 2.14. □

{cweavedef}

7.3 Definition. Suppose that δ is an ordinal, γ is an infinite cardinal and Φ is a subset of Ψ_δ. A weaving $\{\phi_a : a \in [\gamma]^{<\omega}\}$ through Φ is *complete* if for all $n \in \omega$ and all $\psi \in \Phi \cap \Psi^n_\delta$, there exist $a \in [\gamma]^n$ and $j \in \mathcal{I}_{n,n}$ such that $\psi = H^n_\delta(\phi_a, j)$;

{cweaverem}

7.4 Remark. As in Remark 6.7, given a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ and a weaving through Φ_δ, the proof of Theorem 7.2 gives a model of $\langle \Phi_\alpha : \alpha < \delta \rangle$, for which the δ-th level of its Scott process is contained in the given Φ_δ. The δ-th level is equal to Φ_δ if and only if the weaving is complete.

Proposition 7.10 below show that if $\mathcal{P} = \langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that Φ_δ amalgamates and has cardinality $\aleph_1$, then there exists a complete weaving through $\mathcal{P}$. To simplify the proof, we introduce a notion of strong weaving (at the cost of limiting the set of models of $\mathcal{P}$ we can construct, see Remark 7.8).

{sweavingdef}

7.5 Definition. Suppose that δ is an ordinal and Φ is a subset of Ψ_δ. A *strong weaving* through Φ is a set

$$\{\phi_a : a \in [\gamma]^{<\omega}\} \subseteq \Phi,$$

for some infinite ordinal γ, satisfying conditions (1) and (2) of Definition 7.1 plus the following condition: for all $a \in [\gamma]^{<\omega}$, and all $\psi \in \Phi \cap \Psi_\delta^{|a|+1}$ such that $H_\delta^{|a|+1}(\psi, i_{|a|}) = \phi_a$, there exist a $b \in [\gamma]^{|a|+1}$ containing a and a $y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})$ such that

$$\psi = H_\delta^{|a|+1}(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\})).$$

7.6 Remark. In condition (3) of Definition 7.1 and in Definition 7.5, the variable y is in fact the unique member of $X_{|b|} \setminus \mathrm{range}(j_{a,b})$.

{betterpick}

Proposition 7.7. *Suppose that δ is an ordinal and Φ is a subset of Ψ_δ. Every strong weaving through Φ is both a weaving and complete.*

Proof. That a strong weaving satisfies condition (3) of Definition 7.1 follows from condition (2b) of Definition 3.1. Completeness for formulas in $\Phi \cap \Psi_\delta^n$ follows by induction on n. □

{limitremark}

7.8 Remark. Let $\mathcal{P} = \langle \Phi_\alpha : \alpha \leq \delta \rangle$ be a Scott process. If there is a strong weaving through Φ_δ, then Φ_δ amalgamates. Moreover, the τ-structures induced by strong weavings through Φ_δ (as in the proof of Theorem 7.2) are, up to isomorphism, the models of $\mathcal{P}$ of and Scott rank at most δ. This is in contrast to threads and weavings : every model of a Scott process $\mathcal{P}$ of successor length is induced (up to isomorphism) by weaving through the last level of $\mathcal{P}$.

A subset S of a collection C of sets is $\subseteq$-*cofinal* in C if every member of C is contained in a member of S.

{cofstrong}

Proposition 7.9. *Suppose that $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that Φ_δ amalgamates. Let $\mathcal{W} = \{\phi_a : a \in [\gamma]^{<\omega}\}$ be a subset of Φ satisfying conditions (1) and (2) of Definition 7.1, such that the set of $a \in [\gamma]^{<\omega}$ for which the condition in Definition 7.5 is satisfied is $\subseteq$-cofinal in $[\gamma]^{<\omega}$. Then $\mathcal{W}$ is a strong weaving.*

Proof. Suppose that we have $a \subseteq b \in [\gamma]^{<\omega}$, and that the condition in Definition 7.5 holds for b. Suppose that $\psi \in \Phi_\delta^{|a|+1}$ is such that $H_\delta^{|a|+1}(\psi, i_{|a|}) = \phi_a$. By Proposition 5.20, there is formula $\theta \in \Phi_\delta^{|b|+1}$ such that $H_\delta^{|b|+1}(\theta, i_{|b|}) = \phi_b$ and $H_\delta^{|b|+1}(\theta, j_{a,b} \cup \{(x_{|a|}, x_{|b|})\}) = \psi$. Then there exist a $\beta \in \gamma \setminus b$ a $y \in X_{|b|+1} \setminus \mathrm{range}(j_{b,b\cup\{\beta\}})$ such that

$$\theta = H_\delta^{|b|+1}(\phi_{b\cup\{\beta\}}, j_{b,b\cup\{\beta\}} \cup \{(x_{|b|}, y)\}),$$

which implies that

$$\psi = H_\delta^{|a|+1}(\phi_{a\cup\{\beta\}}, j_{a,a\cup\{\beta\}} \cup \{(x_{|a|}, y)\}),$$

for y the unique element of $X_{|a|+1} \setminus \mathrm{range}(j_{a,a\cup\{\beta\}})$. □

The proof of Proposition 7.10 implements the final part of Harrington's argument as it appears in Marker's slides [14].

{existweaving}

Proposition 7.10. *If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that Φ_δ amalgamates and has cardinality $\aleph_1$, then there is a strong weaving through Φ_δ.*

Proof. We recursively pick suitable formulas ϕ_a, for $a \in [\omega_1]^{<\omega}$. To begin with, let ϕ_n ($n \in \omega$) be any elements of Φ_δ with the property that $H_\delta^n(\phi_n, i_m) = \phi_m$, for all $m \leq n < \omega$. Suppose now that we have $\alpha < \omega_1$ and that ϕ_a has been chosen for each finite subset of α (note that a choice of ϕ_a determines a choice of ϕ_b for each subset of b, where a is a finite subset of ω_1) and no other subsets of ω_1. Using some wellordering of $[\omega_1]^{<\omega} \times \Phi_\delta$ in ordertype ω_1, we fix the least pair $a \in [\alpha]^{<\omega}$, $\psi \in \Phi_\delta^{|a|+1}$ as in Definition 7.5 for which the corresponding condition has not been met (which must exist since Φ_δ is uncountable), and let $\phi_{a\cup\{\alpha\}}$ be this ψ. Fixing a bijection $\pi\colon \omega \to (\alpha \setminus a)$, we now successively choose the formulas $\phi_{a\cup\{\alpha\}\cup\pi[n]}$. For each positive n, the choice of $\phi_{a\cup\{\alpha\}\cup\pi[n]}$ requires amalgamating $\phi_{a\cup\{\alpha\}\cup\pi[n-1]}$ with $\phi_{a\cup\pi[n]}$, which have already been chosen. The fact that Φ_δ amalgamates (via Proposition 5.20) implies that there exists a suitable choice for $\phi_{a\cup\{\alpha\}\cup\pi[n]}$. Since $\phi_{a\cup\{\pi(n-1)\}}$ did not satisfy the third condition of Definition 7.5 with respect to a and ψ, this choice of $\phi_{a\cup\{\alpha\}\cup\pi[n]}$ does not require identifying $\pi(n-1)$ and α. Proceeding in this fashion completes the construction of the desired strong weaving. □

Putting together Theorems 6.8 (for the case where Φ_δ is countable) and 7.2 with Propositions 7.7 and 7.10, we have the following.

{omega1model}

Theorem 7.11. *If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process, Φ_δ amalgamates and $|\Phi_\delta| \leq \aleph_1$, then $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has a model of Scott rank at most δ.*

{noomega2}

7.12 Remark. One might naturally try to adapt the proof of Theorem 7.2 to build a model of size $\aleph_2$ by assigning a formula from Φ_δ to each finite tuple from ω_2. Doing this in the manner of the proof of Theorem 7.2, one finds oneself with an uncountable $\alpha < \omega_2$ such that formulas have been assigned for all finite subsets of α, but not for $\{\alpha\}$. Choosing formulas for all finite subsets of $\alpha + 1$, one comes to a point where, for some countably infinite $B \subseteq \alpha$, formulas have been chosen for all sets of the form $\{\alpha\} \cup b$, for b a finite subset of B. Then, for some $\beta \in \alpha \setminus B$, one would like to chose a formula for some finite superset c of $\{\alpha, \beta\}$ intersecting B. Finally, consider $\gamma \in B \setminus c$. We have at this point that formulas have been chosen for $\{\alpha, \gamma\}$, $\{\beta, \gamma\}$ and c, but not for $\{\alpha, \beta, \gamma\}$, and our assumptions do not give us suitable choice for $\{\alpha, \beta, \gamma\}$ that extends the choices already made. One can naturally define a notion of 3-amalgamation such that this construction could succeed under the assumption that this property holds.

{namalgremark} **7.13 Remark.** Given an ordinal δ and $n \in \omega \setminus 2$, say that a set $\Phi \subseteq \Psi_\delta$ *n-amalgamates* if for all $m \in \omega$ and all

$$\{\phi_a : a \in [(m+n) \setminus m]^{n-1}\} \subseteq \Phi \cap \Psi_\delta^{m+n-1},$$

if (using the notation $j_{a,b}$ from the beginning of this section, restricted to finite subsets of ω),

$$H_\delta^{m+n-1}(\phi_a, j_{m \cup (a \cap b), m \cup a}) = H_\delta^{m+n-1}(\phi_b, j_{m \cup (a \cap b), m \cup b})$$

for all $a, b \in [(m+n) \setminus m]^{n-1}$ then there exists $\theta \in \Phi \cap \Psi_\delta^{m+n}$ such that

$$H_\delta^{m+n}(\theta, j_{m \cup a, m+n}) = \phi_a$$

for all a. The proof of Proposition 7.10 then gives that for all $n \in \omega \setminus 2$ and all Scott processes

$$P = \langle \Phi_\alpha : \alpha \leq \delta \rangle$$

such that $|\Phi_\delta| \leq \aleph_{n-1}$, if Φ_δ n-amalgamates then $\mathcal{P}$ has a model of cardinality at most $\aleph_{n-1}$ with Scott rank at most δ. We leave the details to the interested reader, as well as the verification that 2-amalgamation is equivalent to amalgamation.

Theorem 7.14. *If $P = \langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that, for all $n \in \omega \setminus 2$, Φ_δ n-amalgamates, then P has a model of cardinality $|\Phi_\delta|$ and Scott rank at most δ*

Proof. Let $\kappa = |\Phi_\delta|$. We build a strong weaving $\{\phi_a : a \in [\gamma]^{<\omega}\}$ through Φ_δ, for some ordinal $\gamma \in [\kappa, \kappa^+)$, where the value of γ is determined by the construction. At each stage of our construction, we will have chosen formulas $\phi_a \in \Phi_\delta^{|a|}$ for all a in some subset of $[\eta]^{<\omega}$ closed under subsets, for some $\eta < \kappa^+$, satisfying condition (2) of Definition 7.1. As always, we let $\phi_{\langle\rangle}$ be the unique member of Φ_δ^0. We have tasks of two types:

- Choosing a formula for each nonempty $a \in [\eta]^{<\omega}$, once we have chosen formula for each element of $[a]^{|a|-1}$. This we can do by $|a|$-amalgamation (with $m = 0$).
- Satisfying Definition 7.5. When doing this for a given $a \in [\eta]^{<\omega}$ (for which ϕ_a has been chosen) and $\psi \in \Phi_\delta^{|a|+1}$ (for which $H_\delta^{|a|+1}(\psi, i_{|a|}) = \phi_a$), we choose the corresponding b from $[\eta]^{|a|+1}$ if possible (given the choices already made). If this is not possible, we let $\phi_{a \cup \{\eta\}} = \psi$ (and increase η by one).

The ordinal γ will then be the supremum of the values of η during the construction. $\square$

Say that a set $\Phi \subseteq \Psi_\delta$ is *rigid* if there is no $\phi \in \Phi \cap \Psi^2_\delta$ such that $H^2_\delta(\phi, i_1) = H^2_\delta(\phi, \{(x_0, x_1)\})$. Intuitively, this says that no two distinct points satisfy the same formula at level δ. A countable τ-structure M of Scott rank δ is rigid in the usual sense if and only if $\Phi_\delta(M)$ is rigid in this sense. If Φ is rigid then Φ satisfies n-amalgamation for all $n \in \omega$. A question of Arnie Miller asks : can there exist a ϕ in $\mathcal{L}_{\aleph_1,\aleph_0}$ with uncountably many rigid models but not perfectly many? Remark 7.13 shows that such a ϕ would have models of all infinite cardinalities. Part (4) of Theorem 11.1 shows that it would have non-rigid models as well. One could ask similar questions for n-amalgamation.

