Common hypercyclic vectors for unilateral weighted shifts on ℓ^2

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Abstract

Each $w \in \ell^\infty$ defines a bounded linear operator $B_w : \ell^2 \to \ell^2$ where $B_w(x)(i) = w(i) \cdot x(i+1)$ for each $i \in \omega$. A vector $x \in \ell^2$ is *hypercyclic* for B_w if the set $\{B_w^k(x) : k \in \omega\}$ of forward iterates of x is dense in ℓ^2. For each such w, the set $\mathsf{HC}(w)$ consisting of all vectors hypercyclic for B_w is G_δ. The set of *common hypercyclic vectors* for a set $W \subseteq \ell^\infty$ is the set $\mathsf{HC}^*(W) = \bigcap_{w \in W} \mathsf{HC}(w)$. We show that $\mathsf{HC}^*(W)$ can be made arbitrarily complicated by making W sufficiently complex, and that even for a G_δ set W the set $\mathsf{HC}^*(W)$ can be non-Borel.

Finally, by assuming the Continuum Hypothesis or Martin's Axiom, we are able to construct a set W such that $\mathsf{HC}^*(W)$ does not even have the property of Baire.

1 Introduction

Given a separable Banach space X and a linear operator $T : X \to X$, one says that an element $x \in X$ is *hypercyclic* for T iff the forward iterates of x under T form a dense subset of X. That is, x is hypercyclic for T iff the set

$$\{T^n(x) : n \in \omega\}$$

is dense in X. In this paper, we will consider the hypercyclic vectors of a specific class of operators on the Hilbert space ℓ^2 of square-summable sequences.

Definition 1.1. Given $w \in \ell^\infty$, define a bounded linear operator $B_w : \ell^2 \to \ell^2$ by

$$B_w(x)(i) = w(i) \cdot x(i+1).$$

Such a B_w is called a *unilateral weighted shift.* If $y \in \ell^2$ is hypercyclic for such a B_w, then y is said to be *hypercyclic for w*. Let $\mathsf{HC}(w)$ denote the collection of all $y \in \ell^2$ which are hypercyclic for w.

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It is routine to check that $\mathsf{HC}(w)$ is a G_δ set for any $w \in \ell^\infty$. The question addressed in the present paper is how much the complexity of $\mathsf{HC}(w)$ can be increased by looking at those sequences which are hypercyclic for many w simultaneously. Concretely, for $W \subseteq \ell^\infty$, let

$$\mathsf{HC}^*(W) = \bigcap_{w \in W} \mathsf{HC}(w).$$

It turns out that $\mathsf{HC}^*(W)$ can be made arbitrarily complicated by making W sufficiently complex (Theorem 1.4). Even for a G_δ set W, however, the set $\mathsf{HC}^*(W)$ can still be non-Borel (Theorem 1.5). Moreover, by assuming the Continuum Hypothesis (or Martin's Axiom), a set W can constructed so that $\mathsf{HC}^*(W)$ fails to have the Baire property.

It is necessary to introduce a few preliminaries and some terminology before proceeding. One of the key descriptive set theoretic concepts in this paper is that of a *pointclass*. There are many variations on the definition of the term "pointclass". For the purposes of the present work, we will use the following definition.

Definition 1.2. A *pointclass* Γ is a collection of subsets of Polish (separable completely metrizable) spaces such that

- Γ is closed under continuous preimages,
- Γ is closed under finite unions and
- Γ is closed under finite intersections.

Given a pointclass Γ, the *dual pointclass* $\bar{\Gamma}$ consists of those Y contained in some Polish space X such that $X \setminus Y \in \Gamma$. A pointclass is *non-self-dual* iff there exist a Polish space X and a set $Y \subseteq X$ such that $Y \in \Gamma$ but $Y \notin \bar{\Gamma}$ (equivalently, $X \setminus Y \notin \Gamma$).

To take a few examples, "closed" and "open" are dual pointclasses as are "F_σ" and "G_δ". All four of these classes are non-self-dual.

The Borel structure of ℓ^∞ (as a Banach space) is fundamentally different from the inherited structure of $\mathbb{R}^\omega$. In fact, since the cellularity of ℓ^∞ is $\mathfrak{c}$, it is possible to construct sets which are closed in ℓ^∞, but not even Borel in $\mathbb{R}^\omega$. Therefore, when considering the topological complexity of sets of weight sequences in ℓ^∞, we will use the Borel structure of $\mathbb{R}^\omega$. In fact, in all of our constructions, the weight sequences used are in $W \subseteq \{1,2\}^\omega$.

The next proposition establishes an upper bound on the complexity of $\bigcap_{w \in W} \mathsf{HC}(w)$ when W is analytic.

Proposition 1.3. *If W be a subset of ℓ^∞ which is analytic in the product topology on $\mathbb{R}^\omega$, the intersection $\bigcap_{w \in W} \mathsf{HC}(w)$ is co-analytic.*

Proof. To see this, observe that, for $y \in \ell^2$,

$$y \in \bigcap_{w \in W} \mathsf{HC}(w) \iff (\forall w \in \ell^\infty)(w \in W \implies y \in \mathsf{HC}(w)).$$

Thus, $\bigcap_{w \in W} \mathsf{HC}(w)$ is co-analytic since the relation

$$P(y, w) \iff y \in \mathsf{HC}(w)$$

is itself G_δ and the $\mathbf{\Sigma}^1_1$ relation "$w \in W$" is in the hypothesis of the conditional statement above. $\square$

The next two theorems show that the upper bound from the last proposition cannot be improved.

Theorem 1.4. *Given a non-self-dual pointclass Γ which contains the closed sets, there is a set $W \subseteq \{1,2\}^\omega$ such that $\bigcap_{w\in W} \mathsf{HC}(w)$ is not in Γ.*

Theorem 1.5. *There is a Borel set $W \subseteq \{1,2\}^\omega$ such that $\bigcap_{w\in W} \mathsf{HC}(w)$ is properly co-analytic, i.e., not analytic.*

The third main theorem of this paper uses Martin's Axiom to construct a set $W \subseteq \{1,2\}^\omega$ such that $\mathsf{HC}^*(W)$ does not have the property of Baire.

Theorem 1.6. *Assuming MA, there exists $W \subseteq \{1,2\}^\omega$ such that $\mathsf{HC}^*(W)$ does not have the property of Baire.*

The proof of Theorem 1.6 is essentially a more complex version of the construction (under CH) of a Bernstein set. Also note that, since MA is a consequence of CH, Theorem 1.6 is a consequence of CH as well. In fact, the proof under CH is somewhat simpler than the Martin's Axiom version.

For the reader unfamiliar with Martin's Axiom, it is (at an intuitive level) a strengthening of the Baire Category Theorem to encompass arbitrary intersections of fewer than continuum many dense open sets. A detailed statement of Martin's Axiom is included in the next section.

2 Preliminaries

2.1 Fundamentals

Before proceeding, it will be helpful to review some standard facts about the Banach spaces of interest in this paper. Let $\|\cdot\|_2$ denote the usual ℓ^2 norm. In what follows, this notation will be used for finite sequences as well, i.e., for $s \in \mathbb{R}^{<\omega}$,

$$\|s\|_2 = \sqrt{s(0)^2 + \ldots + s(n)^2}$$

assuming s is of length $n+1$.

The notation $|s|$ will be used to denote both the length of a string (if $s \in 2^{<\omega}$) and the length of an interval (if $s \subseteq \omega$ is an interval). The notation $\|x\|_\infty$ will denote the ℓ^∞- or sup-norm of x. Again, this definition makes sense for any string x – either finite or infinite.

