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SOME RESULTS ABOUT (+) PROVED BY ITERATED FORCING

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Abstract. We shall show that the consistency of CH $+\neg(+)$ and CH $+(+)+$there are no club guessing sequences on ω_1. We shall also prove that $\diamondsuit^+$ does not imply the existence of a strong club guessing sequence on ω_1.

§0. Introduction. The principle (+) and its variations were first considered by the second author in [2]. They are very weak club guessing principles. The properties of the principles were largely unknown until recently. While J. Moore proved that MRP implies the negation of (+), it was not known whether the negation of (+) has any large cardinal strength, or CH implies (+).

The first main result is to show that just from ZFC, we can build a model of CH $+\neg(+)$. Hence, it answers both questions in the previous paragraph. We also build a model of CH $+(+)$ in which there is no club guessing sequence on ω_1. This is the first model satisfying these properties.

The last part of this paper is devoted to construct a model of $\diamondsuit^+$ in which there is no strong club guessing sequence on ω_1. It answers the question asked by the first author in [1]. The proof in fact builds a model of $\diamondsuit^+$ in which the "strong" version of (+) fails. This demonstrates how effective the use of variations of (+) is in the investigation of guessing principles.

The structure of this paper is as follows. In Section 1, we shall give the definitions of $(+)_k$, $(+)_{<\omega}$, and related notions. In Section 2, the results by S. Shelah and J. Moore about the iteration adding no new reals are described. It will be used repeatedly in the later sections. Then, we shall build a model of CH $+\neg(+)$ in Section 3. In Section 4, we shall prove some lemmas about the internalization. They are slightly improved from the ones in [1]. By using these lemmas, we shall prove the consistency of CH $+(+)_{<\omega}+$ there is no club guessing sequence on ω_1 in Section 5, and $\diamondsuit^+ +$ 'there is no strong club guessing sequence on ω_1' in Section 6.

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§1. The principle $(+)$**.** The following principle was introduced by the second author in [2].

Definition 1.1. Let $k < \omega$ and S a stationary subset of $\omega_1 \cap \mathrm{Lim}$. $(+)_k(S)$ is defined as the principle that asserts the existence of a stationary subset T of $[H(\omega_2)]^{\aleph_0}$ such that for every $N \in T$, $N \cap \omega_1 \in S$ and for $N_0, \dots, N_{k-1} \in T$ with $N_i \cap \omega_1 = N_0 \cap \omega_1 \in S$ for every $i < k$, if $D_i \in N_i$ is a club subset of ω_1 for every $i < k$, then $\bigcap_{i<k} D_i \cap N_0 \neq \varnothing$. $(+)(S)$ denotes $(+)_2(S)$.

$(+)_{<\omega}(S)$ is defined as the principle that asserts the existence of a stationary subset T of $[H(\omega_2)]^{\aleph_0}$ that witnesses $(+)_k(S)$ for every $k < \omega$, i.e. for every $N \in T$, $N \cap \omega_1 \in S$ and for every finite subset $\{N_0, \dots, N_{k-1}\}$ of T with $N_i \cap \omega_1 = N_0 \cap \omega_1 \in S$ for every $i < k$, if $D_i \in N_i$ is a club subset of ω_1 for every $i < k$, then $\bigcap_{i<k} D_i \cap N_0 \neq \varnothing$.

Trivially, $(+)_{<\omega}(S)$ implies $(+)_k(S)$ for every $k < \omega$ and for every $k < \omega$, $(+)_{k+1}(S)$ implies $(+)_k(S)$.

Definition 1.2. We say that $\langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ is a $(+)_{<\omega}$-sequence if and only if

(i) for every $\delta \in \omega_1 \cap \mathrm{Lim}$, F_δ is a filter on δ such that every cobounded subset of δ belongs to F_δ, and
(ii) for every club subset D of ω_1, there exists a $\delta \in \omega_1 \cap \mathrm{Lim}$ such that $D \cap \delta \in F_\delta$.

Definition 1.3. Let $k < \omega$. We say that $\langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ is a $(+)_k$-sequence if and only if

(i) for every $\delta \in \omega_1 \cap \mathrm{Lim}$, F_δ is a family of subsets of δ such that F_δ is closed under superset and contains all cobounded subsets of δ and for every $x_0, \dots, x_{k-1} \in F_\delta$, $\bigcap_{i<k} x_k$ is unbounded in δ, and
(ii) for every club subset D of ω_1, there exists a $\delta \in \omega_1 \cap \mathrm{Lim}$ such that $D \cap \delta \in F_\delta$.

We omit k when $k = 2$.

J. Moore showed that $(+)_{<\omega}$ holds if and only if there exists a $(+)_{<\omega}$-sequence and $(+)_k$ holds if and only if there exists a $(+)_k$-sequence.

§2. Iteration adding no new reals. For every set X, let $\bar{\theta}_X$ be the least regular θ cardinal such that $\mathcal{P}(X) \in H(\theta)$ and $\theta_X = \left(2^{|H(\bar{\theta}_X)|}\right)^+$. Notice that if P is a forcing notion, then for every regular cardinal $\theta \geq \theta_P$, if $G \subseteq P$ is generic, then $H(\theta)^{V[G]} = H(\theta)^V[G]$.

Definition 2.1. Let P be a forcing notion. If P is proper and adds no new reals, then we say that P is *totally proper*. Let N be a set (typically a countable elementary submodel of some $H(\theta)$). We say that a condition $p \in P$ is *totally* (N, P)*-generic* if and only if p is (N, P)-generic and p decides all open dense subsets of P lying in N.

We say that a condition $p \in P$ is *finitely* (N, P)*-generic* if and only if p is (N, P)-generic and for every maximal antichain $\mathcal{A}$ of P lying in N, there are at most finitely many $a \in \mathcal{A}$ that is compatible with p.

Clearly, for every proper forcing notion P, P is ω^ω-bounding if and only if whenever N is a countable elementary submodel of $H(\theta_P)$, $P \in N$, and $p \in P \cap N$, there exists a $q \leq p$ that is finitely (N, P)-generic.

The following lemma is due to S. Shelah and proved in [4].

Lemma 2.2. *Let P be an ω^ω-bounding proper forcing notion and $\dot{Q}$ a P-name for an ω^ω-bounding proper forcing notion. Let $\theta \geq \theta_{P*\dot{Q}}$ be a regular cardinal. N_0 and N_1 two countable elementary submodels of $H(\theta)$ with $P, \dot{Q} \in N_0 \in N_1$. Suppose that p is finitely (N_0, P)-generic and (N_1, P)-generic, $\dot{q} \in N_0$ is a P-name for an element of $\dot{Q}$. Then, there exists a P-name $\dot{q}'$ such that $(p, \dot{q}')$ is finitely $(N_0, P * \dot{Q})$-generic and $(N_1, P * \dot{Q})$-generic.*

The main point of the previous lemma is that if p is strong enough, then we do not have to extend p to find a finitely $(N_0, P * \dot{Q})$-generic condition. This is a key lemma to prove the preservation of ω^ω-bounding forcing by countable support iteration.

It was pointed out by S. Shelah in [4] that a countable-support iteration of totally proper forcing notions may add a new real. In the same book, he gave several conditions that guarantees that the iteration adds no new reals. The following is one of them.

Lemma 2.3. *Suppose that $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \eta \rangle$ is a countable support iteration such that for every $\alpha < \eta$, P_α forces that*

(i) *$\dot{Q}_\beta$ is $\mathbb{D}$-complete with respect to some simple 2-completeness system $\mathbb{D}$, and*
(ii) *$\dot{Q}_\beta$ is proper in every totally proper extension.*

Then, P_η adds no new reals.

Instead of completeness systems, we shall use the notion of completely proper forcing, introduced by J. Moore in [3]. Consider the language of ZFC with a predicate P for a distinguished forcing. Let ZFC^P be the axioms of ZFC with the power set axiom replaced by "$\mathcal{P}(\mathcal{P}(P))$ exists". The objects of the category $\mathfrak{M}$ are countable transitive sets M together with a distinguished element P^M such that M satisfies ZFC^P when P is interpreted as P^M.

An *arrow* $\overrightarrow{MN}$ in $\mathfrak{M}$ is an elementary embedding $\varepsilon : M \to N$ with the property that $\varepsilon \in N$ and $N \models$ '$M = \mathrm{dom}(\varepsilon)$ is countable'. We write $M \to N$ to mean that $\overrightarrow{MN}$ is an arrow in $\mathfrak{M}$. We usually consider commutative diagrams in $\mathfrak{M}$, so there will be at most one arrow between two given objects.

Definition 2.4. Suppose that $\hat{N}$ is a model of ZFC^P and $\hat{M}$ is an elementary submodel of $\hat{N}$ such that $\hat{M} \in \hat{N}$ and $\hat{N} \models$'$\hat{M}$ is countable'. Let M and N be the transitive collapses of $\hat{M}$ and $\hat{N}$ respectively. Then, there is a unique *induced arrow* $\overrightarrow{MN}$ that commutes with the collapsing map (See Figure 1).

Definition 2.5. Let θ be a regular cardinal. Then, a (P, θ)-*diagram* is a diagram in $\mathfrak{M}$ such that there exist a minimum M in the order induced from the arrows and an elementary embedding $\varepsilon : M \to H(\theta)$ such that $\varepsilon(P^M) = P$. Let $\hat{M}$ denote $\varepsilon''M$. A P-diagram means a (P, θ)-diagram for some θ.

$$\begin{tikzcd}
\hat{M} \arrow[r, "id"] \arrow[d, "col"] & \hat{N} \arrow[d, "col"] \\
M \arrow[r, "\overrightarrow{MN}"] & N
\end{tikzcd}$$

FIGURE 1. Induced arrow

$$\begin{tikzcd}
 & N_0 \arrow[r] & \tilde{N}_0 \ni G, \text{bounded} \\
M \arrow[ur] \arrow[dr] & & \\
 & N_1 \arrow[r] & \tilde{N}_1 \ni G, \text{bounded}
\end{tikzcd}$$

FIGURE 2. Completely proper forcing notion

DEFINITION 2.6. Let $\overrightarrow{MN}$ be an arrow in $\mathfrak{M}$. Let $G \subseteq P^M$. We say that G is $\overrightarrow{MN}$-*prebounded* if it satisfies the following condition: for every $\tilde{N}$ in $\mathfrak{M}$, if $G \in \tilde{N}$ and there is an arrow from N to $\tilde{N}$, we have $\tilde{N} \models$'G is bounded in $P^{\tilde{N}}$'.