{addthisrem}

7.15 Remark. The natural attempt to combine the proofs of Theorem 6.12 (in the simplified case where Φ^* satisfies amalgamation) and Proposition 7.10 to produce a version of Theorem 6.12 for models of size $\aleph_1$ runs into a problem similar to the one in Remark 7.12. In this case, we have a Scott process $\langle \Phi_\alpha : \alpha \leq \beta \rangle$, for some $\beta \in [\omega_1, \omega_2)$ such that, letting Φ^* be the set of isolated threads in Φ_β,

- Φ^* is a proper subset of Φ_β,
- the extension of $\langle \Phi_\alpha : \alpha < \beta \rangle$ by Φ^* gives a Scott process.

We could then try to build a strong weaving $\{\phi_a : a \in [\omega_1]^{<\omega}\}$ through Φ_β, and an an uncountable set $Y \subseteq \omega_1$ such that $\{\phi_a : a \in [Y]^{<\omega}\}$ is a strong weaving through Φ^* (or, more precisely, induces one via some bijection between Y and ω_1). Carrying out this construction, we come to a point where, for some infinite $\gamma < \omega_1$, ϕ_a has been chosen for every finite subset of γ, and for $\{\gamma\} \cup a$, for some finite $a \subseteq \gamma$ intersecting Y as so far constructed, but not contained in it. At some stages it will also be that this γ has been put into Y. Now suppose that δ is in $Y \cap \gamma$, as constructed so far, but that no formula for $\{\delta, \gamma\}$ has been chosen. Then we need to choose a formula for $a \cup \{\delta, \gamma\}$ such that the induced formula for $(a \cap Y) \cup \{\delta, \gamma\}$ is in Φ^*. Since Φ_δ amalgamates, we can choose a formula for $a \cup \{\delta, \gamma\}$, but we can't guarantee that the induced formula for $(a \cap Y) \cup \{\delta, \gamma\}$ will be in Φ^*. Similarly, since Φ^* amalgamates we can choose a formula for $(a \cap Y) \cup \{\delta, \gamma\}$ in Φ^*. Then we have the same 3-amalgamation issue as in Remark 7.12, as we would then need to amalgamate the chosen formulas for $(a \cap Y) \cup \{\delta, \gamma\}$, $a \cup \{\delta\}$ and $a \cup \{\gamma\}$ in Φ_β.

8 Finite existential blocks

{qsec}

The function E defined in Definition 2.4 corresponds to a single existential quantifier. In this section we extend E to the function F which corresponds to finite blocks of existential quantifiers. The analysis of F in this section is used in the following section. Most of this section consists of consequences of Proposition 8.5, which gives an alternate characterization of F.

{qdef}

8.1 Definition. For each ordinal β, each $m \in \omega$ and each $\phi \in \Psi^m_\beta$, $F(\phi)$ is the set of ψ such that for some $n \in \omega$ and some ordinal α with $\alpha+n \leq \beta$, $\psi \in \Psi^{m+n}_\alpha$ and there exist $\psi_0, \ldots, \psi_n$ such that

- $\psi_0 = \psi$;
- for all $p \in \{0, \ldots, n-1\}$, $\psi_p \in E(\psi_{p+1})$;
- $\psi_n = V_{\alpha+n,\beta}(\phi)$.

8.2 Remark. Suppose that α, β, m, ϕ and $\psi_0, \ldots, \psi_n$ are as in Definition 8.1. Then by condition (1b) of Definition 3.1, each ψ_i is in $\Psi^{m+n-i}_{\alpha+i}$.

{fshift}

8.3 Remark. Given α, β, n, ϕ and ψ as in Definition 8.1, the issue of whether or not ψ is in $F(\phi)$ depends only on $V_{\alpha+n,\beta}(\phi)$ (as opposed to ϕ). It follows that if $\psi \in F(\phi)$ then $\psi \in F(\theta)$ for any formula $\theta \in \Psi^m_\gamma$ (for some ordinal $\gamma \geq \alpha+n$) such that $V_{\alpha+n,\gamma}(\theta) = V_{\alpha+n,\beta}(\phi)$.

8.4 Remark. An iterated application of condition (1b) of Definition 3.1 gives that if $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, $\alpha < \beta$ are in δ, $\phi \in \Phi_\beta$ and $\psi \in F(\phi)$, then $\psi \in \Phi_\beta$.

Fix for rest of this section a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$.

{qlem}

Proposition 8.5. *Suppose that $m, n \in \omega$ and $\alpha, \beta < \delta$ are such that $\alpha+n \leq \beta$. Let ϕ and ψ be elements of Φ^m_β and Φ^{m+n}_α, respectively. Then $\psi \in F(\phi)$ if and only if there is a formula $\theta \in \Phi^{m+n}_\beta$ such that $H^{m+n}_\beta(\theta, i_m) = \phi$ and $V_{\alpha,\beta}(\theta) = \psi$.*

Proof. By induction on n. In the case $n = 1$, $\psi \in F(\phi)$ if and only if $\psi \in E(V_{\alpha+1,\beta}(\phi))$. In this case, the proposition is Proposition 4.4. The induction step from $n = p$ to $n = p+1$ follows from the induction hypothesis in the cases $n = p$ and $n = 1$ (applied twice). ☐

Remarks 8.6 and 8.7 and Propositions 8.8, 8.9 and 8.10 list several useful properties of the function F.

{flat}

8.6 Remark. Applying Propositions 2.15 and 8.5, and condition (1e) of Definition 3.1, we get that if $m, n, p \in \omega$ and $\alpha, \beta < \delta$ are such that $\alpha+n+p \leq \beta$, and if $\phi \in \Phi^m_\beta$, then for each $\psi \in \Phi^{m+n}_\alpha \cap F(\phi)$ there exists a $\rho \in \Phi^{m+n+p}_\alpha \cap F(\phi)$ such that $H^{m+n+p}_\alpha(\rho, i_{m+n}) = \psi$.

{permutef}

8.7 Remark. Fix $m, n \in \omega$ and suppose $\alpha, \beta < \delta$ are such that $\alpha + n \leq \beta$. Let ϕ be an element of Φ^{m+n}_β, let f be an element of $\mathcal{I}_{m,m+n}$ and let g be any element of $\mathcal{I}_{m+n,m+n}$ extending f. Then, by Proposition 8.5 and part (2) of Remark 2.14, $V_{\alpha,\beta}(H^{m+n}_\beta(\phi, g))$ is in $F(H^{m+n}_\beta(\phi, f))$.

Proposition 8.8 follows from Proposition 8.5 and Remark 2.8. It shows, for suitable ϕ and α, that $F(\phi)$ is closed under $V_{\gamma,\alpha}$ for all $\gamma \leq \alpha$.

{qdownshift}

Proposition 8.8. *Fix $\phi \in \Phi^m_\beta$, for some $\beta < \delta$ and $m \in \omega$. Let $\psi \in \Phi^{m+n}_\alpha$ be an element of $F(\phi)$, for some $n \in \omega$ and some ordinal α with $\alpha + n \leq \beta$. Then for all $\gamma < \alpha$, $V_{\gamma,\alpha}(\psi) \in F(\phi)$.*

Propositions 2.15 and 8.5 imply that members of $F(\phi)$ project horizontally to vertical projections of ϕ.

{fhorproj}

Proposition 8.9. *Suppose that $\alpha < \beta < \delta$, $m \leq n \in \omega$, $\phi \in \Phi^m_\beta$ and $\psi \in \Phi^n_\alpha \cap F(\phi)$. Then $H^n_\alpha(\psi, i_m) = V_{\alpha,\beta}(\phi)$.*

Proposition 8.10 is used in the proof of Theorem 9.9.

{qlift}

Proposition 8.10. *Suppose that $m, n \in \omega$, $\alpha < \beta$ are such that $\beta + n < \delta$, $\phi \in \Phi^m_{\beta+n}$ and $\psi \in \Phi^{m+n}_\alpha \cap F(\phi)$. Then there exists a $\psi' \in V^{-1}_{\alpha,\beta}[\{\psi\}] \cap F(\phi)$.*

Proof. By Proposition 8.5, there is a $\theta \in \Phi^{m+n}_{\beta+n}$ such that $V_{\alpha,\beta+n}(\theta) = \psi$ and $H^{m+n}_{\beta+n}(\theta, i_m) = \phi$. By Proposition 8.5 again, $V_{\beta,\beta+n}(\theta) \in F(\phi)$. □

Proposition 8.11 is a version of condition (2c) of Definition 3.1 for $F(\phi)$.

{qext}

Proposition 8.11. *For all $m, n, p \in \omega$, all $\alpha, \beta < \delta$ such that $\beta \geq \alpha+n+p$, and all $\phi \in \Phi^m_\beta$, $\psi \in \Phi^{m+n}_\alpha \cap F(\phi)$ and $\theta \in \Phi^{m+p}_\alpha \cap F(\phi)$, there exist $j \in \mathcal{I}_{m+p,m+n+p}$ and $\rho \in \Phi^{m+n+p}_\alpha \cap F(\phi)$ such that*

- *$j \circ i_m = i_m$;*
- *$H_\alpha(\rho, i_{m+n}) = \psi$;*
- *$H_\alpha(\rho, j) = \theta$.*

Proof. This can be proved by induction on p, for all m and n simultaneously. In the case where $p = 0$ there is nothing to show (since then $\theta = V_{\alpha,\beta}(\phi)$), so suppose that p is positive. Since $\theta \in F(\phi)$, there is a $\theta' \in \Phi^{m+p-1}_{\alpha+1} \cap F(\phi)$ such that $\theta \in E(\theta')$. By Proposition 8.10, there is a $\psi' \in \Phi^{m+n}_{\alpha+1} \cap F(\phi)$ such that $V_{\alpha,\alpha+1}(\psi') = \psi$. Let $\rho' \in \Phi^{m+p+p-1}_{\alpha+1}$ be the result of applying the induction hypothesis to ψ' and θ'. Since $\theta \in E(\theta')$, the desired ρ can be found in $E(\rho')$ by applying condition (3) of Definition 2.11. □

Proposition 8.12 shows that the set $F(\phi)$ is closed under suitable (horizontal) restrictions. The proposition follows immediately from Propositions 2.15 and 8.5.

{qhdown}

Proposition 8.12. *Suppose that $\alpha < \beta < \delta$, $m, n \in \omega$, $\phi \in \Phi^m_\beta$ and $\psi \in \Phi^{m+n}_\alpha$ are such that $\alpha + n \leq \beta$ and $\psi \in F(\phi)$. Fix $p \in [m, m+n]$ and let $j \in \mathcal{I}_{p,m+n}$ be such that $j \restriction X_m = i_m$. Then $H^{m+n}_\alpha(\psi, j) \in F(\phi)$.*

9 Extending a process of limit length

{ellsec}

Suppose that δ is a limit ordinal and $\mathcal{P}$ is a Scott process. Must there be a Scott process properly extending $\mathcal{P}$? We show in this section that there exists such a proper extension if δ has countable cofinality and each level of $\mathcal{P}$ is countable (Theorem 9.9). We also derive a positive answer from various scatteredness conditions on $\mathcal{P}$ (see Proposition 9.13, for instance). In general the question remains open, as far as we know.

9.1 Definition. Given a limit ordinal β and a sequence $\langle \Phi_\alpha : \alpha < \beta \rangle$ such that each Φ_α is a subset of Ψ_α, a *path through* $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a formula ϕ in Ψ_β such that $V_{\alpha,\beta}(\phi) \in \Phi_\alpha$ for each $\alpha < \beta$.

9.2 Remark. For each limit ordinal α, Ψ_α is the set of paths through the sequence $\langle \Psi_\beta : \beta < \alpha \rangle$. If $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, and $\beta < \delta$ is a limit ordinal, then each member of Φ_β is a path through $\langle \Phi_\alpha : \alpha < \beta \rangle$.

The issue of extending a given Scott process $\mathcal{P}$ of limit length then is whether there exists a set of paths through $\mathcal{P}$ large enough to satisfy conditions (1c) and (2b) of Definition 3.1 while also satisfying condition and (2c) (conditions (1d) and (1e) can then be achieved by closing under horizontal projections).

Proposition 9.3 implies in particular that every path through a Scott process of limit length determines the entire process (recall from Definition 8.1 that the set $F(\phi)$ depends only on ϕ, and not on a particular Scott processes containing ϕ).

{fdetermine}

Proposition 9.3. *Suppose that $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process. Fix $\gamma < \beta$, $n \in \omega$ and $\phi \in \Phi^n_\gamma$. Then for each $\alpha < \beta$ and $m \in \omega$ such that $\alpha + m \leq \gamma$, the set Φ^m_α is equal to $\{H^n_\alpha(\psi, j) : \psi \in F(\phi) \cap \Phi^{n+m}_\alpha, j \in \mathcal{I}_{m,n+m}\}$.*

Proof. Let ψ be a member of Φ^m_α. By condition (1c) of Definition 3.1, there is a $\psi' \in \Phi^m_\gamma$ such that $V_{\alpha,\gamma}(\psi') = \psi$. By condition (2c) of Definition 3.1, there exist $j \in \mathcal{I}_{m,n+m}$ and $\theta \in \Phi^{n+m}_\gamma$ such that $H^{n+m}_\gamma(\theta, i_n) = \phi$ and $H^{n+m}_\gamma(\theta, j) = \psi'$. Then $V_{\alpha,\gamma}(\theta)$ is in $F(\phi)$ by Proposition 8.5, and is as desired by Proposition 2.15. $\square$

Proposition 9.3 implies that "according to $\langle \Phi_\alpha : \alpha < \beta \rangle$ " is unnecessary in the following definition, if $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process.