Since the topology of ℓ^2 is a strict refinement of the product topology on $\mathbb{R}^\omega$, a couple minor lemmas are required to permit the use of some "product topology intuition" when working in ℓ^2. The first of these lemmas indicates a relationship between the 2-norm and the sup-norm of a finite string which will be quite useful.

Lemma 2.1. *If s is a finite string of real numbers, having length n,*

$$\|s\|_2 \leq n^{1/2}\|s\|_\infty.$$

Proof. Suppose that $s \in \mathbb{R}^n$ and $\|s\|_\infty = n^{-1/2} \cdot \varepsilon$ for some positive ε. In other words, $|s(i)| \leq n^{-1/2}\varepsilon$ for all $i < n$. It follows that

$$\begin{aligned} \|s\|_2 &= \sqrt{s(0)^2 + \ldots + s(n-1)^2} \\ &\leq \sqrt{n \cdot (n^{-1/2}\varepsilon)^2} \\ &= \varepsilon \\ &= n^{1/2}\|s\|_\infty \end{aligned}$$

This proves the lemma. □

2.2 Topology in ℓ^2

In much of this paper, it will be helpful to use an alternative topological basis for ℓ^2. Given a finite nonempty string $q \in \mathbb{Q}^{<\omega}$ of rationals and a (rational) number $\varepsilon > 0$, let

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|(x \restriction |q|) - q\|_\infty < \varepsilon|q|^{-1/2} \text{ and } \|x \restriction [|q|, \infty)\|_2 < \varepsilon\}.$$

In the case that $q = \langle\ \rangle$ is the empty sequence, simply let

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|x\|_2 < \varepsilon\}.$$

First note that each $U_{q,\varepsilon}$ is open. In order to check that the $U_{q,\varepsilon}$ form a basis for ℓ^2, fix a basic open ball

$$V = \{x \in \ell^2 : \|x - x_0\|_2 < \varepsilon\}$$

where $x_0 \in \ell^2$ and $\varepsilon > 0$ are fixed. Let $n \in \omega$ be such that

$$\|x_0 \restriction [n, \infty)\|_2 < \varepsilon/4$$

and choose $q \in \mathbb{Q}^n$ such that

$$\|x_0 \restriction n - q\|_\infty < (\varepsilon/4) \cdot n^{-1/2}.$$

First of all, it follows from the definition of $U_{q,\varepsilon}$ that $x_0 \in U_{q,\varepsilon/4}$. To see that $U_{q,\varepsilon/4} \subseteq V$, observe that if $x \in U_{q,\varepsilon/4}$,

$$\begin{aligned} \|x - x_0\|_2 &\leq \|(x - x_0) \restriction n\|_2 + \|(x - x_0) \restriction [n, \infty)\|_2 \\ &\leq n^{1/2}\|(x - x_0) \restriction n\|_\infty + \|x \restriction [n, \infty)\|_2 + \|x_0 \restriction [n, \infty)\|_2 \\ &< n^{1/2}(\|(x \restriction n) - q\|_\infty + \|(x_0 \restriction n) - q\|_\infty) + \varepsilon/4 + \varepsilon/4 \\ &< n^{1/2}((\varepsilon/4)n^{-1/2} + (\varepsilon/4)n^{-1/2}) + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

As $x \in U_{q,\varepsilon/4}$ was arbitrary, it follows that $U_{q,\varepsilon/4} \subseteq V$. Since V was an arbitrary open ball, this shows that the $U_{q,\varepsilon}$ form a topological basis for ℓ^2.

The next lemma relates the continuity of functions into ℓ^2 with respect to two different topologies: the subspace topology inherited from $\mathbb{R}^\omega$ and the inherent Banach space topology induced by the 2-norm.

Lemma 2.2. *If A is a countable set and $f : 2^A \to \ell^2$ is such that*

1. *f is continuous with respect to the product topology on 2^A and the subspace topology on ℓ^2 inherited from $\mathbb{R}^\omega$, and*
2. *there exists $y \in \ell^2$ such that $|f(x)(i)| \leq y(i)$ for all $x \in 2^A$ and $i \in \omega$,*

then f is continuous with respect to the norm-topology on ℓ^2.

Proof. Let $y \in \ell^2$ be as in the statement of the lemma. Towards the goal of showing that f is ℓ^2-continuous, fix $\varepsilon > 0$ and let n be such that

$$\|y \restriction [n, \infty)\|_2 < \varepsilon/4.$$

Since f is continuous into the subspace topology on ℓ^2 (from $\mathbb{R}^\omega$), and 2^A is compact, there exists a finite $F \subseteq A$ such that, for $x_1, x_2 \in 2^A$, if $x_1 \restriction F = x_2 \restriction F$, then

$$|f(x_1)(i) - f(x_2)(i)| < n^{-1/2}\varepsilon/2$$

for all $i < n$. In particular, $x_1 \restriction F = x_2 \restriction F$ guarantees

$$\|((f(x_1) - f(x_2)) \restriction n)\|_2 < \varepsilon/2$$

by Lemma 2.1. It now follows that, whenever $x_1, x_2 \in 2^A$ and $x_1 \restriction F = x_2 \restriction F$,

$$\begin{aligned} \|f(x_1) - f(x_2)\|_2 &\leq \|((f(x_1) - f(x_2)) \restriction n)\|_2 + \|f(x_1) \restriction [n, \infty)\|_2 \\ &\qquad + \|f(x_2) \restriction [n, \infty)\|_2 \\ &< \varepsilon/2 + 2\|y \restriction [n, \infty)\|_2 \\ &< \varepsilon/2 + 2\varepsilon/4 = \varepsilon. \end{aligned}$$

Since ε was arbitrary this completes the proof. Note that since 2^A is compact, f is in fact uniformly continuous with respect to the standard ultrametric on 2^A. □

2.3 Hypercyclic vectors

The next two lemmas are standard. See, e.g., Theorems 1.2 and 1.40 in [1].

Lemma 2.3. *If $W \subseteq \{1, 2\}^\omega$ and $\mathsf{HC}^*(W)$ is nonempty, then $\mathsf{HC}^*(W)$ is dense.*

Proof. Suppose that $y \in \mathsf{HC}^*(W)$. Fix an open set $U \subseteq \ell^2$ and let $s \in R^{<\omega}$ be such that $y + s^\frown\bar{0} \in U$. To see that $y + s^\frown\bar{0} \in \mathsf{HC}^*(W)$, fix $w \in W$ and an open set $V \subseteq \ell^2$. Since ℓ^2 is T_1, there are infinitely many $k \in \omega$ such that $B_w^k(y) \in V$. In particular, there exists $k \geq |s|$ with this property. For such a $k \geq |s|$,

$$B_w^k(y + s^\frown\bar{0}) = B_w^k(y) \in V.$$

As w and V were arbitrary, it follows that $y + s^\frown\bar{0} \in \mathsf{HC}^*(W)$. Since U was arbitrary as well, it follows that $\mathsf{HC}^*(W)$ is dense in ℓ^2. □

Lemma 2.4. *If $w \in \{1, 2\}^\omega$ is such that that there are infinitely many i with $w(i) = 2$, then $\mathsf{HC}(w)$ is comeager.*

Proof. First of all, to see that $\mathsf{HC}(w)$ is nonempty, let U_{q_n,ε_n} enumerate the basic open neighborhoods defined above. Given a sequence $k_0 < k_1 < \ldots$, define for each n, a string $\bar{q}_n \in \mathbb{Q}^{<\omega}$ by

$$\bar{q}_n(i) = 2^{-|\{j\in[i,i+k_n):w(j)=2\}|} \cdot q_n(i)$$

(for $i < |q_n|$). Note that if $y \in \ell^2$ has a copy of $\bar{q}_n$ starting at the k_nth bit of y, then $B_w^{k_n}(y)$ begins with a copy of q_n. All that remains is to choose a specific sequence $k_0 < k_1 < \ldots$ which grows quickly enough that, for each n,

- $k_{n+1} - k_n \geq |\bar{q}_n|$ and
- $\|\bar{q}_n\|_2 \leq 2^{-n-1-k_n} \cdot \min\{\varepsilon_j : j \leq n\}$.