Note that G is not necessarily in N, so we cannot ask if N satisfies 'G is bounded in P^N'. The point of the previous definition is that despite of this fact, no matter how we pick the expansion $\tilde{N}$ of N with $G \in \tilde{N}$, $\tilde{N}$ satisfies 'G is bounded in $P^{\tilde{N}}$'.

DEFINITION 2.7. A forcing notion P is *completely proper* if there is a regular cardinal θ such that for every (P, θ)-diagram of the form $M \to N_i$ $(i < 2)$ and $p \in P^M$, there exists a (M, P^M)-generic filter $G \subseteq P^M$ such that n $p \in G$, and G is $\overrightarrow{MN_i}$-prebounded for $i < 2$.

Figure 2 depicts the definition of completely proper forcing notions. Suppose that we are given the diagram of M, N_0, N_1. Then, for every $p \in P^M$, there exists a (M, P^M)-generic filter such that $p \in G$ and for both $i < 2$, whenever $\tilde{N}_i$ is in $\mathfrak{M}$ and $G \in \tilde{N}_i$, $\tilde{N}_i$ satisfies 'G is bounded in $P^{\tilde{N}_i}$'. Note that G may not belong to either N_0 or N_1.

In [3], J. Moore proved the following lemma.

LEMMA 2.8. *Every completely proper forcing notion is $\mathbb{D}$-complete with respect to some simple 2-completeness system $\mathbb{D}$.*

The following lemma easily follows from Lemma 2.3 and Lemma 2.8.

LEMMA 2.9. *Let $P = \langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \eta \rangle$ be a countable support iteration such that for every $\alpha < \eta$, P_α forces that*

(i) *$\dot{Q}_\alpha$ is completely proper, and*
(ii) *$\dot{Q}_\alpha$ is proper in every totally proper extension.*

Then, P is totally proper.

§3. The negation of $(+)$ is consistent with CH. This section is devoted to the construction of the model of CH $+\neg(+)$.

DEFINITION 3.1. Let $\vec{F} = \langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ be a sequence on ω_1 that satisfies (i) of Definition 1.3, i.e. for every $\delta \in \omega_1 \cap \mathrm{Lim}$, F_δ is a filter on δ such that every cobounded subset of δ belongs to F_δ. Let $P(\vec{F})$ be the forcing notion defined as $p \in P(\vec{F})$ if and only if p is a closed bounded subset of ω_1 so that for every $\delta \in p \cap \mathrm{Lim}$, $p \cap \delta \notin F_\delta$. $P(\vec{F})$ is ordered by end-extension.

LEMMA 3.2. *Let $\vec{F} = \langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ be a sequence on ω_1 that satisfies (i) of Definition 1.3. Then, $P(\vec{F})$ is proper. Moreover, $P(\vec{F})$ is proper in every totally proper extension.*

PROOF. First, we shall show that $P(\vec{F})$ is proper. Let $P = P(\vec{F})$. Let $\theta \geq \theta_P$ be a regular cardinal, N a countable elementary submodel of $H(\theta)$ with $\vec{F}, P \in N$, and $p \in P \cap N$. Define $\delta = N \cap \omega_1$. It is easy to build two generic sequence $\langle p^0_n : n < \omega\rangle$ and $\langle p^1_n : n < \omega\rangle$ for N such that $p^0_0 = p^1_0 = p$ and for every $n < \omega$, $p^0_n \cap p^1_n = p$. Define $q_0 = \bigcup_{n<\omega} p^0_n \cup \{\delta\}$ and $q_1 = \bigcup_{n<\omega} p^1_n \cup \{\delta\}$. If $q_0 \in P$, then clearly q_0 is (N,P)-generic. Suppose not. Then, $q_0 \cap \delta \in F_\delta$. Since every pair of elements in F_δ must have unbounded intersection in δ, $q_1 \cap \delta \notin F_\delta$. It follows that $q_1 \in P$ and hence q_1 is (N,P)-generic.

To see that $P(\vec{F})$ is proper in every totally proper extension, let W be any totally proper extension of V. Note that since $\mathcal{P}(\omega)^W = \mathcal{P}(\omega)^V$, $\vec{F}$ satisfies (i) of Definition 1.3 in W. Thus, we can consider $P(\vec{F})^W$. By using the same proof as above, we can show that $P(\vec{F})^W$ is proper. However, it is easy to see $P(\vec{F})^W = P(\vec{F})^V$. Therefore, $P(\vec{F})^V$ is proper in W. ⊣

LEMMA 3.3. *Let $\vec{F} = \langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ be a sequence on ω_1 that satisfies (i) of Definition 1.3. Then, $P(\vec{F})$ is completely proper.*

PROOF. Let $P = P(\vec{F})$. Let M, N_0, N_1 be countable transitive sets such that M is the transitive collapse of a countable elementary submodel $\hat{M}$ of $H(\theta)$ with $P \in \hat{M}$, $M \to N_0$, $M \to N_1$. Let $p \in P^M$. We need to show that there is a M-generic filter $G \subseteq P^M$ which is $\overrightarrow{MN_i}$-prebounded. Let $\delta = {\omega_1}^M$.

We can easily build three generic sequence $\langle p^k_n : n < \omega\rangle$ $(k < 3)$ such that $p^0_0 = p^1_0 = p^2_0 = p$ and for every $k < l < 3$ and $n < \omega$, $p^k_n \cap p^l_n = p$. Then for each $i = 0, 1$, for at most one $k < 3$, $\bigcup_{n<\omega} p^k_n \in {F_\delta}^{N_i}$. Thus, there exists a $k < 3$ such that $\bigcup_{n<\omega} p^k_n \notin {F_\delta}^{N_i}$ for $i = 0, 1$. Let $q = \bigcup_{n<\omega} p^k_n \cup \{\delta\}$ and $G = \{p' \in P^M : q \leq p'\}$. Then for $i = 0, 1$, if $N_i \to \tilde{N}_i$ and $G \in \tilde{N}_i$, then we have $q \in \tilde{N}_i$ and q is a lower bound of G. Thus, G is $\overrightarrow{MN_i}$-prebounded for $i = 0, 1$. ⊣

THEOREM 3.4. *It is consistent with* GCH *that* $(+)$ *fails.*

PROOF. We begin with the model of GCH. Let P be the countable support iteration of length ω_2 of the bookkeeping of all forcing notions of the form $P(\vec{F})$ where $\vec{F}$ is a $(+)$-sequence. By Lemma 3.3, $P(\vec{F})$ is completely proper. It is easy to see that $P(\vec{F})$ is proper in every extension that has the same ω_1 and countable subsets of ω_1. Therefore, P is proper and by Lemma 2.9, P adds no new reals. Thus, P forces that CH holds and $(+)$ fails. $\dashv$

§4. Internalization. In this section, we shall prove the lemmas that are slighly improved from the ones proved by the first author in [1], which are necessary for the next section.

First, we shall state the rough idea behind the lemmas. Let θ', θ be uncountable regular cardinals such that $\theta' < \theta$. Let $\mathfrak{A}$ be a structure expanding $\langle H(\theta), \in, \leq_{H(\theta)} \rangle$ where $\leq_{H(\theta)}$ is a fixed well-ordering of $H(\theta)$. Then, we can find a structure $\mathfrak{B}$ on $H(\theta')$ such that whenever N is a countable elementary substructure of $\mathfrak{B}$, we have $\mathrm{Sk}^{\mathfrak{A}}(N) \cap H(\theta') = N$.

In order to deal with $(+)_{<\omega}$, we need to use a tower of structures instead of a single structure.

Definition 4.1. A sequence $\langle N_\beta : \beta < \eta \rangle$ is called a *tower* if and only if

(i) (increasing) for every $\gamma < \beta < \eta$, $N_\gamma \subseteq N_\beta$,
(ii) (continuous) for every $\beta < \eta$, if β is a limit ordinal, then $N_\beta = \bigcup_{\gamma < \beta} N_\gamma$, and
(iii) for every $\delta < \eta$ with $\delta + 1 < \eta$, $\langle N_\gamma : \gamma \leq \delta \rangle \in N_{\delta+1}$.

Typically, we also assume that each N_γ is a model of ZFC^-. Here, ZFC^- denotes the axioms of ZFC without the power set axiom. Then, for example, $N_\gamma \in N_{\gamma+1}$ when $\gamma + 1 < \eta$.

However, if $\langle N_\gamma : \gamma < \delta \rangle$ is a tower of countable elementary substructures of $\mathfrak{B}$, $\langle \mathrm{Sk}^{\mathfrak{A}}(N_\gamma) : \gamma < \delta \rangle$ is not a tower in general. The reason is that since the operation of taking the Skolem hull is not definable over $\mathfrak{A}$, $\langle \mathrm{Sk}^{\mathfrak{A}}(N_\xi) : \xi \leq \gamma \rangle$ may not belong to $N_{\gamma+1}$. So, we would like to define a good closure operation that is definable over a reasonable structure and gives you back a nice substructure to work on. This is exactly the motivation for Lemma 4.3.

We temporarily say that a set A is *good* if and only if A is a transitive, $\omega_1 \in A$, $\langle A, \in \rangle \models \mathrm{ZFC}^-$, and A is closed under countable sequences, i.e. $A^{\aleph_0} \subseteq X$.