{minimalsetdef}

9.4 Definition. Let β be a limit ordinal β and let $\langle \Phi_\alpha : \alpha < \beta \rangle$ be such that each Φ_α is a subset of Ψ_α. Let ϕ be a path through $\langle \Phi_\alpha : \alpha < \beta \rangle$, and let $n \in \omega$ be such that $\phi \in \Psi^n_\beta$. The *minimal set* of ϕ according to $\langle \Phi_\alpha : \alpha < \beta \rangle$ is the set of paths θ through $\langle \Phi_\alpha : \alpha < \beta \rangle$ for which there exist

- $m \in \omega \setminus n$;
- $p \in m + 1$;
- $\alpha_0 < \beta$;

- $\psi_0 \in \Phi^m_{\alpha_0} \cap F(\phi)$;
- $j \in \mathcal{I}_{p,m}$;

such that for all $\alpha \in [\alpha_0, \beta)$ and all $\psi \in \Phi^m_\alpha \cap F(\phi)$ such that $V_{\alpha_0,\alpha}(\psi) = \psi_0$, $H^m_\alpha(\psi, j) = V_{\alpha,\beta}(\theta)$.

We write $\mathrm{ms}(\phi)$ for the minimal set of ϕ.

{mschar1}

9.5 Remark. In the case where $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process and each element of $\bigcup\{\Phi_\alpha : \alpha < \beta\}$ is extended by a path through $\mathcal{P}$ (for instance, if β has countable cofinality, or $\mathcal{P}$ is scattered (see Definition 9.15)), the part of Definition 9.4 after the itemized list can equivalently be replaced by "such that for all $\psi \in \Psi_\beta$ such that $V_{\alpha_0,\beta}(\psi) = \psi_0$ and $H^m_\beta(\psi, i_n) = \phi$, $H^m_\beta(\psi, j) = \theta$."

{msdown}

9.6 Remark. Let $p \leq n$ be elements of ω, let j be an element of $\mathcal{I}_{p,n}$, let β be a limit ordinal, and let ϕ and ψ be paths through a Scott process $\langle \Phi_\alpha : \alpha < \beta \rangle$, with $\psi \in \mathrm{ms}(\phi)$ and $\psi \in \Psi^n_\beta$. Then $H^n_\beta(\psi, j)$ is an element of $\mathrm{ms}(\phi)$.

{weakminimal setrem}

9.7 Remark. Let the *weakly minimal set* of a formula ϕ (in the context of Definition 9.4) be the set of formulas $\upsilon \in \Psi^p_\beta$ for which membership in $\mathrm{ms}(\phi)$ is witnessed with $j = i_p$. One obtains an equivalent definition of the minimal set of ϕ by taking the closure of the weakly minimal set under permutations of free variables (i.e., including all formulas of the form $H^p_\beta(\theta, j)$, where $\theta \in \Psi^p_\beta$ is in the weakly minimal set of ϕ and j is in $\mathcal{I}_{p,p}$. This follows from the second part of Remark 2.2, and condition (1d) of Definition 3.1.

{love}

9.8 Remark. Suppose that β is a limit ordinal, $\langle \Phi_\alpha : \alpha \leq \beta \rangle$ is a Scott process and ρ is an element of Φ_β. Then $\mathrm{ms}(\rho) \subseteq \Phi_\beta$. This follows from Proposition 8.5.

{limitext}

Theorem 9.9. *Suppose that δ is a limit ordinal of countable cofinality and $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process such that each Φ_α is countable. Let ρ be a path through $\langle \Phi_\alpha : \alpha < \delta \rangle$. Then there exists a countable $\Phi_\delta \subseteq \Psi_\delta$ such that $\rho \in \Phi_\delta$ and $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process.*

Furthermore, if Υ is a countable subset of Ψ_δ disjoint from $\mathrm{ms}(\rho)$*, Φ_δ can be chosen to be disjoint from Υ.*

Proof. In order to make $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ a Scott process, we need to pick Φ_δ so that conditions (1c), (1d), (1e), (2b) and (2c) of Definition 3.1 are satisfied. Let $\langle \gamma^0_p : p < \omega \rangle$ be an increasing sequence cofinal in δ. We will recursively pick formulas θ_p $(p < \omega)$, a nondecreasing sequence of ordinals γ_p $(p < \omega)$ below δ and a nondecreasing unbounded sequence of integers n_p $(p < \omega)$ such that $\rho \in \Psi^{n_0}_\delta$ and such that, for each $p \in \omega$,

- $\gamma_p \geq \gamma^0_p$;
- $\theta_p \in \Phi^{n_p}_{\gamma_p} \cap F(\rho)$;
- $H^{n_{p+1}}_{\gamma_p}(V_{\gamma_p,\gamma_{p+1}}(\theta_{p+1}), i_{n_p}) = \theta_p$.

These conditions imply that $\theta_0 = V_{\gamma_0,\delta}(\rho)$.

Having chosen the θ_p's, for each $n \in \omega$ we let ϕ_n be the path through $\langle \Phi_\alpha : \alpha < \delta \rangle$ determined by $\{H^{n_p}_{\gamma_p}(\theta_p, i_n) \mid p \in \omega, n_p \geq n\}$. Then for all $m \leq n \in \omega$ we will have that $\phi_m = H_\delta(\phi_n, i_m)$, and we will let

$$\Phi_\delta = \bigcup_{n<\omega} \{H^n_\delta(\phi_n, j) : m \leq n, j \in \mathcal{I}_{m,n}\}.$$

This is enough to ensure that conditions (1d), (1e) and (2c) from Definition 3.1 are met. For condition (1d) this is immediate. For condition (1e), the right-to-left containment follow from condition (1d). For the other direction, fix $m \leq n$ in ω. An arbitrary formula $\psi \in \Phi^m_\delta$ has the form $H^q_\delta(\phi_q, j)$, for some $q \in \omega \setminus m$ and some $j \in \mathcal{I}_{m,q}$. Since $\phi_n = H^p_\delta(\phi_p, i_n)$ for all $p \geq n$ in ω, we may assume that $q \geq n$. Letting $j' \in \mathcal{I}_{n,q}$ be such that $j \restriction X_m = j' \restriction X_m$, we have that $H^q_\delta(\phi_q, j') \in \Phi^n_\delta$, and that $\psi = H^n_\delta(H^q_\delta(\phi_q, j'), i_m)$, by part (2) of Remark 2.14.

To see that condition (2c) holds, fix $n, m \in \omega$, $\phi \in \Phi^n_\delta$ and $\psi \in \Phi^m_\delta$. Then there exist $p, q \in \omega$, $j \in \mathcal{I}_{n,p}$ and $k \in \mathcal{I}_{m,q}$ such that $\phi = H^p_\delta(\phi_p, j)$ and $\psi = H^q_\delta(\phi_q, k)$. Since

$$\phi_p = H^{\max\{p,q\}}_\delta(\phi_{\max\{p,q\}}, i_p)$$

and

$$\phi_q = H^{\max\{p,q\}}_\delta(\phi_{\max\{p,q\}}, i_q),$$

we may assume by part 2 of Remark 2.14 that $p = q$. Similarly, we may assume that $p \geq m+n$. Let A be a subset of X_p of size $m+n$ which contains the ranges of both j and k. Let $j' : X_{m+n} \to A$ be a bijection such that $j = j' \circ i_n$. Then

$$\phi = H^p_\delta(\phi_p, j' \circ i_n) = H^{m+n}_\delta(H^p_\delta(\phi_p, j'), i_n),$$

by part 2 of Remark 2.14, and $H^p_\delta(\phi_p, j') \in \Phi_\delta$. Finally, let $k' \in \mathcal{I}_{m,m+n}$ be such that $k = j' \circ k'$. Then $\psi = H^{m+n}_\delta(H^p_\delta(\phi_p, j'), k')$, as desired.

To complete the proof, we show how to choose the formulas θ_p so that conditions (1c) and (2b) of Definition 3.1 are satisfied, and also so that no member of Υ is in Φ_δ. We let $\theta_0 = V_{\gamma_0,\delta}(\rho)$, as above. Suppose that $p \in \omega$ is such that θ_p has been chosen, but θ_{p+1} has not.

To satisfy condition (1c), let γ_{p+1} be the least member of $\{\gamma^0_q : q \in \omega\}$ which is at least as big as both γ_p and γ^0_{p+1}, and suppose that ψ is an element of Φ^m_α, for some $\alpha \leq \gamma_{p+1}$ and some $m \in \omega$. By Proposition 8.5, we can find a formula $\theta'_p \in \Phi^{n_p}_{\gamma_{p+1}+m+n_p}$ such that $V_{\gamma_p,\gamma_{p+1}+m+n_p}(\theta'_p) = \theta_p$ and

$$H^{n_p}_{\gamma_{p+1}+m+n_p}(\theta'_p, i_{n_0}) = V_{\gamma_{p+1}+m+n_p,\delta}(\rho).$$

By condition (1c), there is a $\psi' \in \Phi^m_{\gamma_{p+1}+m+n_p}$ such that $V_{\alpha,\gamma_{p+1}+m+n_p}(\psi') = \psi$. Applying condition (2c) of Definition 3.1, we can choose $\theta''_p \in \Phi^{m+n_p}_{\gamma_{p+1}+m+n_p}$ and $j \in \mathcal{I}_{m,n_p+m}$ such that

$$H^{m+n_p}_{\gamma_{p+1}+m+n_p}(\theta''_p, i_{n_p}) = \theta'_p$$

and

$$H^{m+n_p}_{\gamma_{p+1}+m+n_p}(\theta''_p, j) = \psi'.$$

Then $\theta_{p+1} = V_{\gamma_{p+1},\gamma_{p+1}+m+n_p}(\theta''_p)$ is as desired, by Propositions 2.15 and 8.5.

To satisfy condition (2b), fix $m \in \omega$. Each element of Φ^m_δ will have the form $H^{n_p}_\delta(\phi_p, j)$ for some $n_p \geq m$ and $j \in \mathcal{I}_{m,n_p}$, in which case $V_{\alpha+1,\delta}(\phi)$ will be $V_{\alpha+1,\gamma_p}(H^{n_p}_{\gamma_p}(\theta_p, j))$. Suppose then that for some $p \in \omega$ we have chosen $n_p \geq m$ and θ_p but not θ_{p+1}, and that $j \in \mathcal{I}_{m,n_p}$ and ψ in $E(V_{\alpha+1,\gamma_p}(H^{n_p}_{\gamma_p}(\theta_p, j)))$ are given. By Proposition 8.10, it suffices to find a $\theta'_p \in \Phi^{n_p+1}_{\gamma_p} \cap F(\rho)$ such that $H^{n_p+1}_{\gamma_p}(\theta'_p, i_{n_p}) = \theta_p$, and such that

$$H^{n_p+1}_\alpha(V_{\alpha,\gamma_p}(\theta'_p), j \cup \{(x_m, y)\}) = \psi$$

for some $y \in X_{n_p+1} \setminus \mathrm{range}(j)$. By Proposition 2.15,

$$V_{\alpha+1,\gamma_p}(H^{n_p}_{\gamma_p}(\theta_p, j)) = H^{n_p}_{\alpha+1}(V_{\alpha+1,\gamma_p}(\theta_p), j).$$

By condition (3) of Definition 2.11, there is a $\psi' \in E(V_{\alpha+1,\gamma_p}(\theta_p))$ such that

$$\psi = H^{n_p+1}_\alpha(\psi', j \cup \{(x_m, y)\})$$

for some $y \in X_{n_p+1} \setminus \mathrm{range}(j)$. By Proposition 8.10, there is a $\theta^*_p \in \Phi^{n_p}_{\gamma_p+1} \cap F(\rho)$ such that $\theta_p = V_{\gamma_p,\gamma_p+1}(\theta^*_p)$. By Proposition 4.3, there is a $\theta'_p \in E(\theta^*_p)$ such that $V_{\alpha,\gamma_p}(\theta'_p) = \psi'$. Then θ'_p is as desired.

Finally let us see how to avoid the members of Υ. Fix $m \leq n_p$, $j \in \mathcal{I}_{m,n_p}$ and $\upsilon \in \Upsilon \cap \Psi^m_\delta$. It suffices to show that we can find γ_{p+1} in the interval $(\max\{\gamma_p, \gamma^0_{p+1}\}, \delta)$ and a $\theta_{p+1} \in \Phi^{n_p}_{\gamma_{p+1}} \cap F(\rho)$ such that $H^{n_p}_\delta(\theta_{p+1}, j) \neq \upsilon$. Since Υ is disjoint from $\mathrm{ms}(\rho)$, there exists such a θ_{p+1} as desired. □

We now turn our attention to extending Scott processes of limit length in the scattered case, which includes the case of counterexamples to Vaught's Conjecture.