Having done this, let $p_n = k_{n+1} - k_n - |\bar{q}_n|$ and

$$y = \bar{q}_0{}^\frown 0^{p_0}{}^\frown \bar{q}_1{}^\frown 0^{p_1}{}^\frown \ldots$$

Then for each n, y has a copy of $\bar{q}_n$ beginning at its k_nth term and so

$$B_w^{k_n}(y) = q_n{}^\frown \alpha$$

for some $\alpha \in \ell^2$. Moreover, by the choice of $k_{n+1} < k_n < \ldots$,

$$\begin{aligned}
\|\alpha\|_2 &\leq \sum_{i>n} 2^{k_n} \cdot \|\bar{q}_i\|_2 \\
&\leq \sum_{i>n} 2^{k_n} \cdot 2^{-i-1-k_i} \cdot \min\{\varepsilon_j : j \leq n\} \\
&\leq \sum_{i>n} 2^{-i-1}\varepsilon_n \\
&= 2^{-n-1}\varepsilon_n < \varepsilon_n
\end{aligned}$$

It follows that $B_w^k(y) \in U_{q_n,\varepsilon_n}$. As n was arbitrary, it follows that $y \in \mathsf{HC}(w)$.

It now follows from Lemma 2.3 (applied to $W = \{w\}$) that $\mathsf{HC}(w)$ is dense. As a dense G_δ set, $\mathsf{HC}(w)$ is thus comeager. □

2.4 Martin's Axiom

Given a partially ordered set $(\mathbb{P}, <)$, a subset $D \subseteq \mathbb{P}$ is *dense* iff, for each $p \in \mathbb{P}$, there is a $q \leq p$ with $q \in D$. A set $G \subseteq \mathbb{P}$ is called a *filter* iff

- $(\forall p, q \in \mathbb{P})(p \geq q \in G \implies p \in G)$ and
- $(\forall p, q \in G)(\exists r \in G)(r \leq p \text{ and } r \leq q)$.

Given a cardinal number $\kappa < \mathfrak{c}$, Martin's Axiom (or MA) is the following assertion.

Martin's Axiom. Given the following:

- a partially ordered set $\mathbb{P}$ with no uncountable antichains, i.e., $\mathbb{P}$ has the ccc, and
- a collection $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$ of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| < \mathfrak{c}$,

there exists a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

A well-known consequence of MA asserts that the intersection of fewer than $\mathfrak{c}$-many comeager sets is still comeager. cite Kunen or Jech

3 Proof of Theorem 1.4

Example 7.1 of Bayart-Matheron [1] shows that there is no $y \in \ell^2$ which is hypercyclic for every $w \in \{1,2\}^\omega$. The argument in this section is motivated by a set theoretic construction which uses the proof of this fact. We briefly sketch the idea:

For each $\epsilon > 0$, the map sending each $y \in \ell^2$ to the least n such that $\|y \restriction [n,\infty)\|_2 < \epsilon$ is Borel (but not continuous). It follows from the argument of Example 7.1 of [1] that there is a Borel function b sending each y to a hypercyclic $w \in \{1,2\}$ for which y is not hypercyclic. Since $\mathsf{HC}(w)$ is comeager whenever it is nonempty, if we let M be a countable transitive model of a sufficient fragment of ZFC, with a Borel code for b in M, and let P be a perfect set of elements of ℓ^2 which are mutually Cohen-generic over M, we get that $x \in \mathsf{HC}(b(y))$, for all distinct $x, y \in P$. The proof below carries out these ideas without using forcing.

Lemma 3.1. *There is a dense G_δ set $G \subseteq \ell^2$ and continuous function $f : G \to \{1,2\}^\omega$ such that*

- *$y \notin \mathsf{HC}(f(y))$ for each $y \in G$,*
- *$\mathsf{HC}(f(y)) \neq \emptyset$ (and is therefore comeager) for each $y \in G$, and*
- *for each open $U \subseteq \ell^2$, the image $f[U \cap G]$ contains at least two elements. (As we will see in the proof, this actually follows from the first two properties.)*

Proof. For each $y \in \ell^2$ and $n \in \omega$, let $i_{n,y} \in \omega$ be least such that

- $i_{n,y} > 2i_{n-1,y}$ and
- $\|y \restriction [i_{n,y},\infty)\|_2 < 2^{-1-n}$.

Let $f : \ell^2 \to \{1,2\}^\omega$ be defined by

$$f(y)(i) = \begin{cases} 2 & \text{if } i = i_{n,y} \text{ for some } n \in \omega, \\ 1 & \text{otherwise,} \end{cases}$$

This function is Borel and hence (by Theorem 8.37 in Kechris [2]), there is a dense G_δ set G such that $f \restriction G$ is continuous.

For each $y \in \ell^2$, the sequence $f(y) \in \{1,2\}^\omega$ has infinitely many 2's and hence (by Lemma 2.4) $\mathsf{HC}(f(y))$ is comeager. In particular, $\mathsf{HC}(f(y))$ is comeager for all $y \in G$.

The next step is to see that $y \notin \mathsf{HC}(f(y))$. Given y, fix any $k \geq i_{0,y}$. Suppose that $i_{n,y} \leq k < i_{n+1,y}$. Since each $i_{p,y}$ is at least $2i_{p-1,y}$, it follows that no interval $I \subseteq [i_{n,y},\infty)$ of length k contains more that n elements i where $f(y)(i) = 2$. Thus, to obtain $B^k_{f(y)}(y)$ from y, each bit of y is multiplied by at most n twos as it is shifted. Therefore,

$$\|B^k_{f(y)}(y)\|_2 \leq 2^n \cdot \|y \restriction [i_n,\infty)\|_2 \leq 1$$

by the choice of i_n. In particular, $B^k_{f(y)}(y) \in \{z \in \ell^2 : \|z\|_2 > 1\}$ for only finitely many k and hence $y \notin \mathsf{HC}(f(y))$.

Thus, the first two conditions in the statement of the Lemma have been satisfied. To prove the third, fix an open set $U \subseteq \ell^2$ and pick any $y \in U \cap G$. Since $\mathsf{HC}(f(y)) \neq \emptyset$ and is thus comeager, there exists $x \in \mathsf{HC}(f(y)) \cap U \cap G$. Since $x \notin f(x)$, the weight sequences $f(x)$ and $f(y)$ must be distinct as otherwise

$$x \in \mathsf{HC}(f(y)) = \mathsf{HC}(f(x))$$

contrary to the choice of f. It follows that $f[U \cap G]$ contains at least two elements. $\square$

The key lemma in the proof of Theorem 1.4 is Lemma 3.2 below. In essence, the proof of Lemma 3.2 is a direct construction of a perfect set of mutually generic Cohen reals. (Recall that a *perfect* set is a closed set with no isolated points.) This will be essential to building a set $W \subseteq \{1,2\}^\omega$ of weight sequences such that $\mathsf{HC}^*(W)$ is of arbitrary definable complexity. For the present purposes, the salient property of perfect sets is that, for any pointclass Γ, a given perfect set has subsets which are not in Γ.