Let A be a good set, and $\leq_A$ a fixed well-ordering on A. Let $\mathfrak{A}$ be a structure expanding $\langle A, \in, \leq_A \rangle$. We shall define two sequences $\langle \mathfrak{A}_\beta : \beta < \omega_1 \rangle$ of expansions of $\mathfrak{A}$ and $\langle F_\beta : \beta < \omega_1 \rangle$ as follows. Let $\mathfrak{A}_0 = \mathfrak{A}$ and $F_0 = \varnothing$.

Suppose that we have defined $\mathfrak{A}_\gamma$ and F_γ for all $\gamma < \beta$. Then, let $F_\beta : \beta \times [A]^{\leq \aleph_0} \to A$ be defined by $F_\beta(\gamma, x) = \mathrm{Sk}^{\mathfrak{A}_\gamma}(x)$. Then, define $\mathfrak{A}_\beta = \langle \mathfrak{A}_0, F_\beta \rangle$. After we have defined $\mathfrak{A}_\beta$ and F_β for all $\beta < \omega_1$, let $F = \bigcup_{\beta < \omega_1} F_\beta$ and $\mathfrak{A}^* = \langle \mathfrak{A}_0, F \rangle$.

The following lemma lists some trivial facts.

Lemma 4.2. (i) *For every $\beta < \omega_1$ and $x \in [A]^{\leq \aleph_0}$, $F(\beta, x) \prec \mathfrak{A}$.*
(ii) *If N is a countable substructure of $\mathfrak{A}^*$ and $\beta \in N \cap \omega_1$, then N is an elementary substructure of $\mathfrak{A}_\beta$.*
(iii) *If N is a countable elementary substructure of $\mathfrak{A}_\beta$ for some $\beta < \omega_1$, and $\gamma \in N \cap \beta$, then N is an elementary substructure of $\mathfrak{A}_\gamma$.*
(iv) *If $\gamma < \beta < \omega_1$ and $\gamma \in x \in [A]^{\leq \aleph_0}$, then $F(\gamma, x) \subseteq F(\beta, x)$.*
(v) *If $x, y \in [A]^{\leq \aleph_0}$, $x \subseteq y$, and $\gamma < \omega_1$, then $F(\gamma, x) \subseteq F(\gamma, y)$.*

(vi) *For every* $\gamma < \omega_1$, $\delta < \omega_1$, *and an* $\subseteq$*-increasing sequence* $\langle x_\beta : \beta < \delta \rangle$ *in* $[A]^{\leq \aleph_0}$, $F(\gamma, \bigcup_{\beta<\delta} x_\beta) = \bigcup_{\beta<\delta} F(\gamma, x_\beta)$.

PROOF. (i) is trivial since for every $\beta < \omega_1$, $\mathfrak{A}_\beta$ is an expansion of $\mathfrak{A}$.

For (ii), let N be a countable substructure of $\mathfrak{A}^*$ and $\beta \in N \cap \omega_1$. Suppose that $\varphi(v, v_1, \dots, v_k)$ be a formula in the language of $\mathfrak{A}_\beta$. Suppose that $a_1, \dots, a_k \in N$ and $\mathfrak{A}_\beta \models (\exists v)\varphi(v, a_1, \dots, a_k)$. Let $a \in A$ be $\leq_A$-least such that $\mathfrak{A}_\beta \models \varphi(a, a_1, \dots, a_k)$. It suffices to show that $a \in N$. Let $x = F(\beta, \{a_1, \dots, a_k\})$. Since N is a substructure of $\mathfrak{A}^*$, N is closed under F. So, $x \in N$. Since x is countable, we have $x \subseteq N$. Since $x \prec \mathfrak{A}_\beta$ and a is definable from $a_1, \dots, a_k$ over $\mathfrak{A}_\beta$, we have $a \in x$. So, $a \in N$.

(iii) can be proved by the same argument as in the previous paragraph.

For (iv), notice that since $\gamma \in F(\beta, x) \prec \mathfrak{A}_\beta$, by (iii) $F(\beta, x) \prec \mathfrak{A}_\gamma$. Since $F(\gamma, x) = \mathrm{Sk}^{\mathfrak{A}_\gamma}(x)$, we have $F(\gamma, x) \subseteq F(\beta, x)$.

(v) and (vi) are clear from general facts about the Skolem hull. $\dashv$

Then, we can prove the following lemma.

LEMMA 4.3. *Let A and B be good sets with $B \in A$. Let $\leq_A$ be a fixed well-ordering on A. Let $\mathfrak{A}$ be a structure expanding $\langle A, \in, \leq_A, B \rangle$. Let $\mathfrak{B}$ be a structure expanding $\langle B, \in, \leq_A \restriction B \rangle$ such that whenever $N \prec \mathfrak{B}$, $\mathrm{Sk}^{\mathfrak{A}^*}(N) \cap B = N$. Let $\langle N_\beta : \beta < \eta \rangle$ be a tower of countable subsets of B. For each $\beta < \eta$, let $\delta_\beta = N_\beta \cap \omega_1$. Then, there exists a tower $\langle M_\beta : \beta < \eta \rangle$ of countable elementary substructures of $\mathfrak{A}$ such that for every $\beta < \eta$ if $N_\beta \prec \mathfrak{B}$, then $M_\beta \cap B = N_\beta$.*

PROOF. $\langle M_\beta : \beta < \eta \rangle$ is defined by induction as follows. Let $M_0 = F(0, N_0)$. When β is a non-zero limit ordinal and $\langle M_\gamma : \gamma < \beta \rangle$ is defined, then let $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$. For every $\beta < \eta$ with $\beta + 1 < \eta$, let $M_{\beta+1} = F(\delta_\beta + 1, N_{\beta+1})$.

Claim 1. $\langle M_\beta : \beta < \eta \rangle$ forms a tower.

$\vdash$ By definition, this sequence is continuous. We shall show that for every $\beta < \eta$ with $\beta + 1 < \eta$, $\langle M_\gamma : \gamma \leq \beta \rangle \in M_{\beta+1}$. Since $\langle N_\gamma : \gamma \leq \beta \rangle \in N_{\beta+1}$, we have $\langle N_\gamma : \gamma \leq \beta \rangle \in M_{\beta+1}$. Since $F \restriction (\delta_\beta + 1) \times [A]^{\leq \aleph_0} = F_{\delta_\beta+1}$, $\langle M_\gamma : \gamma \leq \beta \rangle$ is definable from $\langle N_\gamma : \gamma \leq \beta \rangle$ over $\mathfrak{A}_{\delta_\beta+1}$. Recall that $M_{\beta+1} = F(\delta_\beta + 1, N_{\beta+1}) \prec \mathfrak{A}_{\delta_\beta+1}$. Thus, $\langle M_\gamma : \gamma \leq \beta \rangle \in M_{\beta+1}$.

For every $\beta < \eta$ with $\beta + 1 < \eta$, since M_β is a countable element of $M_{\beta+1}$, we have $M_\beta \subseteq M_{\beta+1}$. So, it is easy to see that $\langle M_\beta : \beta < \eta \rangle$ is increasing. $\dashv$ (Claim 1)

Claim 2. For every $\beta < \eta$, $M_\beta \prec \mathfrak{A}$.

$\vdash$ If β is 0 or a successor ordinal, this is trivial from the definition. If β is a nonzero limit ordinal, then M_β is a union of an increasing sequence of elementary substructures of $\mathfrak{A}$. Hence, $M_\beta \prec \mathfrak{A}$. $\dashv$ (Claim 2)

Claim 3. For every $\beta < \eta$, if $N_\beta \prec \mathfrak{B}$, then $M_\beta \cap B = N_\beta$.

$\vdash$ Suppose that $\beta < \eta$ and $N_\beta \prec \mathfrak{B}$. By the definition of $\mathfrak{B}$, we have $\mathrm{Sk}^{\mathfrak{A}^*}(N_\beta) \cap B = N_\beta$. So, it suffices to show that $M_\beta \subseteq \mathrm{Sk}^{\mathfrak{A}^*}(N_\beta)$.

If $\beta = 0$, then $M_0 = F(0, N_0) = \mathrm{Sk}^{\mathfrak{A}_0}(N_0)$. By Lemma 4.2 (ii), $\mathrm{Sk}^{\mathfrak{A}^*}(N_0) \prec \mathfrak{A}_0$. Thus, $M_0 \subseteq \mathrm{Sk}^{\mathfrak{A}^*}(N_0)$.

If β is a successor ordinal, let γ be its predecessor. Then, $M_\beta = F(\delta_\gamma+1, N_\beta) = \mathrm{Sk}^{\mathfrak{A}_{\delta_\gamma+1}}(N_\beta)$. Since $\langle N_\xi : \xi \leq \gamma\rangle \in N_\beta$, we have $\delta_\gamma + 1 \in N_\beta$. So, by Lemma 4.2 (ii), $\mathrm{Sk}^{\mathfrak{A}^*}(N_\beta) \prec \mathfrak{A}_{\delta_\gamma+1}$. Therefore, $M_\beta \subseteq \mathrm{Sk}^{\mathfrak{A}^*}(N_\beta)$.

Finally, assume β is a limit ordinal. Let $x \in M_\beta$. Since $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$, there exists $\gamma < \beta$ such that $x \in M_{\gamma+1} = F(\delta_\gamma + 1, N_{\gamma+1})$. Since $\delta_\gamma + 1 \in N_{\gamma+1} \subseteq N_\beta$ and $N_{\gamma+1} \in N_{\gamma+2} \subseteq N_\beta$, we have $M_{\gamma+1} \in \mathrm{Sk}^{\mathfrak{A}^*}(N_\beta)$. Since $M_{\gamma+1}$ is countable, we have $M_{\gamma+1} \subseteq \mathrm{Sk}^{\mathfrak{A}^*}(N_\beta)$. Hence, $x \in \mathrm{Sk}^{\mathfrak{A}^*}(N_\beta)$. $\dashv$ (Claim 3)

$\dashv$

§5. CH+$(+)_{<\omega}$+there is no club guessing sequence on ω_1. The following property is needed for our proof of the preservation lemma for a $(+)_{<\omega}$-sequence.