{ecdef}

9.10 Definition. Given a limit ordinal β and sets Φ_β $(\alpha < \beta)$ such that each Φ_α is a subset of Ψ_α, a path $\bigwedge\{\psi_\alpha : \alpha < \beta\}$ through $\langle \Phi_\alpha : \alpha < \beta \rangle$ is *isolated* (with respect to $\langle \Phi_\alpha : \alpha < \beta \rangle$) if for some $\alpha_0 < \beta$, $|V^{-1}_{\alpha_0,\alpha}[\{\phi_{\alpha_0}\}] \cap \Phi_\alpha| = 1$ for all $\alpha \in (\alpha_0, \beta)$.

Proposition 9.3 shows that the term "with respect to $\langle \Phi_\alpha : \alpha < \beta \rangle$" is unnecessary in Definition 9.10, if $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process.

{ecover}

9.11 Remark. Suppose that β is a limit ordinal, and $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process. Suppose that $m \leq n$ are elements of ω, $j \in \mathcal{I}_{m,n}$ and $\phi \in \Psi^n_\beta$ is an isolated path through $\mathcal{P}$. Then $H^n_\beta(\phi, j)$ is isolated. To see this, note first of all that the case $m = n$ follows from part (1) of Remark 2.14. This fact allows us to reduce to the case where $j = i_m$. Then a proof by induction reduces to the case where $n = m + 1$. This case follows part (2) of Proposition 5.1.

9.12 Remark. Given a limit ordinal β and sets Φ_β $(\alpha < \beta)$ such that each Φ_α is a subset of Ψ_α, the isolated paths through $\langle \Phi_\alpha : \alpha < \beta \rangle$ are exactly the minimal set of the sentence formed by taking the conjunction of the unique members of each set Φ^0_α. This follows from Remark 9.11 and Proposition 8.5.

In Proposition 9.13, we do not require δ to have countable cofinality (whereas we did for Theorem 9.9).

{limitecmodels}

Proposition 9.13. *Suppose that δ is a limit ordinal, and that $\mathcal{P} = \langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process such that each element of $\bigcup \{\Phi_\alpha : \alpha < \delta\}$ is extended by an isolated path through $\mathcal{P}$. Letting Φ_δ be the set of isolated paths through $\mathcal{P}$, $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process. Furthermore, Φ_δ then satisfies amalgamation, and every Scott process properly extending $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has rank at most δ.*

Proof. Checking that Φ_δ induces a Scott process involves checking conditions (1e), (2b) and (2c) of Definition 3.1. Remark 9.11 gives one direction of (1e). The other conditions can be shown by applying the corresponding fact at levels above the ordinal α_0 witnessing that the formulas in question are isolated.

That Φ_δ amalgamates also follows from the definition of the functions $H^n_{\alpha+1}$ $(n \in \omega)$ for any ordinal α above the ordinal α_0 witnessing that the formulas in question are isolated. By Proposition 5.19, it also follows from the fact that some Scott properly extending $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has rank δ, which follows from the next paragraph.

To see that every Scott process $\langle \Phi_\alpha : \alpha \leq \delta + 1 \rangle$ extending $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has rank δ, suppose that we have $n \in \omega$, $\phi \in \Phi^n_{\delta+1}$ and $\psi \in \Phi^{n+1}_\delta$ such that $H^{n+1}_\delta(\psi, i_n) = V_{\delta,\delta+1}(\phi)$. Let $\beta < \delta$ be such that $V_{\delta,\delta+1}(\phi)$ and ψ are the unique members of $V^{-1}_{\beta,\delta}[\{V_{\beta,\delta+1}(\phi)\}]$ and $V^{-1}_{\beta,\delta}[\{V_{\beta,\delta}(\psi)\}]$ respectively. Then

$$H^{n+1}_{\beta+1}(V_{\beta+1,\delta}(\psi), i_n) = V_{\beta+1,\delta+1}(\phi)$$

by Proposition 2.15, so $V_{\beta,\delta}(\psi) \in E(V_{\beta+1,\delta+1}(\phi))$ by condition (2a) of Definition 3.1. Then conditions (2a) and (2b) of Definition 3.1 imply that $\psi \in E(\phi)$. $\square$

9.14 Definition. A *Scott subprocesses* is a set of the form $\langle \Phi_\alpha : \alpha \in I \rangle$, for some Scott process $\{\Phi_\alpha : \alpha < \beta\}$ and $I \subseteq \beta$.

{scattereddef}

9.15 Definition. Given partial orders $(P, \leq_P)$ and $(Q, \leq_Q)$, let us say that $(P, \leq_P)$ *contains a copy of* $(Q, \leq_Q)$ if there is a function π from Q to P such that for all q_1, q_2 in Q, $q_1 \leq_Q q_2$ if and only if $\pi(q_1) \leq \pi(q_2)$. We say that a partial order is *scattered* if it does not contain a copy of $2^{<\omega}$, ordered by extension. We say that a Scott subprocess $\langle \Phi_\alpha : \alpha \in I \rangle$ is *scattered* if $(\bigcup_{\alpha \in I} \Phi_\alpha, \leq_V)$ is scattered in this sense (recall Definition 2.7). We say that a Scott process $\langle \Phi_\alpha : \alpha < \beta \rangle$ is *eventually scattered* if $\langle \Phi_\alpha : \alpha \in I \rangle$ is scattered for some cofinal $I \subseteq \beta$.

{CBrem0}

9.16 Remark. If a Scott process $\langle \Phi_\alpha : \alpha < \beta \rangle$ of limit length is eventually scattered, then there is a $\gamma < \beta$ such that $\langle \Phi_\alpha : \alpha \in (\gamma, \beta) \rangle$ is scattered.

{CBrem}

9.17 Remark. Whether or not a given partial order is scattered is absolute between wellfounded models of ZFC containing the partial order. Suppose that $(T, \leq_T)$ is a tree ordering (a partial order such that the predecessors of any point are wellordered by $\leq_T$) with $T \subseteq L[T]$. The Cantor-Bendixon analysis (iteratively removing nodes without incompatible extensions) shows that if $(T, \leq_T)$ is scattered then every maximal linearly ordered subset of T is a member of $L[T]$. The Cantor-Bendixon *rank* of $(T, \leq_T)$ is the (possibly transfinite) number of steps it takes for this analysis to terminate. If a Scott subprocess $\mathcal{P} = \langle \Phi_\alpha : \alpha \in I \rangle$ is scattered, we call the Cantor-Bendixon rank of the partial order $(\bigcup\{\Phi_\alpha : \alpha \in I\}, \leq_V)$ the *Cantor-Bendixon rank* of $\mathcal{P}$. If an ordinal γ is greater than the Cantor-Bendixon rank of $\mathcal{P}$, then every path through $\mathcal{P}$ is an element of $L_{\beta+\gamma}[\mathcal{P}]$.

9.18 Definition. We say that a τ-structure N is *Scott rank atomic* if, letting δ be the Scott rank of N, δ is a limit ordinal, and every element of $\Phi_\delta(N)$ is isolated in $\langle \Phi_\alpha(N) : \alpha < \delta \rangle$.

Combining Proposition 9.13 with Theorems 1.2, 6.8 and 7.11, we get the following model-existence result. Recall that for any ordinal γ, $\mathrm{Col}(\omega, \gamma)$ is the partial order which adds a function (generically, a surjection) from ω to γ by finite pieces, ordered by inclusion.

{alllimits}

Theorem 9.19. *Let ϕ be a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$ and let η be the quantifier depth of ϕ. Let $\beta \in (\eta, \omega_2)$ be an ordinal such that ϕ has a model of Scott rank β, but only countably many models of Scott rank γ for each countable ordinal γ in the interval (η, β). Then for every limit ordinal $\delta \in (\eta, \beta)$, ϕ has a Scott rank atomic model of Scott rank δ.*

Proof. Let M be a model (which by taking the transitive collapse of a suitable elementary substructure if necessary we may assume to be of cardinality at most $\aleph_1$) of ϕ of Scott rank β, and fix a limit ordinal $\delta < \beta$. Let

$$\mathcal{P} = \langle \Phi_\alpha(M) : \alpha \in (\eta, \delta) \rangle.$$

We claim first that $\mathcal{P}$ is scattered. To see this, suppose to the contrary that π embeds $2^{<\omega}$ into $(\bigcup\{\Phi_\alpha(M) : \alpha \in (\eta, \delta)\}, \leq_V)$. Let X be a countable elementary submodel of $L_{\omega_3}[\mathcal{P}]$ with $\eta \subseteq X$ and $\pi \in X$. Let γ be the ordertype of $X \cap \delta$. Let Q be the transitive collapse of X, and let $\mathcal{P}' = \langle \Phi_\alpha : \alpha \in (\eta, \gamma) \rangle$ and π' be the images of $\mathcal{P}$ and π under this collapse. For each $\alpha \leq \eta$, let Φ_α be $\Phi_\alpha(M)$, and let $\mathcal{P}^*$ be $\langle \Phi_\alpha : \alpha < \gamma \rangle$. Let g be Q-generic for $\mathrm{Col}(\omega, \omega_1^Q)$, the partial order adding a surjection from ω to ω_1^Q by finite pieces.

For each $x \in 2^\omega$, $\langle \pi'(x \restriction n) : n \in \omega \rangle$ gives a path through an initial segment of $\mathcal{P}^*$ properly extending $\langle \Phi_\alpha(M) : \alpha \leq \eta \rangle$. Continuum many $x \in 2^\omega$ are generic over $Q[g]$ for Cohen forcing. For each such x, the corresponding path is a formula ϕ_x which by Theorem 9.9 is part of a Scott process in $Q[g][x]$ of successor length properly extending $\langle \Phi_\alpha(M) : \alpha \leq \eta \rangle$ and having a countable top level. By Theorem 6.8, each of these formulas has a countable model N_x

(of Scott rank rank less than ω_2^Q) in the corresponding $Q[g][x]$, and by Theorem 1.2 they are all models of ϕ. Each ϕ_x then has the form $\phi^{N_x}_{\bar{a},\zeta}$, for some ordinal ζ and some finite tuple $\bar{a}$ from N_x. Since the formulas $\phi_x$are pairwise $\leq_V$-incompatible, no τ-structure can satisfy more than one of them with the same finite tuple, so no countable τ-structure can satisfy more than countably many of them. It follows then that there exist continuum many models of ϕ of Scott rank less than ω_2^Q, which is countable, giving a contradiction.

Now Proposition 9.13 and Theorem 7.11 give a model of ϕ of Scott rank δ, as desired (the model cannot have Scott rank less than δ, since $\mathcal{P}$ is nonterminating). □

{scatteredtree}

9.20 Remark. Suppose that $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process of countable length, with all levels countable, having only countably many models of Scott rank γ for each countable ordinal γ. Let T be the (class-sized) tree of Scott processes extending $\mathcal{P}$, ordered by extension. A minor variation of the first paragraph of the proof of Theorem 9.19 shows that T is scattered.

9.21 Definition. Let $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ be a Scott process of limit length, let $m \leq n$ be elements of ω and let $\phi \in \Psi^m_\beta$ and $\psi \in \Psi^n_\beta$ be paths through $\mathcal{P}$. Let f be an element of $\mathcal{I}_{m,n}$. We say that ψ is (f, ϕ)-*isolated* if there exists a $\gamma < \beta$ such that, for all $\delta \in (\gamma, \beta)$, $V_{\delta,\beta}(\psi)$ is the unique $\theta \in \Phi^n_\delta$ such that $V_{\gamma,\delta}(\theta) = V_{\gamma,\beta}(\psi)$ and $H^n_\delta(\theta, f) = V_{\delta,\beta}(\phi)$.

{mschar2}

9.22 Remark. If $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ is an eventually scattered Scott process of limit length, $k \in \omega$ and $\phi \in \Psi^k_\beta$ is a path through $\mathcal{P}$, then $\mathrm{ms}(\phi)$ is the set of formulas of the form $H^n_\beta(\psi, g)$, where for some $m \leq n$ in ω (with $n \geq k$), $\psi \in \Phi^n_\beta$ is (i_k, ϕ)-isolated and g is in $\mathcal{I}_{m,n}$. That this set is contained in $\mathrm{ms}(\phi)$ follows from Remarks 9.5 and 9.6. The other direction follows from the usual argument that in an eventually scattered partial order every node is extended by an isolated path, applied to $F(\phi)$.