Lemma 3.2. *Given $f : G \to \{1,2\}^\omega$ as in Lemma 3.1, there is a perfect set $P \subseteq \ell^2$ such that*

- $x \notin \mathsf{HC}(f(x))$ *for all $y \in P$ (this is from Lemma 3.1)*
- $x \in \mathsf{HC}(f(y))$ *for all distinct $x, y \in P$.*

Proof. First of all, let $f : \ell^2 \to \{1,2\}^\omega$ be as in Lemma 3.1, i.e.,

- f is continuous on a comeager G_δ set $G \subseteq \ell^2$
- $y \notin \mathsf{HC}(f(y))$ for each $y \in G$
- $\mathsf{HC}(f(y))$ is comeager for each $y \in G$
- f is not constant on $U \cap G$ for each open $U \subseteq \ell^2$

Let $\mathsf{HC} = \{(y,w) \in \ell^2 \times \{1,2\}^\omega : y \in \mathsf{HC}(w)\}$. Since HC and G are both dense G_δ sets, there exist dense open sets $D_n \subseteq \ell^2 \times \{1,2\}^\omega$ such that

$$\mathsf{HC} \cap (G \times \{1,2\}^\omega) \supseteq \bigcap_n D_n.$$

and $D_0 \supseteq D_1 \supseteq D_2 \supseteq \ldots$.

The key is to build Cantor schemes $(P_s)_{s\in 2^{<\omega}}$ (in ℓ^2) and $(Q_s)_{s\in 2^{<\omega}}$ (in $\{1,2\}^\omega$) such that

- Each P_s is open and Q_s is a basic clopen set.
- If s is an initial segment of t, then $P_s \supseteq \overline{P}_t$ and $Q_s \supseteq Q_t$.
- If s and t have no common extensions, then $\overline{P}_s \cap \overline{P}_t = \emptyset$ and $Q_s \cap Q_t = \emptyset$.
- $f[\overline{P}_s \cap G] \subseteq Q_s$.
- If $s, t \in 2^n$ are distinct, then $P_s \times Q_t \subseteq D_n$.

The construction is by induction on $|s|$. Suppose that P_s and Q_s are given with the properties above for all $s \in 2^n$.

Step 1. By the properties of f (Lemma 3.1) each $f[P_s \cap G]$ has at least two elements. Therefore, for all $s \in 2^n$, the continuity of $f \restriction G$ implies that there are open sets $U_{s^\frown 0}$ and $U_{s^\frown 1}$ such that

- $\overline{U}_{s^\frown 0}, \overline{U}_{s^\frown 1} \subseteq P_s$
- $f[U_{s^\frown 0} \cap G] \cap f[U_{s^\frown 1} \cap G] = \emptyset$

The second property above (together with the properties of the Q_s) implies that for all distinct $s, t \in 2^{n+1}$, the sets $f[U_s \cap G]$ and $f[U_t \cap G]$ are disjoint.

Now fix $s \in 2^{n+1}$ and pick $x \in U_s \cap G$. By the properties of f, the set of hypercyclic vectors for $f(x)$ is comeager and hence has nonempty intersection with all $U_t \cap G$ (for $t \in 2^{n+1}$), i.e., $((U_t \cap G) \times \{f(x)\}) \cap \mathsf{HC} \neq \emptyset$. In particular, shrinking the neighborhoods U_t if necessary (for $t \neq s$), there must be a neighborhood Q_s of $f(x)$ such that $U_t \times Q_s \subseteq D_{n+1}$ for each $t \in 2^{n+1}$ with $t \neq s$. Finally, by the continuity of $f \restriction G$, it is possible to shrink U_s to ensure that $f[\overline{U}_s \cap G] \subseteq Q_s$.

Repeat the process above for each $s \in 2^{n+1}$. Each time this process is repeated, a new Q_s is produced and each U_t (for $t \in 2^{n+1} \setminus \{s\}$) shrinks finitely many times. To complete the induction, let $P_s = U_s$ (after it has been shrunk as above). The P_s and Q_s now satisfy the desired properties above.

Step 2. Let P be the perfect set associated with the Cantor scheme $(P_s)_{s \in 2^{<\omega}}$, i.e.,

$$P = \bigcup_{\alpha \in 2^\omega} \bigcap_n P_{\alpha \restriction n}.$$

It follows from the definition of f that $y \notin \mathsf{HC}(f(x))$ for each $x \in P$. On the other hand, suppose that $x, y \in P$ are distinct. Let $\alpha, \beta \in 2^\omega$ be such that

$$\{x\} = \bigcap_n P_{\alpha \restriction n} \qquad \text{and} \qquad \{y\} = \bigcap_n P_{\beta \restriction n}.$$

If $n \in \omega$ is large enough that $\alpha \restriction n \neq \beta \restriction n$, the properties of the P_s and Q_s guarantee that

$$(x, f(y)) \in P_{\alpha \restriction n} \times Q_{\beta \restriction n} \subseteq D_n.$$

Since this holds for all but finitely many n, it follows that $(x, f(y)) \in \mathsf{HC}$, i.e., $x \in \mathsf{HC}(f(y))$ as desired. This completes the proof. $\square$

Proof of Theorem 1.4. Let $f : P \to \{1,2\}^\omega$ be as in Lemma 3.2, i.e., for all $x, y \in P$,

- $y \notin \mathsf{HC}(f(y))$ and
- $x \in \mathsf{HC}(f(y))$ if $x \neq y$.

Let Γ be any pointclass as in the statement of Theorem 1.4. Choose any $A \subseteq P$ with $A \in \Gamma \setminus \bar{\Gamma}$. Consider the common hypercyclic vectors of $f[A]$, i.e., the set $\mathsf{HC}^*(f[A])$. For $x, y \in P$,

$$x \in \mathsf{HC}(f(y)) \iff x \neq y$$

and so $\mathsf{HC}^*(f[A]) \cap P = P \setminus A$. In particular, $\mathsf{HC}^*(f[A]) \cap P \notin \Gamma$ and hence $\mathsf{HC}^*(f[A]) \notin \Gamma$ either since Γ contains the closed subsets of ℓ^2.

This completes the proof. $\square$

4 Proof of Theorem 1.5

Given a countable set A, a subset a of A may be identified with with its characteristic function in 2^A. In what follows, we will freely make use of this identification and regard $\mathcal{P}(A)$ (the power set of A) as being equipped with the usual product-of-discrete topology of 2^A.

The following technical lemma is the key to the proof of Theorem 1.5. In fact, it can be used to prove Theorem 1.4 as well.

Lemma 4.1. *Given a countable set A, it is possible to assign to each $a \subseteq A$, sequences $y_a \in \ell^2$ and $w_a \in \{1, 2\}^\omega$ such that*

1. *for all $a, b \subseteq A$, we have $y_a \in \mathsf{HC}(w_b) \iff b \nsubseteq a$, and*
2. *the maps $a \mapsto y_a$ and $a \mapsto w_a$ are homeomorphisms between 2^A and their ranges.*

Proof. Let $\pi \colon \omega \to \mathbb{Q}^{<\omega}$ be a surjection. Let A be the fixed countable set from the statement of the lemma. For coding purposes, fix a bijection

$$\langle \cdot, \cdot, \cdot \rangle : \omega \times (\mathbb{Q} \cap (0, 1)) \times A \to \omega.$$

Given $n \in \omega$, let $p_n \in \omega$, $\varepsilon_n > 0$ and $i_n \in A$ be such that

$$n = \langle p_n, \varepsilon_n, i_n \rangle.$$

Finally, let

$$\rho_n = \min\{\varepsilon_r : r \leq n\}.$$

The first step of the proof is to choose a suitable partition

$$I_0, J_0, I_1, J_1, \ldots$$

of ω into consecutive intervals, i.e., such that $\min(J_n) = \max(I_n) + 1$ and $\min(I_{n+1}) = \max(J_n) + 1$. For convenience, we let $I_0 = \{0\}$. Each J_n will be chosen with $|J_n| = |\pi(p_n)|$. The lengths of the I_n (for $n > 0$) will be chosen recursively and, for concreteness, of minimal length satisfying

1. $|I_n| \geq |I_{n-1}|$,
2. $|I_n| > \max(J_{n-1})$ and
3. $2^{-|I_n|} \cdot \|\pi(p_n)\|_2 \leq 2^{-n-1} \cdot \rho_n \cdot 2^{-\max(J_{n-1})} \cdot 2^{-|I_{n-1}|}$.

for $n > 1$. The length of I_0 is arbitrary – I_0 can even be the empty interval.