Definition 5.1. We say that a $(+)_{<\omega}$-sequence $\langle F_\xi : \xi \in \omega_1 \cap \mathrm{Lim}\rangle$ is *p-point like* if and only if for every $\xi \in \omega_1 \cap \mathrm{Lim}$,

(i) there exists an $x \in F_\xi$ such that $\mathrm{otp}(x) = \omega$,
(ii) for every $y \subseteq \xi$, either $y \in F_\xi$ or $\xi \setminus y \in F_\xi$, and
(iii) whenever $\langle x_n : n < \omega\rangle$ is a $\subseteq$-decreasing sequence in F_ξ, there exists an $x \in F_\xi$ such that $x \subseteq^* x_n$.

While we are not sure whether $(+)_{<\omega}$ implies a p-point like $(+)_{<\omega}$-sequence, it can be easily built from a club guessing sequence on ω_1 and a p-point as follows.

Lemma 5.2. *Suppose that there is a club guessing sequence on ω_1 and there is a p-point. Then, there exists a p-point like $(+)_{<\omega}$-sequence.*

Proof. Let $\langle C_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ be a tail club guessing sequence on ω_1 and U a p-point. Without loss of generality, we may assume that for every $\delta \in \omega_1 \cap \mathrm{Lim}$, $\mathrm{otp}(C_\delta) = \omega$. For every $n < \omega$ and $\delta \in \omega_1 \cap \mathrm{Lim}$, let $C_\delta(n)$ denote the $(n+1)$-st element of C_δ. For each $\delta \in \omega_1 \cap \mathrm{Lim}$, define F_δ to be the filter on δ generated by the sets of the form $\{C_\delta(n) : n \in z\}$ for some $z \in U$. It is easy to check that $\langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ is a p-point like $(+)_{<\omega}$-sequence. $\dashv$

We shall prove a preservation theorem for a class of forcing notions that preserve a p-point like $(+)_{<\omega}$-sequence.

Definition 5.3. Let P be a forcing notion, A a set, and $\vec{F} = \langle F_\xi : \xi \in \omega_1 \cap \mathrm{Lim}\rangle$ a $(+)_{<\omega}$-sequence. We say that P is $(+)_{<\omega}$-proper for $\vec{F}$ on A if and only if there exists a club subset E of $[A]^{\aleph_0}$ such that whenever

(i) $\langle N_\gamma : \gamma \leq \delta + 1\rangle$ is a tower of countable elementary substructures of $\langle A, \in\rangle$ with $N_\delta \cap \omega_1 = \delta$,
(ii) N_δ and $N_{\delta+1}$ belong to E,
(iii) $\vec{F}, P \in N_0$,
(iv) $p \in P \cap N_0$,
(v) $x \in F_\delta$ with $\mathrm{otp}(x) = \omega$,
(vi) for every $\gamma \in x$, $N_\gamma \cap \omega_1 = \gamma$ and $N_\gamma \in E$,
(vii) for every $y \in F_\delta \cap N_{\delta+1}$, $x \subseteq^* y$,

there exists a $q \leq p$ such that q is (N_γ, P)-generic for every $\gamma \in x$. We say that P is $(+)_{<\omega}$-proper for $\vec{F}$ if and only if P is $(+)_{<\omega}$-proper for $\vec{F}$ on $H(\bar{\theta}_P)$.

First of all, we shall show that as long as θ is a sufficiently large regular cardinal, the choice of θ does not matter in the definition of $(+)_{<\omega}$-properness.

LEMMA 5.4. *Let $\vec{F}$ be a $(+)_{<\omega}$-sequence and P a forcing notion. The following are equivalent.*

(i) *P is $(+)_{<\omega}$-proper for $\vec{F}$.*
(ii) *For some $\theta \geq \theta_P$, P is $(+)_{<\omega}$-proper for $\vec{F}$ on $H(\theta)$.*
(iii) *For every $\theta \geq \theta_P$, P is $(+)_{<\omega}$-proper for $\vec{F}$ on $H(\theta)$. Moreover, it is witnessed by $E = \{N \in [H(\theta)]^{\aleph_0} : P \in N$ and $P \prec H(\theta)\}$.*

PROOF. First we shall prove (i) implies (iii). Suppose that P is $(+)_{<\omega}$-proper for $\vec{F}$. Suppose that $\langle N_\gamma : \gamma \leq \delta+1\rangle$ is a tower of countable subsets of $H(\theta)$ with $\delta = N_\delta \cap \omega_1$, N_δ and $N_{\delta+1}$ are elementary submodels of $H(\theta)$, $\vec{F}, P \in N_0$, $p \in P \cap N_0$, $x \in F_\delta$ with $\mathrm{otp}(x) = \omega$, for every $\gamma \in x$, N_γ is an elementary submodel of $\langle H(\theta), \in\rangle$, and for every $y \in F_\delta \cap N_{\delta+1}$, $x \subseteq^* y$. We shall show that there exists a $q \leq p$ such that q is (N_γ, P)-generic for every $\gamma \in x$.

Note that $\bar{\theta}_P \in N_0$. Since $\vec{F}, P, \bar{\theta}_P \in N_0$ and P is $(+)_{<\omega}$-proper for $\vec{F}$, there exists a club subset $\bar{E} \in N_0$ of $[H(\bar{\theta}_P)]^{\aleph_0}$ that witnesses the $(+)_{<\omega}$-properness of P for $\vec{F}$. For every $\gamma \leq \delta+1$, define $\bar{N}_\gamma = N_\gamma \cap H(\bar{\theta}_P)$. It is easy to see that $\langle \bar{N}_\gamma : \gamma \leq \delta+1\rangle$ is a tower of countable subsets of $H(\bar{\theta}_P)$, $\delta = \bar{N}_\delta \cap \omega_1$, $\bar{N}_\delta$ and $\bar{N}_{\delta+1}$ belong to $\bar{E}$, $\vec{F}, P \in \bar{N}_0$, for every $\gamma \in x$, $\bar{N}_\gamma$ is an elementary submodel of $H(\bar{\theta}_P)$, and for every $y \in F_\delta \cap \bar{N}_{\delta+1}$, $x \subseteq^* y$. Hence, there exists a $q \leq p$ such that q is $(\bar{N}_\gamma, P)$-generic for every $\gamma \in x$. However, for every $\gamma \in x$, since $\mathcal{P}(P) \subseteq H(\bar{\theta}_P)$, $\bar{N}_\gamma \cap \mathcal{P}(P) = N_\gamma \cap \mathcal{P}(P)$. So, q is (N_γ, P)-generic.

Clearly (iii) implies (ii).

So it suffices to show that (ii) implies (i). Let $\theta \geq \theta_P$ be a regular cardinal such that P is $(+)_{<\omega}$-proper for $\vec{F}$ on $H(\theta)$ witnessed by a club subset E of $[H(\theta)]^{\aleph_0}$. Then, there exists a structure $\mathfrak{A}$ expanding $\langle H(\theta), \in, \leq_{H(\theta)}, P\rangle$ such that for every $N \in [H(\theta)]^{\aleph_0}$, $\mathrm{Sk}^{\mathfrak{A}}(N) \in E$. Let $\mathfrak{B}$ be a structure expanding $\langle H(\bar{\theta}_P), \in, \leq_{H(\theta)} \restriction H(\bar{\theta}_P), P\rangle$ such that for every countable $N \prec \mathfrak{B}$, $\mathrm{Sk}^{\mathfrak{A}^*}(N) \cap B = N$.

We shall show that $E' = \{N \in [H(\bar{\theta}_P)]^{\aleph_0} : N \prec \mathcal{B}\}$ witnesses P is $(+)_{<\omega}$-proper for $\vec{F}$. Let $\langle N_\gamma : \gamma \leq \delta+1\rangle$ be a tower of countable elementary substructures of $\langle H(\theta), \in\rangle$, N_δ and $N_{\delta+1}$ belong to E', $\vec{F}, P \in N_0$, $p \in P \cap N_0$, $x \in F_\delta$ with $\mathrm{otp}(x) = \omega$, for every $\gamma \in x$, $N_\gamma \cap \omega_1 = \gamma$ and $N_\gamma \in E'$. By Lemma 4.3, there exists a tower $\langle M_\gamma : \gamma \leq \delta+1\rangle$ of countable elementary substructures of $\mathfrak{A}$ such that for every ordinal $\gamma \leq \delta+1$, if $N_\gamma \prec \mathcal{B}$, then $M_\gamma \cap H(\bar{\theta}_P) = N_\gamma$. In particular, for every $\gamma \in x$, since $N_\gamma \prec \mathcal{B}$, we have $M_\gamma \cap H(\bar{\theta}_P) = N_\gamma$ and hence $M_\gamma \cap \omega_1 = N_\gamma \cap \omega_1 = \gamma$. Since P is $(+)_{<\omega}$-proper on $H(\theta)$ witnessed by E, there exists a $q \leq p$ such that q is (M_γ, P)-generic for every $\gamma \in x$. However, for every $\gamma \in x$, we have $M_\gamma \cap H(\bar{\theta}_P) = N_\gamma$. So, q is also (N_γ, P)-generic. ⊣

The main point of the $(+)_{<\omega}$-properness is that it preserves $\vec{F}$ as a $(+)_{<\omega}$-sequence.