{rhoremark}

9.23 Remark. Theorem 9.9 shows that if δ is a limit ordinal and $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process with just countably many paths, then for each such path ρ, letting Φ_δ be $\mathrm{ms}(\rho)$ we get a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$. Since $\mathrm{ms}(\phi)$ and being scattered are absolute to forcing extensions, we get the same conclusion from the assumption that $\langle \Phi_\alpha : \alpha < \delta \rangle$ is eventually scattered, without any countability assumption. In this context, then, since $\mathrm{ms}(\rho)$ is the smallest set one can add to $\langle \Phi_\alpha : \alpha < \delta \rangle$ to get a Scott processes with ρ in its last level, it follows (again, in the case where $\langle \Phi_\alpha : \alpha < \delta \rangle$ is eventually scattered) that if ϕ and ψ are paths through $\langle \Phi_\alpha : \alpha < \delta \rangle$ with $\phi \in \mathrm{ms}(\psi)$, then $\mathrm{ms}(\phi)$ is a subset of $\mathrm{ms}(\psi)$.

In the following proposition, the countability assumption on the sets Φ_α can be replaced by the assumption that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is eventually scattered, using Remark 9.23 (recall that pre-rank was defined in Definition 5.12).

{laterbound}

Proposition 9.24. *Let β be an ordinal, and let γ be the least limit ordinal greater than or equal to β. Suppose that $\langle \Phi_\alpha : \alpha \leq \gamma+1 \rangle$ is a Scott process of pre-rank β, and that Φ_α is countable for all $\alpha < \gamma$. Then the rank of $\langle \Phi_\alpha : \alpha \leq \gamma+1 \rangle$ is at most γ.*

Proof. Since Φ_α is countable for all $\alpha < \gamma$, β is countable. By the definition of pre-rank, $\langle \Phi_\alpha : \alpha \leq \gamma \rangle$ is the unique Scott process of length $\gamma + 1$ extending $\langle \Phi_\alpha : \alpha < \gamma \rangle$. By Theorem 9.9, $\langle \Phi_\alpha : \alpha < \gamma \rangle$ has only countably many paths. By Proposition 9.13, all of them are isolated, and $\langle \Phi_\alpha : \alpha \leq \gamma + 1 \rangle$ has rank at most γ. □

10 Absoluteness

{fabsection}

In this section we record various standard absoluteness results concerning counterexamples to Vaught's Conjecture. We assume here that our relational vocabulary τ is countable. The set of τ-structures with domain ω is then naturally seen as a Polish space X_τ, where a basic open set is given by the set of structures in which $R(i_0, \ldots, i_{n-1})$ holds, for R an n-ary relation symbol from τ and $i_0, \ldots, i_{n-1} \in \omega$ (see Section 11.3 of [4], for instance). Given a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, the set of models of ϕ with domain ω is a Borel subset of X_τ. By a theorem of Lopez-Escobar [11], every Borel subset of X_τ which is closed under isomporphism is also the set of models (with domain ω) of some $\mathcal{L}_{\omega_1,\omega}(\tau)$ sentence.

Let us call the following (false) statement the *analytic Vaught Conjecture*: for every countable relational vocabulary τ, every analytic subset of X_τ (closed under isomorphism) which contains uncountably many nonisomorphic structures contains a perfect set of nonisomorphic structures. Steel [19] presents two examples of analytic counterexamples to Vaught's Conjecture (for certain relational vocabularies), one due to H. Friedman and the other to K. Kunen. In this section we use a forcing-absoluteness argument to prove the following, which was presumably well-known. As mentioned in the introduction, the forcing-absoluteness arguments in this section appear in essentially identical form in Section 1 of [6].

{sec10thrm1}

Theorem 10.1. *Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Fix $M \in \mathcal{A}$, and let β be an ordinal. Then $\langle \Phi_\alpha(M) : \alpha < \beta \rangle \in L[x]$.*

Applying this theorem in forcing extensions of V we get the following ostensibly stronger fact.

{sec10cor2}

Corollary 10.2. *Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Let M be a member of the reinterpreted version of $\mathcal{A}$ in a forcing extension of V, and let β be an ordinal. Then $\langle \Phi_\alpha(M) : \alpha < \beta \rangle \in L[x]$.*

Before beginning the proof of Theorem 10.1 (which is short), we make a few observations.

10.3 Remark. Remarks 9.17 and 9.20, along with Theorem 1.2, show that if ϕ is a counterexample to (the usual) Vaught's Conjecture, γ is a limit ordinal greater than the quantifier depth of ϕ and M is an inner model of ZFC containing $\langle \Phi_\alpha(N) : \alpha < \beta \rangle$ for each τ-structure $N \models \phi$ and each $\beta < \gamma$, then M contains $\langle \Phi_\alpha(N) : \alpha < \gamma \rangle$ for each τ-structure $N \models \phi$.

In what follows we will talk of sufficient fragments of ZFC. The theory ZFC$^\circ$ from [2] is one such fragment.

{smallin}

10.4 Remark. As we are assuming that τ is countable, we can associate each atomic or negated atomic formula from τ to an element of ω, and each Scott process over τ of length 1 to a subset of $\mathcal{P}(\omega)$. For a countable τ-structure, this subset of $\mathcal{P}(\omega)$ will be countable. Let $=^+$ be the equivalence relation on functions from $\omega \times \omega$ to 2 defined by setting $f =^+ g$ if

$$\{\{m : f(n,m) = 1\} : n \in \omega\} = \{\{m : g(n,m) = 1\} : n \in \omega\}.$$

Then $=^+$ is easily seen to be a Borel equivalence relation, and it follows for instance from Silver's theorem on coanalytic equivalence relations (Theorem 5.3.5 of [4]) that every analytic set $A \subseteq {}^{\omega\times\omega}2$ containing representatives of uncountably many equivalence classes contains a perfect $=^+$-inequivalent set (for $=^+$ this can be proved more easily, considering separately the cases where

$$\{\{m : f(n,m) = 1\} : n \in \omega, f \in A\}$$

is countable or uncountable, and in the former case arguing as in the proof of Theorem 4.6 of [2]).

Let $\mathcal{A}$ be an analytic family of τ-structures on ω. It follows from the previous paragraph that if the set of Scott processes of length 1 corresponding to structures in $\mathcal{A}$ is uncountable, then there exists a perfect subset of ${}^{\omega\times\omega}2$ coding distinct elements of this set, and, via the proofs of Theorems 6.4 and 6.5, a perfect subset of ${}^{\omega}\omega$ coding distinct structures in $\mathcal{A}$. Working by induction, essentially the same analysis (breaking into successor and limit cases) shows that if $\beta > 0$ is a countable ordinal and the set of Scott processes of length of less than β corresponding to structures in $\mathcal{A}$ is countable, then if there are uncountably many Scott processes of length of β corresponding to structures in $\mathcal{A}$, then there is a perfect subset of ${}^{\omega}\omega$ coding distinct elements of $\mathcal{A}$. If $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture, then, the set of Scott processes length β corresponding to structures in $\mathcal{A}$ is countable for each $\beta < \omega_1$.

For any analytic family $\mathcal{A}$ of τ-structures, and any countable (possibly empty) set of Scott processes of length $\beta < \omega_1$, the assertion that there exists a member of the family whose Scott process up to length β is not in this countable set is Σ^1_1 in codes for β, the family and the countable set, and thus absolute to any model of (a sufficient fragment of) ZFC that contains them. Furthermore, if such a model thought that there were uncountably many Scott processes of length β corresponding to structures in $\mathcal{A}$, it could find a perfect subset of ${}^{\omega}\omega$ coding distinct Scott processes in this family. It follows that if $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture then any inner model N of (a sufficient fragment of) ZFC containing a real parameter code for $\mathcal{A}$ contains all sequences of the form $\langle \Phi_\alpha(M) : \alpha < \beta \rangle$, for $M \in \mathcal{A}$ and $\beta < \omega_1^N$. This gives Theorem 10.1 for initial segments of Scott processes of length less than $\omega_1^{L[x]}$.

Proof of Theorem 10.1. Let $\theta > \beta$ be a regular cardinal of $L[x]$ such that $L_\theta[x]$ satisfies a sufficient fragment of ZFC as in Remark 10.4 (for instance, let θ be a regular cardinal of V greater than $2^{2^{(|\beta|+\omega_1)}}$). Let X be a countable (in V) elementary submodel of $L_\theta[x]$ containing $\{x, \langle \Phi_\alpha : \alpha < \beta \rangle\} \cup \beta$. Let γ be such that the transitive collapse of X is $L_\gamma[x]$. By the last paragraph of Remark 10.4, whenever g is an $L_\gamma[x]$-generic filter for $\mathrm{Col}(\omega, \beta)$, $\langle \Phi_\alpha : \alpha < \beta \rangle$ is in $L_\gamma[x][g]$. This means that $\langle \Phi_\alpha : \alpha < \beta \rangle$ is in $L_\gamma[x]$ (this is a classical forcing fact due to Solovay; the point is that otherwise one could choose a generic filter while ensuring that each name in $L_\gamma[x]$ realizes to some value other than $\langle \Phi_\alpha : \alpha < \beta \rangle$). □

{manyranks}

10.5 Remark. Let $\mathcal{A}$ be an analytic family of τ-structures on ω. The assertion that $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture is Π^1_2 in a real parameter x for $\mathcal{A}$, and therefore absolute to $L[x]$.[1] It follows (assuming that $\mathcal{A}$ a counterexample to the analytic Vaught Conjecture) that for every ordinal γ, there are cofinally many ordinals below $(|\gamma|^+)^{L[x]}$ which are the Scott rank of a countable structure in $\mathcal{A}$, in any forcing extension of $L[x]$ via the partial order $\mathrm{Col}(\omega, \gamma)$ (all levels of the Scott processes of these structures are then countable in the corresponding forcing extensions). Applying Theorem 9.19, this gives (in the case where $\mathcal{A}$ is Borel) that this set of ordinals (in such a forcing extension) includes coboundedly many limit ordinals below $(\kappa^+)^{L[x]}$.

Theorems 7.11 and 9.19, along with Corollary 10.2 and Remark 10.5, give the following unpublished theorem of Leo Harrington from the 1970's. The arguments we have given here give a slightly stronger version of Harrington's theorem than the one in [14]. A similar result (for countable models) appears in [17]. Theorem 11.2 gives non-Scott-rank-atomic models (as does [17]).

{Harringtonplus}

Theorem 10.6 (Harrington)**.** *Suppose that τ is a countable relational vocabulary and that $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ gives a counterexample to Vaught's Conjecture. Let α be the quantifier depth of ϕ. Then for every limit ordinal δ in the interval $[\alpha, \omega_2)$, ϕ has a Scott rank atomic model of Scott rank δ.*

Proof. By Theorem 9.19, it suffices to show that for cofinally many $\beta < \omega_2$, ϕ has a model of Scott rank at least β. Fix such a β. By Remark 10.5, in some forcing extension by the partial order $\mathrm{Col}(\omega, \beta)$, ϕ has a countable model with Scott rank in the interval (β, ω_2). Let γ be the Scott rank of this model. By Corollary 10.2, the Scott process of this model of length $\gamma + 1$ exists already in V, and since the levels of this Scott process are countable in the $\mathrm{Col}(\omega, \beta)$

[1]There exist perfectly many nonisomorphic structures in $\mathcal{A}$ if and only if some wellfounded countable model of a sufficient fragment of ZFC thinks there exist perfectly many nonisomorphic structures in $\mathcal{A}$ (see the proof of Theorem 4.6 of [2], for instance), and this later statement is easily seen to be Σ^1_2. The statement that there are countable models in $\mathcal{A}$ of unboundedly many Scott ranks below ω_1 is easily seen to be Π^1_2.

extension, they have cardinality at most $\aleph_1$ in V. By Proposition 5.19, the top level of this Scott process amalgamates. By Theorem 7.11, a model of this Scott process exists. By Theorem 1.2, this model is a model of ϕ. $\square$

Standard arguments show that if there is a counterexample to Vaught's Conjecture then there is one of quantifier depth at most ω, in an expanded language. We given a new proof of this fact in Section 12.

{hmremark}

10.7 Remark. The proof of Theorem 1 of [5] can be rephrased in terms of the arguments given here, showing that any counterexample to Vaught's Conjecture can be strengthened to a minimal counterexample. The point again is that if $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau)$ is a counterexample to Vaught's Conjecture, and α is the quantifier depth of σ, then there is a sentence $\sigma' \in \mathcal{L}_{\omega_1,\omega}(\tau)$ which is the unique member of $\Phi^0_\alpha(M)$ for uncountably many countable models M satisfying σ. Then all models of σ' are models of σ, by Theorem 1.2, and σ' is also a counterexample to Vaught's Conjecture. Let S be the set of all countable length Scott processes which have σ' as their unique sentence at level α and are initial segments of the Scott process of some model of uncountable Scott rank. Since σ' is a counterexample to Vaught's Conjecture, S is not empty, by Theorem 10.6. On the other hand, since σ' does not have perfectly many countable models, there will be a $\mathcal{P}$ in S without incompatible extensions in S. Since any extension of $\mathcal{P}$ in S will have the same property, there is such a member of S with successor length. Let ϕ be the unique sentence in the last level of this process. Then ϕ is a counterexample to Vaught's Conjecture, and all uncountable models of ϕ satisfy the same $\mathcal{L}_{\omega_1,\omega}(\tau)$-theory.