The next step is to define the desired y_a and w_a for each $a \subseteq A$. For $n = \langle p, \varepsilon, i \rangle$, define y_a on I_n and J_n by

1. $y_a \restriction I_n = \bar{0}$,
2. $y_a \restriction J_n = \bar{0}$ if $i \in a$, and
3. $y_a \restriction J_n = 2^{-|I_n|} \cdot \pi(p)$ if $i \notin a$.

The first important observation about the map $a \mapsto y_a$ is that it is continuous. To see this, first observe that every initial segment of y_a is determined by an initial segment of a. This implies that $a \mapsto y_a$ is continuous into the product topology on ℓ^2 (which it inherits from $\mathbb{R}^\omega$). Now invoke Lemma 2.2 and use the fact that y_a is always termwise bounded by $y_\emptyset \in \ell^2$. It now follows that $a \mapsto y_a$ is in fact continuous with respect to the norm-topology on ℓ^2.

It also follows from the definition of y_a that the function $a \mapsto y_a$ is injective. As the domain of this map (2^A) is compact, $a \mapsto y_a$ must therefore be a homeomorphism with its range.

Now define $w_a \in \{1,2\}^\omega$ (for $a \subseteq A$) by making sure that the restrictions $w_a \restriction I_n \cup J_n$ satisfy

1. $(\forall n)(i_n \notin a \implies w_a \restriction I_n \cup J_n = \bar{1})$,
2. $(\forall n)(i_n \in a \implies (\forall j \in J_n)(|\{t \in [j, \min(J_n)+j) : w_a(t) = 2\}| = |I_n|)$ and
3. if $i, j \in I_n$ with $i < j$ and $w_a(j) = 2$, then $w_a(i) = 2$.

The continuity of $a \mapsto w_a$ follows from the fact that initial segments of w_a are completely determined by initial segments of a.

The next three claims will complete the proof. The proofs of these three claims all follow similar arguments using the definitions of the y_a and w_a.

Claim. Each y_a is in ℓ^2.

It suffices to show that the ℓ^2 norm of y_a is finite. Indeed, by the triangle inequality and the third part of the definition of y_a,

$$\begin{aligned}
\|y_a\|_2 &\leq \sum_{n\in\omega} \|y_a \restriction J_n\|_2 \\
&\leq \sum_{n\in\omega} 2^{-|I_n|} \cdot \|\pi(p_n)\|_2 \\
&\leq \sum_{n\in\omega} 2^{-n-1} \cdot \rho_n \cdot 2^{-\max(J_{n-1})} \cdot 2^{-|I_{n-1}|} \\
&\leq \sum_{n\in\omega} 2^{-n-1} \\
&\leq 1
\end{aligned}$$

This proves the claim.

Claim. If $a, b \subseteq A$ with $b \subseteq a$, then $y_a \notin \mathsf{HC}(w_b)$.

For this claim, it suffices to show that $\|B^k_{w_b}(y_a)\|_2 \leq 1$ or $B^k_{w_b}(y_a)(0) = 0$ for each $k \in \omega$. This will establish that there is no $k \in \omega$ such that $B^k_{w_b}(y_a)$ is in the open set

$$U = \{y \in \ell^2 : \|y\|_2 > 1 \text{ and } y(0) \neq 0\}.$$

To this end, fix $k \in \omega$ and let $n \in \omega$ be such that $k \in I_n \cup J_n$. First of all, if $i_n \in a$, then $y_a \restriction I_n \cup J_n = \bar{0}$ and hence

$$B^k_{w_b}(y_a)(0) = w_b(0) \cdot \ldots \cdot w_b(k-1) \cdot y_a(k) = 0.$$

On the other hand, if $i_n \notin a \supseteq b$, then $w_b \restriction I_n \cup J_n = \bar{1}$ and hence

$$|\{j < k : w_b(j) = 2\}| \leq \max(J_{n-1}).$$

To obtain an estimate of $\|B^k_{w_b}(y_a)\|_2$, a couple preliminary observations will be useful. Suppose $t \in \omega$ is such that $k + t \in I_r$ for some $r \in \omega$. In this case,

$$B^k_{w_b}(y_a)(t) = 0$$

since $y_a(k+t) = 0$. If $k + t \in J_n$ (where $k \in I_n \cup J_n$), then

$$|B^k_{w_b}(y_a)(t)| \leq 2^{\max(J_{n-1})} \cdot |y_a(k+t)|$$

since $w_b \restriction I_n \cup J_n = \bar{1}$. Finally, if $k + t \in J_r$ for some $r > n$, then

$$\begin{aligned} |B^k_{w_b}(y_a)(t)| &\leq 2^k \cdot |y_a(k+t)| \\ &\leq 2^{\max(J_{r-1})} \cdot |y_a(k+t)| \end{aligned}$$

since $k \leq \max(J_n) \leq \max(J_{r-1})$. It now follows by the triangle inequality that

$$\begin{aligned} \|B^k_{w_b}(y_a)\|_2 &\leq \sum_{r \geq n} 2^{\max(J_{r-1})} \cdot \|y_a \restriction J_r\|_2 \\ &\leq \sum_{r \geq n} 2^{\max(J_{r-1})} \cdot 2^{-r-1} \cdot \rho_r \cdot 2^{-\max(J_{r-1})} \cdot 2^{-|I_{r-1}|} \\ &\leq \sum_{r \geq n} 2^{-r-1} \\ &\leq 1 \end{aligned}$$

This completes the proof of the claim.

Claim. If $a, b \subseteq A$ with $b \nsubseteq a$, then $y_a \in \mathsf{HC}(w_b)$.