LEMMA 5.5. *Let $\vec{F} = \langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim} \rangle$ be a p-point like $(+)_{<\omega}$-sequence and P a forcing notion that is $(+)_{<\omega}$-proper for $\vec{F}$. Then, P forces that $\vec{F}$ generates a $(+)_{<\omega}$-sequence.*

PROOF. Let $p \in P$ and $\dot{D}$ a P-name for a club subset of ω_1. Let $\theta = \theta_P$. Pick a sufficiently large regular cardinal θ. Build a tower $\langle N_\gamma : \gamma < \omega_1 \rangle$ of countable elementary submodels of $\langle H(\theta), \in \rangle$ with $P, \vec{F}, p \in N_0$. Define $E = \{\gamma < \omega_1 : N_\gamma \cap \omega_1 = \gamma\}$. Then, E is a club subset of ω_1. Since $\vec{F}$ is a $(+)_{<\omega}$-sequence, there exists a $\delta \in E$ such that $E \cap \delta \in F_\delta$. Since $\vec{F}$ is p-point like, there exists an $x \in F_\delta$ such that $\mathrm{otp}(x) = \omega$, $x \subseteq E \cap \delta$, and $x \subseteq^* y$ for every $y \in F_\delta \cap N_{\delta+1}$. Then, we can apply $(+)_{<\omega}$-properness of P for $\vec{F}$ to $\langle N_\gamma : \gamma \leq \delta + 1 \rangle$, p, and x to get $q \leq p$ such that q is (N_γ, P)-generic for every $\gamma \in x$. It implies that for every $\gamma \in x$, $q \Vdash$'$\gamma = N_\gamma \cap \omega_1 \in \dot{D}$'. Thus, $q \Vdash$'$x \subseteq \dot{D} \cap \delta$ and hence $\dot{D} \cap \delta \in F_\delta$'. $\dashv$

The following forcing notion, defined by S. Shelah in [4], is the most obvious one to force that $\vec{C}$ is not a tail club guessing sequence.

DEFINITION 5.6. Let $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \mathrm{Lim} \rangle$ be a guessing sequence on ω_1. Then, let $P(\vec{C})$ be the forcing notion defined as $p \in P(\vec{C})$ if and only if p is a closed bounded subset of ω_1 such that for every $\delta \in p \cap \mathrm{Lim}$, $C_\delta \not\subseteq^* p \cap \delta$. $P(\vec{C})$ is ordered by end-extension.

$P(\vec{C})$ is $(+)_{<\omega}$-proper for any p-point like $(+)_{<\omega}$-sequence. In particular, it preserves any p-point like $(+)_{<\omega}$-sequence.

LEMMA 5.7. *Let $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \mathrm{Lim} \rangle$ be a tail club guessing sequence on ω_1 and $\vec{F} = \langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim} \rangle$ a p-point like $(+)_{<\omega}$-sequence. Then, $P(\vec{C})$ is $(+)_{<\omega}$-proper for $\vec{F}$.*

PROOF. Set $\theta = \theta_{P(\vec{C})}$ and let $\langle N_\gamma : \gamma \leq \delta + 1 \rangle$, p, and x be as in the assumption of the definition of $(+)_{<\omega}$-properness. For every $\gamma < \delta$, let $\delta_\gamma = N_\gamma \cap \omega_1$. Let $\langle \gamma_n : n < \omega \rangle$ be the increasing enumeration of x. Notice that by (vi) of Definition 5.3, for every $n < \omega$, $\delta_{\gamma_n} = N_{\gamma_n} \cap \omega_1 = \gamma_n$. It is easy to build a decreasing sequence $\langle p_n : n < \omega \rangle$ such that

(i) $p_0 = p$, and
(ii) for every $n < \omega$,
  (a) $p_n \in N_{\gamma_n}$,
  (b) p_{n+1} is (N_{γ_n}, P)-generic,
  (c) if $(\gamma_n, \gamma_{n+1}) \cap C_\delta \neq \varnothing$, then $(\gamma_n, \gamma_{n+1}) \cap (C_\delta \setminus p_{n+2}) \neq \varnothing$.

We claim $C_\delta \setminus x$ is unbounded in δ. Since $P \in N_0 \subseteq N_{\delta+1}$ and $\delta \in N_{\delta+1}$, we have $C_\delta \in N_{\delta+1}$. If $C_\delta \notin F_\delta$, then since $\vec{F}$ is p-point like, $\delta \setminus C_\delta \in F_\delta \cap N_{\delta+1}$. By assumption, $x \subseteq^* \delta \setminus C_\delta$. Thus, we have $C_\delta \cap x$ is bounded in δ and hence $C_\delta \setminus x$ is unbounded in δ. Suppose that $C_\delta \in F_\delta$. Since $\vec{F}$ is p-point like, there exists a $y \in F_\delta \cap N_{\delta+1}$ such that y is an unbounded co-unbounded subset of C_δ. By assumption, $x \subseteq^* y$. Thus, $C_\delta \setminus x$ is unbounded in δ.

Let $q = \bigcup_{n<\omega} p_n \cup \{\delta\}$. Since $C_\delta \setminus x$ is unbounded in δ, there are unboundedly many $n < \omega$ such that $(\gamma_n, \gamma_{n+1}) \cap C_\delta \neq \varnothing$ and hence $(\gamma_n, \gamma_{n+1}) \cap (C_\delta \setminus q) \neq \varnothing$. Therefore, $C_\delta \not\subseteq^* q$, which implies $q \in P$. Thus, q witnesses the lemma. $\dashv$

The following lemma is the reason why we defined the $(+)_{<\omega}$-properness. It shows that not only $P(\vec{C})$ preserves p-point like $(+)_{<\omega}$-sequences, but also the iteration of the forcing notions of the form $P(\vec{C})$ preserves them.

LEMMA 5.8. *Let $\vec{F} = \langle F_\delta : \delta \in \omega_1 \cap \mathrm{Lim}\rangle$ be a $(+)_{<\omega}$-sequence. Suppose that $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \eta\rangle$ is a countable support iteration such that*

(i) *for every $\alpha < \eta$, $\mathbf{1}_{P_\alpha} \Vdash$ '$\dot{Q}_\alpha$ is $(+)_{<\omega}$-proper for $\vec{F}$', and*
(ii) *P_η adds no new real.*

Then, P_η is $(+)_{<\omega}$-proper for $\vec{F}$.

PROOF. For every regular cardinal $\theta \geq \theta_{P_\eta}$, and $\beta < \alpha \leq \eta$, let $\varphi(\theta, \beta, \alpha)$ be the following assertion: For every $\langle N_\gamma : \gamma \leq \delta + 2\rangle$, $\dot{p}$, q, and x, if

(i) $\langle N_\gamma : \gamma \leq \delta + 2\rangle$ is a tower of countable subsets of $H(\theta)$ with $N_\delta \cap \omega_1 = \delta$,
(ii) N_δ, $N_{\delta+1}$, and $N_{\delta+2}$ are elementary submodels of $\langle H(\theta), \in\rangle$,
(iii) $\vec{F}, P_\eta, \beta, \alpha \in N_0$,
(iv) $\dot{p} \in N_0$ is a P_β-name for an element of $P_{\beta,\alpha}$,
(v) $x \in F_\delta$ with $\mathrm{otp}(x) = \omega$,
(vi) for every $\gamma \in x$, $N_\gamma \cap \omega_1 = \gamma$ and N_γ is an elementary submodel of $\langle H(\theta), \in\rangle$,
(vii) for every $y \in F_\delta \cap N_{\delta+1}$, $x \subseteq^* y$, and
(viii) $q \in P_\beta$ is finitely (N_γ, P_β)-generic for every $\gamma \in x$, finitely (N_δ, P_β)-generic, finitely $(N_{\delta+1}, P_\beta)$-generic, and $(N_{\delta+2}, P_\beta)$-generic,

then, there exists a $q' \in P_\alpha$ such that $q' \restriction \beta = q$, $q' \Vdash$'$q' \restriction [\beta, \alpha) \leq \dot{p}$', and q' is finitely (N_γ, P_α)-generic for every $\gamma \in x$, finitely (N_δ, P_α)-generic, finitely $(N_{\delta+1}, P_\alpha)$-generic, and $(N_{\delta+2}, P_\alpha)$-generic.

Let $\varphi'(\theta, \beta, \alpha)$ denote the assertion that under the same assumption as in $\varphi(\theta, \beta, \alpha)$, there exists a $q' \in P_\alpha$ such that $q' \restriction \beta = q$, $q' \Vdash$'$q' \restriction [\beta, \alpha) \leq \dot{p}$', and q' is (N_γ, P_α)-generic for every $\gamma \in x$.

Note that $\langle N_\gamma : \gamma \in x\rangle$ does not belong to $N_{\delta+1}$. Let $\varphi(\beta, \alpha)$ denote $\varphi(\theta_{P_\eta}, \beta, \alpha)$ and $\varphi'(\beta, \alpha)$ denote $\varphi'(\theta_{P_\eta}, \beta, \alpha)$.

By the same argument as in 5.4, we can prove the following claim.

Claim 1. For every regular $\theta \geq \theta_{P_\eta}$, $\varphi(\theta, \beta, \alpha)$ if and only if $\varphi(\beta, \alpha)$. Moreover, $\varphi'(\theta, \beta, \alpha)$ if and only if $\varphi'(\beta, \alpha)$.

The following claim is trivial.

Claim 2. For every $\beta < \gamma < \alpha \leq \omega_2$, if both $\varphi(\beta, \gamma)$ and $\varphi(\gamma, \alpha)$ hold, then so does $\varphi(\beta, \alpha)$.

To prove $\varphi(\beta, \alpha)$, the seemingly weaker conclusion $\varphi'(\beta, \alpha)$ suffices.

Claim 3. For every $\beta < \alpha \leq \omega_2$, $\varphi(\beta, \alpha)$ is equivalent to $\varphi'(\beta, \alpha)$.