{Hjorthremark}

10.8 Remark. Hjorth [7] showed that if there exists a counterexample to Vaught's Conjecture, then there is one with no model of cardinality $\aleph_2$. Recently, this has been extended by Baldwin, S. Friedman, Koerwien and Laskowski [1], who showed (among other things) that if there exists a counterexample to Vaught's Conjecture, then there is one with with the property that for some countable $\mathcal{L}_{\aleph_1,\aleph_0}$-fragment T, no model of cardinality $\aleph_1$ has a T-elementary extension. Roughly speaking, Hjorth's argument (as reformulated by [1]), finds an absolutely definable method for taking any countable structure M in a relational language and building a structure $H(M)$ in such a way that (1) the Scott sentence of $H(M)$ cannot have a model of cardinality $\aleph_2$; (2) $H(M)$ contains a copy of M; (3) if M and N isomorphic then so are $H(M)$ and $H(N)$. It follows from (3) that for any structure M in a relational vocabulary (countable or not), the structure $H(M)$ has the same Scott process in each forcing extension in which M is countable. From the classical fact due to Solovay mentioned in the proof of Theorem 10.1 it follows that every initial segment of the Scott process of $H(M)$ (as constructed in any such forcing extension) exists in V. It follows from (1) and (2) that if M has Scott rank $\gamma \geq \omega_2$, then the initial segment of the Scott process of $H(M)$ of length γ does not have a model in V (so the condition $|\Phi_\delta| \leq \aleph_1$ in the statement of Theorem 7.11 is necessary).

11 A local condition for amalgamation

{localsec}

In this section we give a sufficient condition for showing that the set of all paths though a Scott process of limit length amalgamates, and use it to produce models of a counterexample to Vaught's Conjecture which are not Scott rank atomic. As in Remark 9.17 above, if γ in the theorem below is greater than the Cantor-Bendixon rank of $\mathcal{P}$, then $\Phi_\beta \subseteq M$. Parts (1) and (5) of the theorem can be phrased more genrally as theorems about scattered trees.

{Barwisetheorem}

Theorem 11.1. *Suppose that $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ is an eventually scattered Scott process, where β is a countable limit ordinal. Let $\gamma > \beta$ be an ordinal, and let $M = L_\gamma[\langle \Phi_\alpha : \alpha < \beta \rangle]$. Suppose that, in M, the cofinality of β is greater than $|\Phi_\alpha|$, for each $\alpha < \beta$. Let Φ_β be the set of all paths through $\mathcal{P}$.*

{Barone}

1. *Let A be a subset of $\bigcup\{\Phi_\alpha : \alpha < \beta\}$, in M, such that A contains a member of Φ_α for cofinally many $\alpha < \beta$. Then there is a $\phi \in \Phi_\beta$ such that, for each $\alpha < \beta$, there exist $\delta \in \beta \setminus \alpha$ and $\psi \in A \cap \Phi_\delta$ such that $V_{\alpha,\beta}(\phi) = V_{\alpha,\delta}(\psi)$.*

{Bartwo}

2. *If $\Phi_\beta \subseteq M$, then $\langle \Phi_\alpha : \alpha \leq \beta \rangle$ is a Scott process.*

{Barthree}

3. *If $\Phi_\beta \subseteq M$, then Φ_β amalgamates.*

{Barfive}

4. *If $\Phi_\beta \subseteq M$ and $\mathcal{P}$ is nonterminating, then no model of $\langle \Phi_\alpha : \alpha \leq \beta \rangle$ of Scott rank β is rigid.*

{Barsix}

5. *Suppose that $\gamma \geq (\beta^+)^M$, $n \in \omega$, A is a stationary subset of β in M, and $\{\psi_{\alpha,i} : \alpha \in A, i < n\}$ is a set in M such that, for each $\alpha \in A$ and $i < n$, $\psi_{\alpha,i} \in \Phi_\alpha$. Then there exist $\phi_i \in \Phi_\beta$ $(i < n)$ such that, in M, for stationarily many $\alpha \in A$, for all $i < n$, $V_{\alpha,\beta}(\phi_i) = \psi_{\alpha,i}$.*

{Barseven}

6. *If $\gamma \geq (\beta^+)^M$, and $\mathcal{P}$ is nonterminating, then Φ_β has a non-isolated path.*

{Bareight}

7. *If $\gamma \geq (\beta^+)^M$, then, for club many $\alpha < \beta$, Φ_α amalgamates.*

Proof. Since $\mathcal{P}$ is eventually scattered, we may work in a forcing extension in which β is countable, as the set of paths through $\mathcal{P}$ is the same in any such extension.

For part (1), let $\langle \gamma_i : i \in \omega \rangle$ be a cofinal increasing sequence in β. Let Θ be the set of θ such that, for some $i \in \omega$, $\theta \in \Phi_{\gamma_i}$, and, for cofinally many $\delta < \beta$, there exists a $\psi \in \Phi_\delta \cap A$ with $V_{\gamma_i,\delta}(\psi) = \theta$. Since in M there is no cofinal function from any Φ_α to β, $\Theta \cap \Phi_{\gamma_0}$ is nonempty, and, for each $i \in \omega$ and each θ in $\Theta \cap \Phi_{\gamma_i}$, there is $\rho \in \Theta \cap \Phi_{\gamma_{i+1}}$ with $V_{\gamma_i,\gamma_{i+1}}(\rho) = \theta$. The existence of a ϕ as desired follows.

For part (2), all parts of Definition 3.1 are immediate (from the assumption that $\mathcal{P}$ is eventually scattered, which implies that every member of each Φ_α $(\alpha < \beta)$ is part of an element of Φ_β), aside from conditions (2b) and (2c). Each of these follow easily from part (1). For condition (2b), fix $\phi \in \Phi^n_\beta$ (for some $n \in \omega$), $\alpha < \beta$ and $\theta \in E(V_{\alpha+1,\beta}(\phi))$. We wish to find a $\psi \in \Phi^{n+1}_\beta$ such that $H^{n+1}_\beta(\psi, i_n) = \phi$ and $V_{\alpha,\beta}(\psi) = \theta$. Applying part (1), it suffices to show that for each $\delta \in (\alpha, \beta)$, there is a $\rho \in \Phi^{n+1}_{\delta}$ such that $H^{n+1}_{\delta}(\rho, i_n) = V_{\delta,\beta}(\phi)$ and $V_{\alpha,\delta}(\rho) = \theta$. The existence of such a ρ follows from condition (2b) applied to $V_{\delta,\beta}(\phi)$. The argument for condition (2c) is similar, but we use the fact that the union of the sets $\mathcal{I}_{m,n}$ is countable.

For part (3), fix $m < n \in \omega$, $\phi \in \Phi^{m+1}_{\beta}$ and $\psi \in \Phi^n_{\beta}$ such that $H^{m+1}_{\beta}(\phi, i_m) = H^n_{\beta}(\psi, i_m)$. Applying the first conclusion, it suffices to show that for each $\alpha < \beta$, there exist $\theta \in \Phi^{n+1}_{\alpha}$ and $y \in X_{n+1} \setminus X_m$ such that

$$H^{n+1}_{\alpha}(\theta, i_m \cup \{(x_m, y)\}) = V_{\alpha,\beta}(\phi)$$

and $H^{n+1}_{\alpha}(\theta, i_n) = V_{\alpha,\beta}(\psi)$. Fix α. Since $V_{\alpha,\beta}(\phi)$ is in $E(V_{\alpha+1,\beta}(H^{m+1}_{\beta}(\phi, i_m)))$ and

$$V_{\alpha+1,\beta}(H^{m+1}_{\beta}(\phi, i_m)) = V_{\alpha+1,\beta}(H^n_{\beta}(\psi, i_m)) = H^n_{\alpha+1}(V_{\alpha+1,\beta}(\psi), i_m),$$

(by Proposition 2.15) there exist $\theta \in E(V_{\alpha+1,\beta}(\psi), i_m)$ and $y \in X_{n+1} \setminus X_m$ such that

$$H^{n+1}_{\alpha}(\theta, i_m \cup \{(x_m, y)\}) = V_{\alpha,\beta}(\phi).$$

Then θ is as desired.

For part (4), let f be $\{(x_0, x_1)\}$. We need to find a $\phi \in \Phi^2_{\beta}$ such that $H^2_{\beta}(\phi, i_1) = H^2_{\beta}(\phi, f)$. Applying the first conclusion, it suffices to show that for each $\alpha < \beta$, there is a $\theta \in \Phi^2_{\alpha}$ such that $H^2_{\alpha}(\phi, i_1) = H^2_{\alpha}(\phi, f)$. Fix α. Since $\mathcal{P}$ is nonterminating, there exist $n \in \omega$ and distinct $\rho_1, \rho_2 \in \Phi^n_{\alpha+1}$ such that $V_{\alpha,\alpha+1}(\rho_1) = V_{\alpha,\alpha+1}(\rho_2)$. By condition (2c) of Definition 3.1, there exists an $\upsilon \in \Phi^{2n}_{\alpha+1}$, and $f, g \in \mathcal{I}_{n,2n}$ such that $H^{2n}_{\alpha+1}(\upsilon, f) = \rho_1$ and $H^{2n}_{\alpha+1}(\upsilon, g) = \rho_2$. There must be some $x_i \in X_n$ then such that $f(x_i) \neq g(x_i)$. Let $h = \{(x_0, f(x_i)), (x_1, g(x_i))\}$. Then $H^{2n}_{\alpha}(V_{\alpha,\alpha+1}(\upsilon), h)$ is as desired.

Part (5) follows from repeated application of the result for the case $n = 1$. We prove this case. Fix $\xi < \beta$ such that $\langle \Phi_{\alpha} : \alpha \in (\xi, \beta) \rangle$ is scattered, and let η be the Cantor-Bendixon rank of $\langle \Phi_{\alpha} : \alpha \in (\xi, \beta) \rangle$. Assuming that there is no ϕ as desired there is a $\zeta \in (\beta + \eta, \beta^+)$ such that, letting $M' = L_{\zeta}[\mathcal{P}]$, there exist, in M' an enumeration $\langle \phi_{\alpha} : \alpha < \beta \rangle$ of Φ_{β} and a club $C \subseteq \beta$ such that for all $\delta \in C \cap A$ and all $\alpha < \delta$, $V_{\delta,\beta}(\phi_{\alpha}) \neq \psi_{\delta}$. Working in M, we can find an elementary submodel X of M' such that $\xi \subseteq X$, $C \in X$ and $X \cap \beta \in C \cap A$. Let $\delta = X \cap \beta$. Then the Cantor-Bendixon rank of $\langle \Phi_{\alpha} : \alpha \in (\xi, \delta) \rangle$ is less than the ordertype of $X \cap \zeta$, which means that every path through $\langle \Phi_{\alpha} : \alpha \in (\xi, \delta) \rangle$, in particular ψ_{δ}, is in the transitive collapse of X. This means that $\psi_{\delta} = V_{\delta,\beta}(\phi_{\alpha})$ for some $\alpha < \delta$, giving a contradiction.

For part (6), the assumption that $\mathcal{P}$ is nonterminating, plus Proposition 9.13, implies that for every limit ordinal $\alpha < \beta$, Φ_{α} has a non-isolated path. Applying part (5) gives a non-isolated ϕ.

For part (7), let A be the set of $\alpha < \beta$ for which Φ_{α} does not amalgmate. Working in M, for each $\alpha \in A$, pick $m_{\alpha} < n_{\alpha}$ in ω, $\phi_{\alpha} \in \Phi^{m_{\alpha}+1}_{\alpha}$ and $\psi_{\alpha} \in \Phi^{n_{\alpha}}_{\alpha}$ such that $H^{m_{\alpha}+1}_{\alpha}(\phi_{\alpha}, i_{m_{\alpha}}) = H^{n_{\alpha}}_{\alpha}(\psi_{\alpha}, i_{m_{\alpha}})$, and for which there is no amalgamating formula in Φ_{α}. Applying part (5), there exist ϕ and ψ in Φ_{β} and a stationary set $B \subseteq A$ such that, for all $\alpha \in B$, $V_{\alpha,\beta}(\phi) = \phi_{\alpha}$ and $V_{\alpha,\beta}(\psi) = \psi_{\alpha}$.

Letting p and n be such that $\phi \in \Phi^p_\beta$ and $\psi \in \Phi^n_\beta$, it follows that $0 < p \leq n$, and $H^p_\beta(\phi, i_{p-1}) = H^n_\beta(\psi, i_{p-1})$. Applying part (3), there exist $\theta \in \Phi^{n+1}_\beta$ and $y \in X_{n+1} \setminus X_{p-1}$ such that $H^{n+1}_\beta(\theta, i_n) = \psi$ and $H^{n+1}_\beta(\theta, i_{p-1} \cup \{(x_{p-1}, y)\}) = \phi$. Then, for any $\alpha \in B$, $V_{\alpha,\beta}(\theta)$ contradicts our assumption that $\alpha \in A$. $\square$

By Theorem 7.11, there is exactly one model of a Scott process

$$\langle \Phi_\alpha : \alpha \leq \beta \rangle$$

of Scott rank β, if $|\Phi_\beta| \leq \aleph_1$, $\langle \Phi_\alpha : \alpha < \beta \rangle$ is nonterminating and Φ_β amalgamantes.