For this final claim, it suffices to show that, for each $q \in \mathbb{Q}^{<\omega}$ and $\varepsilon > 0$, there is a $k \in \omega$ such that $B^k_{w_b}(y_a)$ is in the open set

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|(x \restriction |q|) - q\|_\infty < \varepsilon |q|^{-1/2} \text{ and } \|x \restriction [|q|, \infty)\|_2 < \varepsilon\}$$

as these open sets form a topological basis for ℓ^2. Indeed, fix $q \in \mathbb{Q}^{<\omega}$ and let $p \in \omega$ be such that $\pi(p) = q$. Fix $i \in b \setminus a$ and let $n = \langle p, \varepsilon, i \rangle$. Since $i \in b$ and $i \notin a$, the second case in the definition of $w_b \restriction I_n \cup J_n$ and the second case in the definition of $y_a \restriction J_n$ are active. In particular, for each $j \in J_n$,

$$|\{t \in [j, \min(J_n) + j) : w_b(t) = 2\}| = |I_n|.$$

It follows that

$$B_{w_b}^{\min(J_n)}(y_a) = \pi(p)^\frown y$$

for some $y \in \ell^2$. To show that $B_{w_b}^{\min(J_n)}(y_a) \in U_{q,\varepsilon}$, it now suffices to show that $\|y\|_2 < \varepsilon$, since $q \prec B_{w_b}^{\min(J_n)}(y_a)$ by the choice of n. Indeed, observe that, again by the triangle inequality,

$$\begin{aligned}
\|y\|_2 &\leq 2^{\min(J_n)} \cdot \sum_{r>n} \|y_a \restriction J_r\|_2 \\
&\leq 2^{\min(J_n)} \cdot \sum_{r>n} 2^{-|I_r|} \cdot \|\pi(p_r)\|_2 \\
&\leq 2^{\min(J_n)} \cdot \sum_{r>n} 2^{-r-1} \cdot \rho_r \cdot 2^{-\max(J_{r-1})} \cdot 2^{-|I_{r-1}|} \\
&\leq 2^{\max(J_n)} \cdot \sum_{r>n} 2^{-r-1} \cdot \rho_n \cdot 2^{-\max(J_n)} \\
&\leq \varepsilon \cdot \sum_{r>n} 2^{-r-1} \\
&< \varepsilon
\end{aligned}$$

since $\rho_n \leq \varepsilon = \varepsilon_n$. This complete the proof of the claim and proves Lemma 4.1. $\square$

Proof of Theorem 1.5. The key to this proof is an application of Lemma 4.1 with the countable set A taken to be $\omega^{<\omega}$. With this in mind, let

$$\mathsf{Wf} = \{T \subseteq \omega^{<\omega} : T \text{ is a well-founded subtree}\}$$

and

$$\mathsf{C} = \{p \subseteq \omega^{<\omega} : p \text{ is a maximal } \prec\text{-chain}\}.$$

In other words, C may be identified with the set of infinite branches through $\omega^{<\omega}$. The set Wf properly co-analytic while C is G_δ. Let $W = \{w_p : p \in \mathsf{C}\}$ and notice that W is also G_δ since $p \mapsto w_p$ is a homeomorphism by Lemma 4.1. By Proposition 1.3,

$$\mathsf{HC}^*(W) = \bigcap_{w \in W} \mathsf{HC}(w)$$

is co-analytic since W is Borel. To see that it is not analytic, recall that any subtree $T \subseteq \omega^{<\omega}$ is well-founded iff T has no infinite branches. In turn, this is equivalent to

$$\begin{aligned}
(\forall p \in \mathsf{C})(p \nsubseteq T) &\iff (\forall p \in \mathsf{C})(y_T \in \mathsf{HC}(w_p)) && \text{(by Lemma 4.1)} \\
&\iff (y_T \in \mathsf{HC}^*(W)).
\end{aligned}$$

It follows that Wf is a continuous preimage of $\mathsf{HC}^*(W)$ under the map $T \mapsto y_T$. In turn, this implies that $\mathsf{HC}^*(W)$ cannot be analytic. $\square$

As was mentioned above, Lemma 4.1 may be used to give another proof of Theorem 1.4.

Alternative proof of Theorem 1.4. Let $P \subseteq 2^\omega$ be a perfect set such that $b \nsubseteq a$ for any two distinct $a, b \in P$. The construction of such a set is a standard inductive argument (similar to the construction of a perfect independent set). Let y_a and w_a be as in Lemma 4.1 for $a \subseteq \omega$. It follows that $y_a \in \mathsf{HC}(w_b)$ iff $a \neq b$ for all $a, b \in P$.

Given a non-self-dual pointclass Γ which contains both the open and closed sets, fix $Y \subseteq P$ with $Y \in \Gamma \setminus \bar{\Gamma}$. Since P is closed, it follows that $P \setminus Y \in \bar{\Gamma} \setminus \Gamma$. Let

$$W = \{w_a : a \in Y\}.$$

Now consider the set

$$\mathsf{HC}^*(W) = \bigcap_{w \in W} \mathsf{HC}(w).$$

For $a \in P$, notice that $y_a \in \mathsf{HC}^*(W)$ iff $a \notin Y$. Hence,

$$\mathsf{HC}^*(W) \cap \{y_a : a \in P\} = \{y_a : a \in P \text{ and } a \notin Y\} = \{y_a : a \in P \setminus Y\}$$

It follows that $\mathsf{HC}^*(W) \notin \Gamma$ since $\{y_a : a \in P\}$ is closed and $\{y_a : a \in P \setminus Y\} \in \bar{\Gamma} \setminus \Gamma$ (because $a \mapsto y_a$ is a homeomorphism). This completes the proof of the theorem. □

5 Proof of Theorem 1.6

The goal of this section is to show that, assuming Martin's Axiom (MA), there is a set $W \subseteq \{1, 2\}$ whose set of common hypercyclic vectors does not have the property of Baire (Theorem 1.6). We expect that the assumption of additional axioms is unnecessary.

5.1 Nicely hypercyclic vectors

Definition 5.1. Given $n, k \in \omega$, we say that function w with domain ω is *n-nice* at k if $w(i) = w(k + i)$ for all $i < n$.

Recall that for $q \in \mathbb{Q}^{<\omega}$ and $\epsilon \in \mathbb{Q}^+$ we have defined the set

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|(x \restriction |q|) - q\|_\infty < \varepsilon |q|^{-1/2} \text{ and } \|x \restriction [|q|, \infty)\|_2 < \varepsilon\}$$

and that the collection of sets of the form $U_{q,\epsilon}$ forms a basis for ℓ^2.

Definition 5.2. Given $k \in \omega$, $w \in \mathbb{R}^\omega$, $y \in \ell^2$, $q \in \mathbb{Q}^{<\omega}$ and $\epsilon \in \mathbb{Q}^+$, we say that B_w^k *maps y nicely* into $U_{q,\epsilon}$ if

1. $B_w^k(y) \in U_{q,\epsilon}$;
2. w is $|q|$-nice at k;
3. $\|y \restriction [k + |q|, \infty)\|_2 < \epsilon 2^{-k}$.

We say that y is *nicely hypercyclic* for w ($y \in \mathsf{NHC}(W)$) if for each $U_{q,\epsilon}$ there is a k such that B_w^k maps y nicely into $U_{q,\epsilon}$, and *nicely hypercyclic* if there is such a w. We write NH for the set of nicely hypercyclic elements of ℓ^2.

Remark 5.3. Condition (3) of Definition 5.2 implies that $\left\|B_v^k(y) \upharpoonright [|q|, \infty)\right\|_2 < \epsilon$ for any $v \in \{1,2\}^\omega$, which is the second condition in the statement $B_v^k(y) \in U_{q,\epsilon}$. Given that y satisfies Condition (3), then, satisfaction of the other two conditions depends only finite initial segments of w and y. In particular, if $w \in \{1,2\}^\omega$, B_w^k maps y nicely into $U_{q,\epsilon}$ and $v \in \{1,2\}^\omega$ is such that $|w^{-1}[\{2\}] \cap k| = |v^{-1}[\{2\}] \cap k|$ and v is $|q|$-nice at k, then B_v^k maps y nicely into $U_{q,\epsilon}$.

We present a convenient sufficient condition for being nicely hypercyclic and use it to show that NH is comeager in ℓ^2.