$\vdash$ Trivially, $\varphi(\beta, \alpha)$ implies $\varphi'(\beta, \alpha)$. Assume $\varphi'(\beta, \alpha)$. Let $\theta' = \theta_{P_\eta}$. Let $\theta = \left(2^{H(\theta')}\right)^+$. By Claim 1, it suffices to show $\varphi(\theta, \beta, \alpha)$. Suppose that θ, $\langle N_\gamma : \gamma \leq \delta + 2\rangle$, $\dot{p}$, q, and x are as in the assumption of $\varphi(\theta, \beta, \alpha)$. For every $\gamma \leq \delta$, let $N'_\gamma = N_\gamma \cap H(\theta')$. Notice that $\langle N'_\gamma : \gamma \leq \delta\rangle$ is a tower of countable subsets of $H(\theta')$.

In $N_{\delta+1}$, pick two countable elementary substructures $N'_{\delta+1}$ and $N'_{\delta+2}$ of $\langle H(\theta'), \in\rangle$ such that $\langle N'_\gamma \leq \delta\rangle \in N'_{\delta+1} \in N'_{\delta+2}$. Define $\mathcal{D}$ to be the set of all $r \in P_\beta$ such that r decides all open dense subsets of P_β lying in $N'_{\delta+2}$.

Note $\mathcal{D} \in N_{\delta+1}$. Let $\mathcal{A} \subseteq \mathcal{D}$ be a maximal antichain in P_β lying in $N_{\delta+1}$. Since q is finitely $(N_{\delta+1}, P_\beta)$-generic, there exists a finite subset $\{r_0, \ldots, r_{k-1}\}$ of $\mathcal{A} \cap N_{\delta+1}$ that is predense below q. Without loss of generality, we may assume that for every $i < k$, r_i is compatible with q.

Fix $i < k$. Since r_i decides all P_β-names for ordinals lying in $N'_{\delta+2}$, r_i is compatible with q, and q is (N'_γ, P_β)-generic for every $\gamma \in x$, r_i is also (N'_γ, P_β)-generic for every $\gamma \in x$. In $N_{\delta+1}$, we can pick a $y'_i \in F_\delta$ such that $\mathrm{otp}(y'_i) = \omega$ and for every $\gamma \in y'_i$, r_i is (N'_γ, P_β)-generic. By assumption, we have $x \subseteq^* y'_i$, i.e. $x \setminus y'_i$ is finite. Thus, $x \cup y'_i \in N_{\delta+1}$. Thus, for every $i < k$, there exists a $y_i \in F_\delta \cap N_{\delta+1}$ such that $x \subseteq y_i$ and r_i is (N'_γ, P_β)-generic for every $\gamma \in y_i$. Let $y = \bigcap_{i<k} y_i$. Then, $y \in F_\delta \cap N_{\delta+1}$, $x \subseteq y$, and for every $i < k$, r_i is (N'_γ, P_β)-generic for every $\gamma \in y$.

Again fix $i < k$. By the definition of $\mathcal{D}$, r_i decides all open dense subsets of P_β lying in $N'_{\delta+2}$. It is easy to see that r_i is totally $(N'_{\delta+1}, P_\beta)$-generic and totally $(N'_{\delta+2}, P_\beta)$-generic. By applying $\varphi'(\beta, \alpha)$ to $\langle N'_\gamma : \gamma \leq \delta + 2\rangle$, $\dot{p}$, y, and r_i, we can pick an $r'_i \in P_\alpha$ such that $r'_i \restriction \beta = r_i$, $r_i \Vdash$'$r'_i \restriction [\beta, \alpha) \leq \dot{p}$', and r'_i is (N'_γ, P_α)-generic for every $\gamma \in y$. Without loss of generality, we may assume $r'_i \in N_{\delta+1}$.

Pick a P_β-name $\dot{r}' \in N_{\delta+1}$ so that $\mathbf{1}_{P_\beta} \Vdash$'$\dot{r}' \in \dot{P}_{\beta,\alpha}$' and for every $i < k$, $r_i \Vdash$'$\dot{r}' = r'_i \restriction [\beta, \alpha)$'. Recall that $\{r_0, \ldots, r_{k-1}\}$ is predense below q and for every $i < k$, $r_i \Vdash$'$r'_i \restriction [\beta, \alpha) \leq \dot{p}$'. Thus, $q \Vdash$'$\dot{r}' \leq \dot{p}$'. Also since for every $i < k$ and $\gamma \in y$, r'_i is (N_γ, P_α)-generic, $q \Vdash$'$\dot{r}'$ is $(N_\gamma[\dot{G}_\beta], P_{\beta,\alpha})$-generic for every $\gamma \in y$'.

By Lemma 2.2, there exists a $q' \in P_\alpha$ such that $q' \restriction \beta = q$, $q \Vdash$'$q' \restriction [\beta, \alpha) \leq \dot{r}'$', and q' is finitely $(N_{\delta+1}, P_\alpha)$-generic, and $(N_{\delta+2}, P_\alpha)$-generic. Then, we have $q \Vdash$'$q' \restriction [\beta, \alpha) \leq \dot{r}' \leq \dot{p}$'. Moreover, for every $\gamma \in x$, since q is (N_γ, P_β)-generic and $q \Vdash$'$\dot{r}'$ is $(N'_\gamma[\dot{G}_\beta], P_{\beta,\alpha})$-generic', q' is (N_γ, P_α)-generic. Since $N_\gamma \subseteq N_{\delta+1}$ and q' is finitely $(N_{\delta+1}, P_\alpha)$-generic, it implies that q' is finitely (N_γ, P_α)-generic. Therefore, q' witnesses $\varphi(\theta, \beta, \alpha)$. $\dashv$ (Claim 3)

Claim 4. For every $\beta < \eta$, $\varphi(\beta, \beta + 1)$ holds.

$\vdash$ By Claim 3, it suffices to show $\varphi'(\beta, \beta + 1)$. Let $\langle N_\gamma : \gamma \leq \delta + 2\rangle$, $\dot{p}$, q, and x be as in the assumption of $\varphi'(\beta, \beta + 1)$. Let $G_\beta \subseteq P_\beta$ be generic with $q \in G_\beta$. Work in $V[G]$. By assumption, $Q_\beta = (\dot{Q}_\beta)^{G_\beta}$ is $(+)_{<\omega}$-proper for $\vec{F}$. Let $s = (\dot{p})^{G_\beta}(\beta)$. For every $\gamma \in x$, since q is (N_γ, P_β)-generic, $N_\gamma[G_\beta]$ is an elementary submodel of $H(\theta)^V[G_\beta]$. So, $\langle N_\gamma[G_\beta] : \gamma \leq \delta + 1\rangle$, s, and x satisfy the assumption of Definition 5.3. Since Q_β is $(+)_{<\omega}$-proper for $\vec{F}$, there exists a $t \leq s$ such that t is $(N_\gamma[G_\beta], Q_\beta)$-generic for every $\gamma \in x$. Let $\dot{t}$ be a P_β-name for t.

Let $q' \in P_{\beta+1}$ be so that $q' \restriction \beta = q$ and $q \Vdash$'$q'(\beta) = \dot{t}$'. Then, clearly q' witnesses $\varphi'(\theta, \beta, \alpha)$. $\dashv$ (Claim 4)

Now, we shall prove the limit case.

Claim 5. Let $\beta < \alpha \leq \eta$. Suppose that for every β', α' with $\beta \leq \beta' < \alpha' < \alpha$, $\varphi(\beta', \alpha')$ holds. Then, so does $\varphi(\beta, \alpha)$.

$\vdash$ By Claim 3, it suffices to show $\varphi'(\beta, \alpha)$. Let θ, $\langle N_\gamma : \gamma \leq \delta + 2\rangle$, $\dot{p}$, x, and q be as in the assumption of $\varphi(\beta, \alpha)$. Let $\langle \delta_n : n < \omega\rangle$ be the increasing enumeration of x. Let $\alpha_0 = \beta$ and for each $m < \omega$, let $\alpha_{m+1} = \sup(\alpha \cap N_{\delta_m})$. Note that for every $m < \omega$, $\alpha_m \in N_{\delta_m}$ and since $N_\delta = \bigcup_{\gamma<\delta} N_\gamma = \bigcup_{m<\omega} N_{\delta_m}$, $\langle \alpha_m : m < \omega\rangle$ is an increasing cofinal sequence in $\alpha \cap N_\delta$.

We shall build sequences $\langle \dot{p}_m : m < \omega\rangle$ and $\langle q_m : m < \omega\rangle$ as follows.

(i) $\dot{p}_0 = \dot{p}$ and $q_0 = q$,

(ii) for every $m < \omega$,
- (a) $\dot{p}_m$ is a P_{α_m}-name for an element of $P_{\alpha_m,\alpha}$,
- (b) $\dot{p}_m \in N_{\delta_m}$,
- (c) $q_m \in P_{\alpha_m}$,
- (d) $q_{m+1} \restriction \alpha_m = q_m$,
- (e) $q_m \Vdash$'$q_{m+1} \restriction [\alpha_m, \alpha_{m+1}) \leq \dot{p}_m \restriction [\alpha_m, \alpha_{m+1})$',
- (f) q_m is finitely (N_γ, P_{α_m})-generic for every $\gamma \in x$, finitely (N_δ, P_{α_m})-generic, finitely $(N_{\delta+1}, P_{\alpha_m})$-generic, and $(N_{\delta+2}, P_{\alpha_m})$-generic
- (g) $q_{m+1} \Vdash$'$\dot{p}_{m+1} \leq \dot{p}_m \restriction [\alpha_{m+1}, \alpha)$', and
- (h) $q_{m+1} \Vdash$'$\dot{p}_{m+1}$ is $(N_{\delta_n}[\dot{G}_{\alpha_{m+1}}], P_{\alpha_{m+1},\alpha})$-generic for every $n \leq m$'.

By definition, $\dot{p}_0 = \dot{p}$ and $q_0 = q$ satisfy the inductive hypothesis. Suppose that $\dot{p}_m$ and q_m have been defined.