Putting Theorem 11.1 together with the results of Section 10, we get Theorem 11.2 below. In conjunction with Theorem 10.6, we have the following theorem of Sacks (see [16, 17]): if ϕ is a counterexample to Vaught's Conjecture, then for club many ordinals α below each of ω_1 and ω_2, ϕ has two nonisomorphic models of Scott rank α.

{twomodelclubs}

Theorem 11.2. *Let τ be a countable relational vocabulary, and suppose that $\phi \in L_{\omega_1,\omega}(\tau)$ is a counterexample to Vaught's Conjecture. Then for club many ordinals α below each of ω_1 and ω_2, ϕ has a model Scott rank α which is not Scott rank atomic.*

Proof. Let κ be either ω or ω_1 (of V). Work in $L[\phi]$. By Corollary 10.2 (and the fact that ϕ remains a counterexample to Vaught's Conjecture in any forcing extension, as discussed in Remark 10.5), there exists a nonterminating Scott process $\mathcal{P} = \langle \Phi_\alpha : \alpha < \kappa^+ \rangle$ such that

- in any forcing extension in which κ^+ is countable, $\mathcal{P}$ is satisfied by a model of ϕ;
- each Φ_α has cardinality less than κ^+.

Letting η be the quantifier depth of ϕ, we have by Theorem 1.2 that any model of $\langle \Phi_\alpha : \alpha \leq \eta \rangle$ is a model of ϕ. By part (7) of Theorem 11.1, Φ_β amalgamates, for club many $\beta < \kappa^+$. Fix such a β. Since $\mathcal{P}$ is a nonterminating Scott process of length greater than β, Φ_β contains a non-isolated path, by Proposition 9.13. By Theorem 7.11, there is a model of $\langle \Phi_\alpha : \alpha \leq \beta \rangle$ of Scott rank β (it cannot be of Scott rank less than β, since $\mathcal{P}$ is nonterminating). $\square$

The following question, a natural follow-up to Theorems 10.6 and 11.2, appears to be open.

11.3 Question. Let τ be a countable relational vocabulary, and suppose that $\phi \in L_{\omega_1,\omega}(\tau)$ is a counterexample to Vaught's Conjecture. Must there be an ordinal α such that ϕ has three nonisomorphic models of Scott rank α?

A positive answer to the previous question would follow from a positive answer to both parts of the following question. The question has several natural variations (for instance, one could strengthen the assumption on γ, as in parts (5) and (6) of Theorem 11.1).

11.4 Question. Suppose that $\mathcal{P} = \langle \Phi_\alpha : \alpha < \beta \rangle$ is a nonterminating scattered Scott process, where β is a limit ordinal. Let $\gamma > \beta$ be an ordinal, and let $M = L_\gamma[\langle \Phi_\alpha : \alpha < \beta \rangle]$. Suppose that, in M, the cofinality of β is greater than $|\Phi_\alpha|$, for each $\alpha < \beta$. Let Φ_β be the set of paths through $\mathcal{P}$, and suppose that $\Phi_\beta \subseteq M$.

1. Must $\mathrm{ms}(\phi)$ amalgamate, for each $\phi \in \Phi_\beta$?
2. Must there be a non-isolated $\phi \in \Phi_\beta$ such that $\mathrm{ms}(\phi) \neq \Phi_\beta$?

12 Isomorphic subprocesses

{Isomsubsec}

Recall that a *Scott subprocesses* is a set of the form $\langle \Phi_\alpha : \alpha \in I \rangle$, for some Scott process $\{\Phi_\alpha : \alpha < \beta\}$ and $I \subseteq \beta$. An *isomorphism* between Scott subprocesses $\langle \Phi_\alpha : \alpha \in I \rangle$ and $\langle \Upsilon_\alpha : \alpha \in J \rangle$ is a bijection

$$\pi \colon \bigcup \{\Phi_\alpha : \alpha \in I\} \to \bigcup \{\Upsilon_\alpha : \alpha \in J\}$$

which commutes with the vertical and horizontal projection functions, i.e., such that there is an order preserving bijection σ from I to J and,

- for all $\alpha \leq \gamma$ in I, and all $\phi \in \Phi_\gamma$, $V_{\sigma(\alpha),\sigma(\gamma)}(\pi(\phi)) = \pi(V_{\alpha,\gamma}(\phi))$;
- for all $\alpha \in I$, $m \leq n$ in ω, $\phi \in \Phi^n_\alpha$ and $j \in \mathcal{I}_{m,n}$,

$$H^n_{\sigma(\alpha)}(\pi(\phi), j) = \pi(H^n_\alpha(\phi, j)).$$

The following theorem shows, among other things, that a Scott process of successor length is essentially determined by how the projection functions act on its first and last levels.

{subisom}

Theorem 12.1. *Let $\langle \Phi_\alpha : \alpha \leq \beta \rangle$ and $\langle \Upsilon_\alpha : \alpha \leq \gamma \rangle$ be nonterminating Scott processes. Let $\delta < \beta$ and $\epsilon < \gamma$ such that the Scott subprocesses $\{\Phi_\delta, \Phi_\beta\}$ and $\{\Upsilon_\epsilon, \Upsilon_\gamma\}$ are isomorphic. Then the intervals $[\delta, \beta]$ and $[\epsilon, \gamma]$ have the same ordertype, and $\langle \Phi_\alpha : \alpha \in [\delta, \beta] \rangle$ and $\langle \Upsilon_\alpha : \alpha \in [\epsilon, \gamma] \rangle$ are isomorphic.*

Proof. Let $\pi \colon \Phi_\delta \cup \Phi_\beta \to \Upsilon_\epsilon \cup \Upsilon_\gamma$ be an isomorphism. Let ζ be such that $\delta + \zeta = \beta$. Without loss of generality, we may assume that $\epsilon + \zeta \leq \gamma$. We define recursively, for $\eta < \zeta$ a $\subseteq$-increasing sequence of isomorphisms

$$\pi_\eta \colon \bigcup \{\Phi_\alpha : \alpha \in [\delta, \delta + \eta] \cup \{\beta\}\} \to \bigcup \{\Upsilon_\alpha : \alpha \in [\epsilon, \epsilon + \eta] \cup \{\gamma\}\},$$

with π_0 as π. When η is a limit ordinal, the existence of a unique extension as desired follows from condition (1c) of Definition 3.1, applied to $\Phi_{\delta+\eta}$, Φ_β, $\Upsilon_{\epsilon+\eta}$. The successor case is similar, but slightly more involved : for each $n \in \omega$ and each $\phi \in \Phi^n_{\delta+\eta+1}$, the formula $V_{\delta+\eta,\beta}(\psi)$ and the set

$$V_{\delta+\eta,\beta}[\{\theta \in \Phi^{n+1}_\beta : H^{n+1}_\beta(\theta, i_n) = \psi\}]$$

are the same for all $\psi \in \Phi_\beta \cap V^{-1}_{\delta+\eta+1,\beta}[\{\phi\}]$. It follows that the formula $V_{\epsilon+\eta,\gamma}(\pi_\eta(\psi))$ and the set

$$V_{\epsilon+\eta,\gamma}[\{\theta \in \Upsilon^{n+1}_\gamma : H^{n+1}_\gamma(\theta, i_n) = \pi_\eta(\psi)\}],$$

and therefore, $V_{\epsilon+\eta+1,\gamma}(\pi_\eta(\psi))$ are the same for all such ψ, by Proposition 4.4. This common value of $V_{\epsilon+\eta+1,\gamma}(\pi_\eta(\psi))$ is the appropriate value for $\pi_{\eta+1}(\phi)$.

Finally, having defined π_η for each $\eta < \zeta$, let π^* be the union of these functions. Then π^* is an isomorphism from $\bigcup\{\Phi_\alpha : \alpha \in [\delta, \beta]\}$ to $\bigcup\{\Upsilon_\alpha : \alpha \in [\epsilon, \epsilon + \zeta), \gamma\}$. It follows then that $V_{\epsilon+\zeta,\gamma}$ is injective on Υ_γ (essentially by the argument just given for constructing the maps π_η). Since $\langle \Upsilon_\alpha : \alpha \leq \gamma\rangle$ was assumed to be nonterminating, we have that $\epsilon + \zeta = \gamma$. □

12.2 Remark. The assumption that the Scott subprocesses in Theorem 12.1 are nonterminating is mostly for notational convenience. Without this assumption, and adding the assumption that the ordertype of the interval $[\delta, \beta]$ is at most that of the interval $[\xi, \gamma]$, the proof gives directly that $\langle \Phi_\alpha : \alpha \in [\delta, \beta]\rangle$ is isomorphic to $\langle \Upsilon_\alpha : \alpha \in [\epsilon, \epsilon + \zeta) \cup \{\gamma\}\rangle$, where ζ is such that $\delta + \zeta = \beta$. In general, if $\langle \Phi_\alpha : \alpha \leq \delta\rangle$ is a Scott process, γ is an element of δ, I is a subset of γ and $V_{\gamma,\delta}\restriction\Phi_\delta$ is injective, then $\langle \Phi_\alpha : \alpha \in I \cup \{\gamma\}\rangle$ is isomorphic to $\langle \Phi_\alpha : \alpha \in I \cup \{\delta\}\rangle$.

The following theorem shows that, up to isomorphism, the tree of Scott processes extending a given a Scott process of successor length is determined by the last two levels of the Scott processes, up to isomorphism.

{recoverytheorem}

Theorem 12.3. *Let $\mathcal{P} = \langle \Phi_\alpha : \alpha \leq \beta + 1\rangle$ and $\mathcal{Q} = \langle \Upsilon_\alpha : \alpha \leq \gamma + 1\rangle$ be Scott processes such that the Scott subprocesses $\{\Phi_\beta \cup \Phi_{\beta+1}\}$ and $\{\Upsilon_\gamma \cup \Upsilon_{\gamma+1}\}$ are isomorphic. Then for each ordinal $\delta > 0$ and each Scott processes $\langle \Phi_\alpha : \alpha \leq \beta + 1 + \delta\rangle$ extending $\mathcal{P}$ there is a unique Scott processes $\langle \Upsilon_\alpha : \alpha < \gamma + 1 + \delta\rangle$ extending $\mathcal{Q}$ such that the Scott subproceses $\langle \Phi_\alpha : \alpha \in [\beta, \beta + 1 + \delta)\rangle$ and $\langle \Upsilon_\alpha : \alpha \in [\gamma, \gamma + 1 + \delta)\rangle$ are isomorphic.*

Proof. By induction on δ. The limit case is immediate. For the case $\delta + 1$, the desired set $\Upsilon_{\gamma+1+\delta+1}$ is induced by the fact that each $\phi \in \Phi_{\beta+1+\delta+1}$ is uniquely determined by $V_{\beta+1+\delta,\beta+1+\delta+1}(\phi)$ and $E(\phi)$. Checking that this induced set $\Upsilon_{\gamma+1+\delta+1}$ gives a Scott process is routine for essentially all the conditions of Definition 3.1. For condition (2b), it follows from the fact that the domain of our given isomorphism contains cofinally many levels below $\gamma + 1 + \delta$. □

{recrem}

12.4 Remark. We needed to start with an isomorphism on a pair of levels in Theorem 12.3, as opposed to just one level, in order to ensure that the successor levels of $\mathcal{Q}$ would satisfy condition (2b) of Definition 3.1. In Theorem 12.5 below this issue does not arise, since we start our construction of $\mathcal{Q}$ at level 0, so there are no instances of condition (2b) to consider.

In combination with the previous theorem, the Theorem 12.5 shows that if there is a counterexample to Vaught's Conjecture, then there is one given by a Scott process of length 2. This gives another proof of the well-known fact that if there is a counterexample to Vaught's Conjecture, then there is one given by a $\mathcal{L}_{\aleph_1,\aleph_0}$ sentence of quantifier depth ω.

{startingover}

Theorem 12.5. *Let β, γ be ordinals, and let $\mathcal{P} = \langle \Phi_\alpha : \alpha \in [\beta, \beta + \gamma) \rangle$ be a Scott subprocess. Then there is a Scott process $\mathcal{Q} = \langle \Upsilon_\alpha : \alpha < \gamma \rangle$ isomorphic to $\mathcal{P}$, over a distinct vocabulary.*

Proof. By Theorem 12.3 and Remark 12.4, it suffices to produce a Scott process of length 1 whose unique level is isomorphic to Φ_β. Let τ be the vocabulary corresponding to $\mathcal{P}$. Let μ the vocabulary consisting of, in addition to $=$, a relation symbol R_ϕ for each $\phi \in \Phi_\beta$, where R_ϕ and ϕ have the same arity. We construct Υ_0 and the desired isomorphism π by defining the formula $\pi(\phi)$ for each $\phi \in \Phi_\beta$.