Lemma 5.4. *An element y of ℓ^2 is nicely hypercyclic if for each $U_{q,\epsilon}$ and each $m \in \omega$ there exist $n \leq k$ in ω with $n \geq m$ and $k - n \geq m$ such that*

$$\|(2^n y \upharpoonright [k, k+|q|)) - q\|_\infty < \epsilon |q|^{-1/2}$$

and

$$\|y \upharpoonright [k+|q|, \infty)\|_2 < \epsilon 2^{-k}.$$

Proof. Assuming that y satisfies the given condition, recursively build a $w \in \{1,2\}^\omega$ for which y is nicely hypercyclic. Given a finite initial segment $w \upharpoonright m$ of the desired w, and a basic open set $U_{q,\epsilon}$, let $n \leq k$ be as given by the hypothesis. Extend $w \upharpoonright m$ to $w \upharpoonright (k+|q|)$ so that $|\{i < k : w(i) = 2\}| = n$ and $w(i) = w(k+i)$ for all $i < |q|$. These choices suffice to meet the challenge given by $U_{q,\epsilon}$. □

Lemma 5.5. *The set of nicely hypercyclic vectors is comeager in ℓ^2.*

Proof. By the previous lemma, it suffices to show that for each $U_{q,\epsilon}$ and $m \in \omega$, for comeagerly many y there exist $n \leq k$ with $n \geq m$ and $k - n \geq m$ such that

$$\|2^n y \upharpoonright [k, k+|q|) - q\|_\infty < \epsilon |q|^{-1/2}$$

and

$$\|y \upharpoonright [k+|q|, \infty)\|_2 < \epsilon 2^{-k}.$$

In fact we will show that the set of such y is dense open. Fix q, ϵ and m, and a basic open set $U_{r,\delta}$.

Let

- n and k be such that $n \geq m$, $2^n > 3\|q\|_2/\delta$, $k \geq n+m$ and $k > |r|$;
- s be $r^\frown t^\frown(2^{-n}q)$, where t is the all-0 sequence of length $k - |r|$;
- $\rho > 0$ be less than $\epsilon 2^{-k}$ and $\delta/3$.

Now let z be an element of $U_{s,\rho}$. Then

$$\|(z \upharpoonright |r|) - r\|_\infty \leq \|(z \upharpoonright |s|) - s\|_\infty < \rho|s|^{-1/2} < \delta|r|^{-1/2}$$

and

$$\begin{aligned}\|z \upharpoonright [|r|, \infty)\|_2 &\leq \|z \upharpoonright [|r|, |s|)\|_2 + \|z \upharpoonright [|s|, \infty)\|_2 \\ &< \|s \upharpoonright [|r|, |s|)\|_2 + \|(z \upharpoonright [|r|, |s|)) - (s \upharpoonright [|r|, |s|))\|_2 + \rho \\ &\leq 2^{-n} \|q\|_2 + (|s| - |r|)^{1/2} \|(z \upharpoonright [|r|, |s|)) - (s \upharpoonright [|r|, |s|))\|_\infty + \rho \\ &< 2^{-n} \|q\|_2 + (|s| - |r|)^{1/2}\rho|s|^{-1/2} + \rho \\ &< \delta/3 + 2\rho < \delta,\end{aligned}$$

so $z \in U_{r,\delta}$. Furthermore,

$$\begin{aligned}\left\|z \upharpoonright [k, k + |q|) - 2^{-n}q\right\|_\infty &= \|z \upharpoonright [k, k + |q|) - s \upharpoonright [k, k + |q|)\|_\infty \\ &\leq \|(z \upharpoonright (k + |q|) - s\|_\infty \\ &< \rho(k + |q|)^{-1/2},\end{aligned}$$

so

$$\|2^n z \upharpoonright [k, k + |q|) - q\|_\infty < 2^n\rho(k + |q|)^{-1/2} < \epsilon|q|^{-1/2}.$$

Since

$$\|z \upharpoonright [k + |q|, \infty)\|_2 < \rho < \epsilon 2^{-k},$$

z, and thus every member of $U_{s,\rho}$, satisfies the given conditions on q, ϵ and m.

□

We define the set of nice weight sequences by adding a nontriviality condition.

Definition 5.6. We say that a $w \in \{1, 2\}^\omega$ is *nice* if $w^{-1}[\{1\}]$ and $w^{-1}[\{2\}]$ are both infinite and, for each $n \in \omega$ there are infinitely many $k \in \omega$ such that w is n-nice at k.

The following remark is not used in the rest of the section, and is stated for its independent interest.

Remark 5.7. If W is a countable set of nice weight sequences then $\mathsf{NHC}^*(W)$ is nonempty. To see this, recursively build a $y \in \mathsf{NHC}^*(W)$ by choosing initial segments $y \upharpoonright m$ while making promises to keep $\|y \upharpoonright [m, \infty)\|_2$ smaller than some sufficiently small ρ_m, letting $\rho_0 = 1$. Given $y \upharpoonright m$, ρ_m, $w \in W$ and a basic open set $U_{q,\epsilon}$, let n and k be such that

- $2^{-n} \|q\|_2 < \rho_m$,
- $k \geq m$,
- $|w^{-1}[\{2\}]| = n$ and
- w is $|q|$-nice at k.

Then we can extend y with all zeros until position k, and let $y \upharpoonright [k, k + |q|) = 2^{-n}q$. We then set $\rho_{k+|q|}$ to be smaller than both $\epsilon 2^{-k}$ and $\rho_m - 2^{-n} \|q\|_2$ and continue the construction.

5.2 The poset of partial witnesses

As a notational convenience, for each finite sequence α in $\{1,2\}^{<\omega}$, let α^+ be the infinite extension of α with all 2's.

Consider the collection of all triples $(\alpha, r, \delta) \in \{1,2\}^{<\omega} \times \mathbb{Q}^{<\omega} \times (\mathbb{Q} \cap (0,1))$ (henceforth referred to as *conditions*) satisfying the following:

- $|\alpha| \geq |r|$ and
- $\delta < 2^{-|\alpha|}$.

Note that these two properties together imply that

$$B^{|\alpha|}_{\alpha^+}[U_{r,\delta}] \subseteq \{y \in \ell^2 : \|y\|_2 < 1\}.$$

Given conditions $p_1 = (\alpha_1, r_1, \delta_1)$ and $p_2 = (\alpha_2, r_2, \delta_2)$, say that p_2 *extends* p_1 (written $p_2 < p_1$) iff

- α_1 is an initial segment of α_2,
- $\overline{U}_{r_2,\delta_2} \subseteq U_{r_1,\delta_1}$ and,
- for all $k \in [|\alpha_1|, |\alpha_2|)$,

$$B^k_{\alpha_2^+}[U_{r_2,\delta_2}] \subseteq \{y \in \ell^2 : \|y\|_2 < 1\}.$$

The next two lemmas are the key to using either CH or Martin's Axiom to construct a set $W \subseteq \{1,2\}^\omega$ of weight sequences such that $\mathsf{HC}^*(W)$ does not have the property of Baire.

Lemma 5.8. *Given a condition $p_1 = (\alpha, r, \delta)$, a dense open set $D \subseteq \ell^2$ and a natural number N, there exists a condition $p_2 = (\beta, s, \eta)$ such that*

1. $p_2 < p_1$
2. $U_{s,\eta} \subseteq D$
3. $|\beta| > N$

Proof. Fix $p_1 = (\alpha, r, \delta)$, a dense open set $D \subseteq \ell^2$ and $N \in \omega$. If necessary, extend p_1 by shrinking the neighborhood $U_{r,\delta}$ and assume that $\delta < 2^{-|\alpha|-1}$.