Let $G \subseteq P_{\alpha_m}$ be generic with $q_m \in G$. Let $p_m = (\dot{p}_m)^G$. Then, $p_m \in P_{\alpha_m,\alpha} \cap N_{\delta_m}[G]$. Since q is (N_{δ_m}, P_β)-generic, $N_{\delta_m}[G]$ is an elementary submodel of $H(\theta)^{V[G]}$. Since $P_{\alpha_m,\alpha}$ is proper, there exists an $p'_m \leq p_m$ that is $(N_{\delta_m}[G], P_{\alpha_m,\alpha})$-generic. Let $\dot{p}_{m+1} \in N_{\delta_{m+1}}$ be a $P_{\alpha_{m+1}}$-name for $p'_m \restriction [\alpha_{m+1}, \alpha)$ and $\dot{s}_m \in N_{\delta_{m+1}}$ a P_{α_m}-name for $p'_m \restriction [\alpha_m, \alpha_{m+1})$.

Apply $\varphi(\alpha_m, \alpha_{m+1})$ to $\langle N_\gamma : \delta_{m+1} \leq \gamma \leq \delta + 2\rangle$, $\dot{s}_m$, q_m, and $x \setminus \delta_{m+1}$ to get $q_{m+1} \in P_{\alpha_{m+1}}$ such that $q_{m+1} \restriction \alpha_m = q_m$, $q_m \Vdash$ '$q_{m+1} \restriction [\alpha_m, \alpha_{m+1}) \leq \dot{s}_m$', and q_{m+1} is finitely $(N_\gamma, P_{\alpha_{m+1}})$-generic for every $\gamma \in x \setminus \delta_{m+1}$, finitely $(N_\delta, P_{\alpha_{m+1}})$-generic, finitely $(N_{\delta+1}, P_{\alpha_{m+1}})$-generic, and $(N_{\delta+2}, P_{\alpha_{m+1}})$-generic.

Most of the inductive hypothesis are clear. We need to show that for every $\gamma \in x \cap \delta_{m+1}$, q_{m+1} is $(N_\gamma, P_{\alpha_{m+1}})$-generic. If $\gamma = \delta_m$, then q_m is $(N_{\delta_m}, P_{\alpha_m})$-generic and $q_m \Vdash$ '$\dot{s}_m$ is $(N_{\delta_m}[\dot{G}_{\alpha_m}], P_{\alpha_m,\alpha_{m+1}})$-generic and $q_{m+1} \restriction [\alpha_m, \alpha_{m+1}) \leq \dot{s}_m$'. Thus, q_{m+1} is $(N_{\delta_m}, P_{\alpha_{m+1}})$-generic. Suppose $n < m$. By inductive hypothesis, q_m is $(N_{\delta_n}, P_{\alpha_m})$-generic and $q_m \Vdash$ '$\dot{p}_m$ is $(N_{\delta_n}[\dot{G}_{\alpha_m}, P_{\alpha_m,\alpha})$-generic'. Since $q_m \Vdash$'$q_{m+1} \restriction [\alpha_m, \alpha_{m+1}) \leq \dot{s}_m \leq \dot{p}_m \restriction [\alpha_m, \alpha_{m+1})$', we have $q_m \Vdash$'$q_{m+1} \restriction [\alpha_m, \alpha_{m+1})$ is $(N_{\delta_n}[\dot{G}_{\alpha_m}], P_{\alpha_m,\alpha_{m+1}})$-generic'. Therefore, q_{m+1} is $(N_{\delta_n}, P_{\alpha_{m+1}})$-generic.

We shall show that $q' = \bigcup_{m<\omega} q_m$ witnesses $\varphi'(\beta, \alpha)$. Clearly we have $q' \restriction \beta = q$. By the construction of q', it is also easy to see $q \Vdash$ '$q' \restriction [\beta, \alpha) \leq \dot{p}$'. We also need to show that q' is (N_{δ_n}, P_α)-generic for every $n < \omega$. Let $n < \omega$. Then, q_{n+1} is $(N_{\delta_n}, P_{\alpha_n})$-generic and $q_{n+1} \Vdash$ '$\dot{p}_{n+1}$ is $(N_{\delta_n}[\dot{G}_{\alpha_{m+1}}], P_{\alpha_{m+1},\alpha})$-generic and $q' \restriction [\alpha_{m+1}, \alpha) \leq \dot{p}_{n+1}$', q' is $(N_{\delta_n}, P_{\alpha_n})$-generic. $\dashv$ (Claim 5)

By combining those claims, we can easily see that $\varphi(\beta,\alpha)$ holds for every $\beta<\alpha\leq\eta$.

Now we shall show that P_η is $(+)_{<\omega}$-proper. Let $\theta=\theta_P$ and $E=\{N\in[H(\theta)]^{\aleph_0}: P\in N$ and $P\prec H(\theta)\}$. Suppose that $\langle N_\gamma:\gamma\leq\delta+1\rangle$, $\vec{F}$, p, and x are as in the assumption of Definition 5.3. Let $N_{\delta+2}$ be a countable elementary submodel of $\langle H(\theta),\in\rangle$ with $N_{\delta+1}\in N_{\delta+2}$. By applying $\varphi(\theta,0,\eta)$, we can obtain a $q\in P_\eta$ such that $q\leq p$ and q is finitely (N_γ,P_η)-generic for every $\gamma\in x$. ⊣

Now, it is easy to show the following theorem.

THEOREM 5.9. *It is consistent that both* CH *and* $(+)_{<\omega}$ *hold and there is no club guessing sequence on* ω_1.

PROOF. Assume that CH, $2^{\aleph_1}=\aleph_2$, and there exists a club guessing sequence on ω_1. By Lemma 5.2, there exists a p-point like $(+)_{<\omega}$-sequence $\vec{F}=\langle F_\delta:\delta\in\omega_1\cap\mathrm{Lim}\rangle$.

Let $P=\langle P_\alpha,\dot{Q}_\beta:\beta<\alpha\leq\omega_2\rangle$ be the countable support iteration of the bookkeeping of all forcing notions of the form $P(\vec{C})$ where $\vec{C}$ is a tail club guessing sequence. S. Shelah showed in [4] that P adds no new reals and hence P forces that CH holds and there is no club guessing sequence on ω_1. By Lemma 5.7, for every $\alpha<\omega_2$, $\mathbf{1}_{P_\alpha}\Vdash$'$\dot{Q}_\alpha$ is $(+)_{<\omega}$-proper for $\vec{F}$'. By Lemma 5.8, P is $(+)_{<\omega}$-proper for $\vec{F}$. By Lemma 5.5, P forces that $\vec{F}$ is a $(+)_{<\omega}$-sequence. ⊣

QUESTION 1. Do we really need to assume that $\vec{F}$ is p-point like?

§6. $\diamondsuit^+$ does not imply the existence of a strong club guessing sequence. In this section, we shall consider the relationship between the following two guessing principles.

DEFINITION 6.1. $\diamondsuit^+$ is the principle that asserts the existence of a sequence $\langle A_\delta:\delta<\omega_1\rangle$ such that

(i) A_δ is a countable subset of $\mathcal{P}(\delta)$, and
(ii) for every subset X of ω_1, there exists a club subset D of ω_1 such that for every $\delta\in D$, $X\cap\delta\in A_\delta$ and $D\cap\delta\in A_\delta$.

DEFINITION 6.2. A sequence $\langle C_\delta:\delta\in\omega_1\cap\mathrm{Lim}\rangle$ is called a *strong club guessing sequence on* ω_1 if and only if

(i) for every $\delta\in\omega_1\cap\mathrm{Lim}$, C_δ is an unbounded subset of δ, and
(ii) for every club subset D of ω_1, there exists a club subset E of ω_1 such that for every $\delta\in E$, $C_\delta\subseteq^* D$.

It is easy to see that $\diamondsuit^+$ implies CH. However, a strong club guessing sequence cannot be killed by ccc forcing. So, by adding many Cohen reals to a model with a strong club guessing sequence on ω_1, we can obtain a model in which there is a strong club guessing sequence on ω_1, but CH fails and hence so does $\diamondsuit^+$. Thus, the first author asked in [1] whether $\diamondsuit^+$ implies the existence of a strong club guessing sequence on ω_1.

In this section, we shall use $(+)$ to show that it is not the case. To this end, we shall first define strong $(+)$.

DEFINITION 6.3. Let $k < \omega$ and S a stationary subset of $\omega_1 \cap \mathrm{Lim}$. Strong $(+)_k(S)$ is defined as the principle that asserts the existence of a club subset T of $[H(\omega_2)]^{\aleph_0}$ such that for every $N \in T$, $N \cap \omega_1 \in S$ and for $N_0, \dots, N_{k-1} \in T$ with $N_i \cap \omega_1 = N_0 \cap \omega_1 \in S$ for every $i < k$, if $D_i \in N_i$ is a club subset of ω_1 for every $i < k$, then $\bigcap_{i<k} D_i \cap N_0 \neq \varnothing$. Strong $(+)(S)$ denotes strong $(+)_2(S)$. If $S = \omega_1$, we simply write strong $(+)_k$.

By the same argument by the second author in [2, Theorem 2.2]. we can show that the existence of a strong club guessing sequence on ω_1 implies strong $(+)$

THEOREM 6.4. *$\diamondsuit^+$ does not imply strong $(+)$, and hence the existence of a strong club guessing sequence.*

PROOF. By Theorem 3.4, it is consistent with GCH that $(+)$ fails. So, we begin with such a model.