Fix $n \in \omega$ and Φ^n_β. The formula $\pi(\phi)$ will be a conjunction consisting of one instance each of the formula $x_i \neq x_j$, for distinct pair x_i, x_j from X_n, and for each formula of the form $R_\psi(y_0, \ldots, y_{m-1})$ (for $m \in \omega$, $\psi \in \Phi^m_\beta$ and $\{y_0, \ldots, y_{m-1}\} \subseteq X_n$) either this formula or its negation. If there exist $i < j < m$ such that $y_i = y_j$ (in particular, if $m > n$) we choose the negation. Otherwise, letting $f \in \mathcal{I}_{m,n}$ be such that $y_i = x_{f(i)}$ for each $i < m$, we choose $R_\psi(y_0, \ldots, y_{m-1})$ if and only if $H^n_\beta(\phi, f) = \psi$. This determines $\pi(\phi)$.

To check that this works, consider $n \in \omega$, $\phi \in \Phi^n_\beta$, $k \leq n$ and $g \in \mathcal{I}_{k,n}$. Let $\theta = H^n_\beta(\phi, g)$. We want to see that $\pi(\theta)$ is the set of conjuncts from $\pi(\phi)$ whose variables are contained in the range of g, with each variable replaced by its g-preimage. For the conjuncts of the form $x_i \neq x_j$ this is clear. Now suppose that we have a formula of the form $R_\psi(y_0, \ldots, y_{m-1})$, for some $m \in \omega$, $\psi \in \Phi^m_\beta$ and $\{y_0, \ldots, y_{m-1}\} \subseteq X_k$. Exactly one of $R_\psi(y_0, \ldots, y_{m-1})$ and its negation is a conjunct of $\pi(\theta)$, and we want to see that $R_\psi(y_0, \ldots, y_{m-1})$ is a conjunct of $\pi(\theta)$ if and only if $R_\psi(g(y_0), \ldots, g(y_{m-1}))$ is a conjunct of ϕ. The case where there exists an $i < j < m$ such that $y_i = y_j$ works out (in each direction), since g is an injection. In the other case, let $f \in \mathcal{I}_{k,m}$ be such that $y_i = x_{f(i)}$ for each $i < m$. Then $H^n_\beta(\phi, g \circ f) = H^k_\beta(\theta, f)$, by part (2) of Remark 2.14, so $H^k_\beta(\theta, f) = \phi$ if and only if $H^n_\beta(\phi, g \circ f) = \psi$, as desired. For the reverse direction, suppose that we have a formula of the form $R_\psi(y_0, \ldots, y_{m-1})$, for some $m \in \omega$ and $\psi \in \Phi^m_\beta$, with $\{y_0, \ldots, y_{m-1}\}$ contained in the range of g. Then the formula $R_\psi(g^{-1}(y_0), \ldots, g^{-1}(y_{m-1}))$ is of the type just considered (i.e., $\{g^{-1}(y_0), \ldots, g^{-1}(y_{m-1})\} \subseteq X_k$), and we are done. $\square$

13 Projection structures

{projstructsec}

The results of section 12 show that if there is a counterexample to Vaught's Conjecture then there is one whose models are essentially the Scott processes of the structures from the given counterexample (note, however, that while the Scott processes of two structures being isomorphic does not imply that the structures themselves are isomorphic, it does imply that their Scott ranks are the same). We make this explicit in this section. However, we leave most of the verification to the reader, as the details are essentially the same as the arguments of the previous section.

We consider in this section structures whose relations satisfy the properties of the projection functions on Scott subprocesses, and whose points play the role of the formulas in a Scott process. These structures could be defined more generally, but we concentrate on a case (i.e., subprocesses with four levels, corresponding to levels 0, 1, γ and $\gamma+1$ of a structure of Scott rank γ) that seems more relevant to Vaught's Conjecture.

We let μ^* be the vocabulary consisting of $=$, unary predictate symbols $\mathtt{P_{n,i}}$ ($n \in \omega$, $i \in 4$) and unary function symbols $\mathtt{v_i}$ for $i \in 4$ and $\mathtt{h_f}$ for f in

$$\bigcup\{\mathcal{I}_{m,n} : m \leq n < \omega\}.$$

If M is a μ^*-structure, we let L_i^M and $L_{\geq i}^M$ denote the sets $\bigcup\{\mathtt{P^M_{n,i}} : \mathtt{n} \in \omega\}$ and $\bigcup\{\mathtt{P^M_{n,j}} : \mathtt{n} \in \omega, \mathtt{j} \in \mathtt{4} \setminus \mathtt{i}\}$ respectively, for each $i \in 4$. Similarly, we let R_m^M and $R_{\geq m}^M$ denote the sets $\bigcup\{\mathtt{P^M_{m,i}} : \mathtt{i} \in \mathtt{4}\}$ and $\bigcup\{\mathtt{P^M_{n,i}} : \mathtt{n} \in \omega \setminus \mathtt{m}, \mathtt{i} \in \mathtt{4}\}$ respectively, for each $m \in \omega$. We say that a *projection structure* is a μ^*-structure M such that the following hold (the following lists established properties of the projection functions, plus parts of the definition of Scott process corresponding to a Scott subprocess whose first two levels are the first two levels of the corresponding process, and such that the vertical projection function from the last level to the second-to-last level is injective).

1. The sets $\mathtt{P^M_{n,i}}$ ($n \in \omega$, $i \in 4$) are nonempty and partition the domain of M. (The $\mathtt{P^M_{n,i}}$'s correspond to Φ^n_α's.)
2. Each $\mathtt{v^M_i}$ has domain $L^M_{\geq i}$ and range L^M_i, and is the identity function on L^M_i. (Each $\mathtt{v^M_i}$ corresponds to a function of the form $\bigcup\{\beta \in I\}V_{\alpha,\beta}$, for I the set of levels at or above α in the corresponding subprocess.)
3. For all $i \leq j < 4$, $\mathtt{v^M_i} \circ \mathtt{v^M_j} = \mathtt{v^M_i}\restriction\mathtt{L^M_j}$. (This corresponds to Remark 2.8.)
4. The function $\mathtt{v^M_3}$ is injective. (The third level of our structures correspond to the γ-th level of a structure of Scott rank γ, and the fourth level to the $(\gamma+1)$-st level.)
5. For all $i \leq j < 4$, and $n \in \omega$, $\mathtt{P^M_{n,i}} = \mathtt{v^M_i}[\mathtt{P^M_{n,j}}]$. (This corresponds to part (1c) of Definition 3.1.)
6. For all $m \leq n$ in ω and all $f \in \mathcal{I}_{m,n} \setminus \bigcup\{\mathcal{I}_{m,p} : p \in [m,n)\}$, $\mathtt{h^M_f}$ has domain $R^M_{\geq n}$ and range R^M_m. (Each $\mathtt{h^M_f}$ corresponds to a function of the form $\phi \mapsto H(\phi, f)$, omitting the subscripts and superscripts on H.)
7. For all $i < j < 4$, $m \leq n$ in ω, $f \in \mathcal{I}_{m,n}$ and $p \in \mathtt{P^M_{n,j}}$,

   $$\mathtt{v_i}[\{\mathtt{q} \in \mathtt{P^M_{m+1,j}} : \mathtt{h^M_{i_m}}(\mathtt{q}) = \mathtt{h^M_f}(\mathtt{p})\}]$$

   is equal to

   $$\{\mathtt{h^M_{i_m \cup \{(x_m,y\}}}(\mathtt{q}) : \mathtt{y} \in \mathtt{X_{n+1}} \setminus \mathtt{X_m}, \mathtt{q} \in \mathtt{v_i}[\{\mathtt{r} \in \mathtt{P^M_{n+1,j}} : \mathtt{h^M_{i_n}}(\mathtt{r}) = \mathtt{p}\}]\}$$

   (This is a combination of part (3) of Definition 2.11 with Proposition 4.4.)

8. For all $i < 4$, all $n \in \omega$, all $f \in \mathcal{I}_{n,n}$ and all $p \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n,i}}$, $\mathtt{h}^{\mathtt{M}}_{\mathtt{f}}(\mathtt{p}) \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n,i}}$. (This corresponds to part (1d) of Definition 3.1.)

9. For all $i < 4$, and all $m \leq n$ in ω, $\mathtt{P}^{\mathtt{M}}_{\mathtt{m,i}} = \mathtt{h}^{\mathtt{M}}_{\mathtt{i_m}}[\mathtt{P}^{\mathtt{M}}_{\mathtt{n,i}}]$. (This corresponds to part (1e) of Definition 3.1.)

10. For all $i < 4$, all $m \leq n \leq p$ in ω, all $p \in \mathtt{P}^{\mathtt{M}}_{\mathtt{p,i}}$, all $f \in \mathcal{I}_{n,p}$ and all $g \in \mathcal{I}_{m,n}$, $\mathtt{h}^{\mathtt{M}}_{\mathtt{g}}(\mathtt{h}^{\mathtt{M}}_{\mathtt{f}}(\mathtt{p})) = \mathtt{h}^{\mathtt{M}}_{\mathtt{f} \circ \mathtt{g}}(\mathtt{p})$. (This corresponds to part (2) of Remark 2.14.)

11. For all $i \leq j < 4$, all $m \leq n$ in ω, all $f \in \mathcal{I}_{m,n}$, and all $p \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n},\beta}$,

$$\mathtt{v}^{\mathtt{M}}_{\mathtt{i}}(\mathtt{h}^{\mathtt{M}}_{\mathtt{f}}(\mathtt{p})) = \mathtt{h}^{\mathtt{M}}_{\mathtt{f}}(\mathtt{v}^{\mathtt{M}}_{\mathtt{i}}(\mathtt{p})).$$

(This corresponds to Proposition 2.15.)

12. For all $n \in \omega$ and all distinct $p, q \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n},1}$,

$$\mathtt{v}^{\mathtt{M}}_{0}[\{\mathtt{r} \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n+1,1}} : \mathtt{h}^{\mathtt{M}}_{\mathtt{i_n}}(\mathtt{r}) = \mathtt{p}\}] \neq \mathtt{v}^{\mathtt{M}}_{0}[\{\mathtt{r} \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n+1,1}} : \mathtt{h}^{\mathtt{M}}_{\mathtt{i_n}}(\mathtt{r}) = \mathtt{q}\}].$$

(This corresponds to the fact that formulas on level 1 are determined by their E-sets, which in turn are definable from the projection functions, by condition (2a) of Definition 3.1.)

13. For all $n \in \omega$, $p \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n},1}$ and $q \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n},2}$ with $\mathtt{v}^{\mathtt{M}}_{1}(\mathtt{q}) = \mathtt{p}$,

$$\mathtt{v}^{\mathtt{M}}_{0}[\{\mathtt{r} \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n+1,2}} : \mathtt{h}^{\mathtt{M}}_{\mathtt{i_n}}(\mathtt{r}) = \mathtt{q}\}] = \mathtt{v}^{\mathtt{M}}_{0}[\{\mathtt{r} \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n+1,1}} : \mathtt{h}^{\mathtt{M}}_{\mathtt{i_n}}(\mathtt{r}) = \mathtt{p}\}].$$

(This corresponds to Proposition 4.4.)

14. For all $i < 4$, n, m in ω, $p \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n,i}}$ and $q \in \mathtt{P}^{\mathtt{M}}_{\mathtt{m,i}}$, there exist $r \in \mathtt{P}^{\mathtt{M}}_{\mathtt{n+m,i}}$ and $f \in \mathcal{I}_{m,n+m}$ such that $p = \mathtt{h}^{\mathtt{M}}_{\mathtt{i_n}}(\mathtt{r})$ and $q = \mathtt{h}^{\mathtt{M}}_{\mathtt{f}}(\mathtt{r})$. (This corresponds to part (2c) of Definition 3.1.)

If $\mathcal{P}$ is a Scott process of length 2 (over a relational vocabulary τ as assumed in this paper), then there is a surjective correspondence between the Scott processes extending $\mathcal{P}$ whose length is two more than their rank and the projection structures (as defined in this section) whose first two levels are isomorphic to $\mathcal{P}$ (with the horizontal and vertical projection functions corresponding to the functions $\mathtt{h_f}$ and $\mathtt{v_i}$). This is essentially Theorem 12.3. This correspondence may not be injective, as non-isomorphic structures may have isomorphic Scott processes (consider for instance a structure M with a unary predicate P, where the restrictions of M to P^M and $|M| \setminus P^M$ are nonisomorphic; the structure with the interpretation of P reversed gives an isomorphic Scott process). It follows that if Vaught's Conjecture is false, then there is a counterexample consisting of all projection structures whose first two levels are isomorphic to some fixed Scott process of length 2 (equivalently, an equivalence class of the class of projection structures under isomorphism of the first two levels).

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