Let $\langle P_\alpha, \dot{Q}_\beta : \beta < \alpha \leq \omega_2 \rangle$ be the standard forcing to add a $\diamondsuit^+$-sequence. So, Q_0 is the set of all functions q such that

(i) $\mathrm{dom}(q) = \delta$ for some $\delta < \omega_1$, and
(ii) for every $\gamma \in \delta$, $q(\gamma)$ is a countable subset of $\mathcal{P}(\gamma)$

Let $\langle \dot{X}_\alpha : 1 \leq \alpha < \omega_2 \rangle$ be a bookkeeping of all good names for subsets of ω_1. Suppose that $\langle P_\beta, \dot{Q}_\gamma : \gamma < \beta \leq \alpha \rangle$ has been defined. To define $\dot{Q}_\alpha$, let $G_\alpha \subseteq P_\alpha$ be generic over V and work in $V[G]$. For every $\delta < \omega_1$, define $A_\delta = p(0)(\delta)$ for some $p \in G$ with $\delta \in \mathrm{dom}(p(0))$. Let $X_\alpha = (\dot{X}_\alpha)^{V[G]}$. Let Q_α be the forcing to shoot a club through $\{\delta < \omega_1 : X_\alpha \cap \delta \in A_\delta\}$. By a standard argument, we can show that P_{ω_2} forces $\diamondsuit^+$. Moreover, P_{ω_2} adds no new reals.

Let $P = P_{\omega_2}$. Define $\tilde{P}$ to be the set of all $p \in P$ such that for every $\alpha \in \mathrm{dom}(p)$, $p \restriction \alpha$ decides $p(\alpha)$.

Claim 1. $\tilde{P}$ is dense in P.

$\vdash$ Let $p \in P$. Let $\theta = \theta_P$ and N a countable elementary submodel of $H(\theta)$ with $P, p \in N$. Set $\delta = N \cap \omega_1$. Let $\langle \mathcal{D}_n : n < \omega \rangle$ be an enumeration of all open dense subsets of P lying in N. We can easily build a decreasing sequence $\langle p_n : n < \omega \rangle$ in P such that $p_0 = p$ and $p_{n+1} \in N \cap \mathcal{D}_n$ for every $n < \omega$. Define $q \in P$ as follows. Let $\mathrm{dom}(q) = N \cap \omega_2$. Let $\mathrm{dom}(q(0)) = \delta + 1$ and $q(0) \restriction \delta = \bigcup_{n<\omega} p_n(0)$. Let $q(0)(\delta)$ be the set of all subsets x of δ such that for some P-name $\dot{X} \in N$ for a subset of ω_1, $x = \{\xi < \delta : \exists n < \omega (p_n \Vdash \text{‘}\xi \in \dot{X}\text{’})\}$.

Suppose that we have defined $q \restriction \alpha$ for some α with $0 < \alpha < \omega_2$. If $\alpha \notin N \cap \omega_2$, then we have nothing to do as $\alpha \notin \mathrm{dom}(q)$. Suppose $\alpha \in N \cap \omega_2$. Let $q(\alpha)$ be a P_α-name such that $q \restriction \alpha \Vdash \text{‘}q(\alpha) = \bigcup_{n<\omega} p_n(\alpha) \cup \{\delta\}\text{’}$.

It is easy to see that $q \in P$ and q is totally (N, P)-generic. Note that for every $n < \omega$ and $\alpha \in N \cap \omega_2$, $p_n(\alpha)$ is a P_α-name lying in N, thus $q \restriction \alpha$ decides it. So, we have $q \in \tilde{P}$ $\dashv$ (Claim 1)

If $p \in \tilde{P}$, then for each $\alpha \in \mathrm{dom}(p)$ with $\alpha > 0$, we identify $p(\alpha)$ with $x \subseteq \omega_1$ such that $p \restriction \alpha \Vdash$‘$p(\alpha) = x$’. For every $\alpha < \omega_2$, let $\tilde{P}_\alpha = P_\alpha \cap \tilde{P}$. It is easy to see that for every $\alpha < \omega_2$, $\tilde{P}_\alpha$ is dense in P_α and $|\tilde{P}_\alpha| = \aleph_1$.

Now, it suffices to show that P forces that strong $(+)$ fails. Suppose that there exist $p \in P$ and a P-name $\dot{E}$ such that $p \Vdash$‘$\dot{E}$ witnesses strong $(+)$’.

That is, p forces that $\dot{E}$ is a club subset of $H(\omega_2)$ and for every $N_0, N_1 \in \dot{E}$, if $N_0 \cap \omega_1 = N_1 \cap \omega_1$, then for every pair $\langle D_0, D_1\rangle$ of club subsets of ω_1 so that $D_0 \in N_0$ and $D_1 \in N_1$, we have $D \cap D_1 \cap N_0 \cap \omega_1 \neq \varnothing$.

Let $\theta = \theta_P$. Since (+) fails in V, there exist N_0, N_1, D_0, and D_1 such that N_0 and N_1 are countable elementary submodels of $H(\theta)$, $N_0 \cap \omega_1 = N_1 \cap \omega_1$, D_0 and D_1 are club subsets of ω_1, $P, p, D_0 \in N_0$, $P, p, D_1 \in N_1$, and $D_0 \cap D_1 \cap N_0 \cap \omega_1 = \varnothing$.

Let $\delta = N_0 \cap \omega_1$. Let $\langle \mathcal{D}^0_n : n < \omega\rangle$ be an enumeration of all open dense subsets of P lying in N_0, and $\langle \mathcal{D}^1_n : n < \omega\rangle$ an enumeration of all open dense subsets of P lying in N_1. Let $\bar{\eta} = \sup(N_0 \cap N_1 \cap \omega_2)$. Since $N_0 \cap \omega_1 = N_1 \cap \omega_1$, we have $N_0 \cap \bar{\eta} = N_1 \cap \bar{\eta}$. Let η_0 be the least ordinal in N_0 above $\bar{\eta}$ and η_1 the least ordinal in N_1 above $\bar{\eta}$.

We shall define two decreasing sequences $\langle p^0_n : n < \omega\rangle$ and $\langle p^1_n : n < \omega\rangle$ such that

(i) $p^0_0 = p^1_0 = p$,
(ii) for every $n < \omega$, $p^0_{n+1} \le p^0_n$ and $p^1_{n+1} \le p^1_n$,
(iii) for every $n < \omega$, $p^0_{n+1} \in \tilde{P} \cap \mathcal{D}^0_n$ and $p^1_{n+1} \in \tilde{P} \cap \mathcal{D}^1_n$, and
(iv) for every $n < \omega$, $p^0_{n+1} \restriction \bar{\eta} \le p^1_n \restriction \bar{\eta} \le p^0_n \restriction \bar{\eta}$.

Suppose that p^0_n and p^1_n has been defined. Note that $p^1_n \restriction \eta_1 \in N_1$. So, $\mathrm{dom}(p^1_n \restriction \eta_1) \in N_1$. Let $\nu^1_n = \sup(\mathrm{dom}(p^1_n \restriction \eta_1))$. Then, $\nu^1_n \in N_1 \cap \bar{\eta} = N_0 \cap \bar{\eta}$. Since $P_{\nu^1_n} \in N_0 \cap N_1$, $|\tilde{P}_{\nu^1_n}| = \omega_1$, and $N_0 \cap \omega_1 = N_1 \cap \omega_1$, we have $\tilde{P}_{\nu^1_n} \cap N_0 = \tilde{P}_{\nu^1_n} \cap N_1$. Therefore, $p^1_n \restriction \eta_1 \in N_0$. Define $\bar{p}^0_n = (p^1_n \restriction \eta_1) \cup (p^0_n \restriction [\eta_0, \omega_2))$. Then, $\bar{p}^0_n \in N_0$. Let $p^0_{n+1} \le \bar{p}^0_n$ be so that $p^0_{n+1} \in \tilde{P} \cap \mathcal{D}^0_n \cap N_0$. By the same argument, we can build p^1_{n+1} that satisfies the inductive hypothesis.

Define q as follows: Let $\mathrm{dom}(q) = \bigcup_{n<\omega}(\mathrm{dom}(p^0_n) \cup \mathrm{dom}(p^1_n))$. Let $\mathrm{dom}(q(0)) = \delta + 1$, $q(0) \restriction \delta = \bigcup_{n<\omega}(p^0_n(0) \cup p^1_n(0))$, and $q(0)(\delta)$ be the set of all subsets x of δ such that $x = \{\xi < \delta : \exists n < \omega(p^i_n \Vdash \text{‘}\xi \in \dot{X}\text{’})\}$ for some $i < 2$ and a P-name $\dot{X} \in N_i$ for a subset of ω_1. For every $\alpha \in \mathrm{dom}(q)$ with $\alpha > 0$, define

$$q(\alpha) = \begin{cases} \bigcup_{n<\omega} p^0_n(\alpha) \cup \{\delta\} & \text{if } \alpha \in \bigcup_{n<\omega} \mathrm{dom}(p^0_n) \\ \bigcup_{n<\omega} p^1_n(\alpha) \cup \{\delta\} & \text{if } \alpha \in \bigcup_{n<\omega} \mathrm{dom}(p^1_n) \end{cases}$$

It is easy to see that $q \in P$.

Moreover, q is (N_0, P)-generic and (N_1, P)-generic. Thus, q forces that both $N_0[\dot{G}]$ and $N_1[\dot{G}]$ belong to $\dot{E}$. But $D_0 \in N_0[\dot{G}]$ and $D_1 \in N_1[\dot{G}]$ are club subsets of ω_1 and $D_0 \cap D_1 = \varnothing$. This is a contradiction. $\dashv$

REFERENCES

[1] T. Ishiu, *Club guessing sequences and filters*, ***J. Symbolic Logic***, vol. 70 (2005), no. 4, pp. 1037–1071.

[2] P. B. Larson, *The canonical function game*, ***Arch. Math. Logic***, vol. 44 (2005), no. 7, pp. 817–827.

[3] Justin Tatch Moore, *ω_1 and $-\omega_1$ may be the only minimal uncountable linear orders*, ***Michigan Math. J.***, vol. 55 (2007), no. 2, pp. 437–457.

[4] Saharon Shelah, ***Proper and improper forcing***, second ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998.

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