Since D is dense, choose a basic open set $U_{s,\eta}$ with

$$\overline{U}_{s,\eta} \subseteq U_{r,\delta} \cap D.$$

Note that

$$B^{|\alpha|}_{\alpha^+}[U_{r,\delta}] \subseteq \{z \in \ell^2 : \|z\|_2 < 1\}$$

(by the definition of "condition") and hence

$$B^{|\alpha|}_{\alpha^+}[U_{s,\eta}] \subseteq \{z \in \ell^2 : \|z\|_2 < 1\}$$

as well. Without loss of generality, assume that $|s| \geq |\alpha|$ and $\eta < 2^{-|\beta|-1}$. Let $\beta \in \{1,2\}^{<\omega}$ have length greater than $\max\{N, |s|\}$ and be of the form

$$\alpha^\frown 1\,1\ldots 1.$$

Now set $p_2 = (\beta, s, \eta)$. To check that $p_2 < p_1$ it remains only to show that

$$B^k_{\beta+}\left[U_{s,\eta}\right] \subseteq \{y \in \ell^2 : \|y\|_2 < 1\}$$

for all $k \in [|\alpha|, |\beta|)$. Towards this end, fix $x \in U_{s,\eta}$. First note that

$$\|x \restriction [|\alpha|, \infty)\|_2 < \delta < 2^{-|\alpha|-1}$$

since $x \in U_{s,\eta} \subseteq U_{r,\delta}$. Second, observe that

$$\|x \restriction [|\beta|, \infty)\|_2 < \eta < 2^{-|\beta|-1}$$

by the definition of $U_{s,\eta}$, the choice of η and the fact that $|\beta| > |s|$. By the triangle inequality and the fact that $\beta^{-1}[\{2\}] = \alpha^{-1}[\{2\}]$, it now follows that

$$\begin{aligned}\|B^k_{\beta+}(x)\|_2 &\leq 2^{|\alpha|} \cdot \|x \restriction [|\alpha|, |\beta|)\|_2 \;+\; 2^{|\beta|} \cdot \|x \restriction [|\beta|, \infty)\|_2 \\ &< 2^{|\alpha|} \cdot 2^{-|\alpha|-1} + 2^{|\beta|} \cdot 2^{-|\beta|-1} \\ &= 1.\end{aligned}$$

This shows that $B^k_{\beta+}\left[U_{s,\eta}\right] \subseteq \{y \in \ell^2 : \|y\|_2 < 1\}$ and completes the proof of the lemma. $\square$

Lemma 5.9. *Given a condition p_1, a nicely hypercyclic $y \in \ell^2$ and a basic neighborhood $U_{q,\varepsilon}$, there is a condition $p_2 = (\beta, s, \eta)$ such that $p_2 < p_1$ and $B^k_{\beta+}$ maps y nicely into $U_{q,\varepsilon}$ for some $k \leq |\alpha|$.*

Proof. Write $p_1 = (\alpha, r, \delta)$ and fix $v \in \{1,2\}^\omega$ such that $y \in \mathsf{NHC}(v)$. Using part 3 of Lemma 5.8, we may extend α if necessary and assume that $|\alpha| \geq |q|$. Now let $k > |\alpha|$ be such that B^k_v maps y nicely into $U_{q,\varepsilon}$, i.e.,

- $B^k_v(y) \in U_{q,\varepsilon}$,
- $v \restriction |q|$ is an initial segment of $v \restriction [k, \infty)$, and
- $\|y \restriction [k + |q|, \infty)\|_2 < \varepsilon 2^{-k}$.

Since there are arbitrarily large such k, it is safe to assume that k is large enough to guarantee

$$\left|v^{-1}[\{1\}] \cap [|\alpha|, k)\right| \geq \left|v^{-1}[\{2\}] \cap |\alpha|\right|.$$

In turn, this means that α can be extended to α_1 with the property that

$$\left|\alpha_1^{-1}[\{2\}] \cap k\right| = \left|v^{-1}[\{2\}]\right|.$$

Let $\beta = \alpha_1{}^\frown (\alpha \restriction |q|)$. Since B^k_v maps y nicely into $U_{q,\varepsilon}$, it follows that $B^k_{\beta+}$ does as well.

Now let $\eta = \min\left\{\frac{\delta}{2}, 2^{-|\beta|-1}\right\}$ and take

$$p_2 = (\beta, r, \eta).$$

It follows that p_2 is a condition and $p_2 < p_1$. $\square$

Equipped with the lemmas above, the next step is to complete the proof of Theorem 1.6.

Theorem 1.6. Assume that Martin's Axiom holds. Let $\langle C_a : a < \mathfrak{c}\rangle$ enumerate all dense G_δ subsets of ℓ^2. The goal is to choose $y_a \in \ell^2$ and $w_a \in \{1,2\}^\omega$ (for $a < \mathfrak{c}$) such that

- $\mathsf{HC}(w_a) \supseteq \{y_b : b < a\}$
- $y_a \in \left(\bigcap_{b\leq a} \mathsf{HC}(w_b)\right) \cap C_a \cap \mathsf{NH}$
- $C_a \setminus \mathsf{HC}(w_a) \neq \emptyset$

Before showing how to find such w_a and y_a, the first step is to verify that the conditions above are sufficient to guarantee the existence of a set of common hypercyclic vectors without the property of Baire. Indeed, let

$$W = \{w_a : a < \mathfrak{c}\}.$$

Notice that the first and second conditions above show that $y_a \in \mathsf{HC}(w_b)$ for all $a, b < \mathfrak{c}$ and so

$$\mathsf{HC}^*(W) \supseteq \{y_a : a < \mathfrak{c}\}.$$

Since $\mathsf{HC}^*(W)$ is closed under changes in finitely many coordinates, if $\mathsf{HC}^*(W)$ has the Baire property then it is either meager or comeager. Therefore, it is sufficient to show that

- $\mathsf{HC}^*(W)$ intersects every dense G_δ subset of ℓ^2 (i.e., it is non-meager) and
- $\mathsf{HC}^*(W)$ contains no dense G_δ subset of ℓ^2 (i.e., it is not comeager).

The first of these statements is witnessed by the fact that $y_a \in C_a \cap \mathsf{HC}^*(W)$ for every $a < \mathfrak{c}$ (see the second condition above). The second is a consequence of the third condition above, i.e., $C_a \setminus \mathsf{HC}(w_a) \neq \emptyset$ for all $a < \mathfrak{c}$.

The construction of the y_a and w_a proceeds in stages $a < \mathfrak{c}$. Suppose that $y_b \in \ell^2$ and $w_b \in \{1,2\}^\omega$ are given for all $b < a$ and satisfy the properties above. Let $D_0 \supseteq D_1 \supseteq \dots$ be dense open sets such that $C_a = \bigcap_n D_n$.

Let $(\mathbb{P}, <)$ be the set of conditions (α, r, δ) ordered by the extension relation $<$. First of all, $\mathbb{P}$ is ccc since it is countable, so Martin's Axiom applies.

For each $b < a$ and basic neighborhood $U_{q,\varepsilon}$, consider the set of conditions

$$E_{b,q,\varepsilon} = \left\{(\alpha, r, \delta) \in \mathbb{P} : (\exists k \leq |\alpha|)\left(B^k_{\alpha^+} \text{ maps } y_b \text{ nicely into } U_{q,\varepsilon}\right)\right\}.$$

It follows from Lemma 5.9 that each $E_{b,q,\varepsilon}$ is dense. For each $n \in \omega$ and $p \in \mathbb{P}$, define

$$\begin{aligned} F_{n,p} = \big\{(\alpha, r, \delta) \in \mathbb{P} : ((|\alpha| \geq n) \wedge ((\alpha, r, \delta) < p) \wedge U_{r,\delta} \subseteq D_n) \\ \text{or } (\alpha, r, \delta) \text{ and } p \text{ have no common extensions}\big\}. \end{aligned}$$

Lemma 5.8 implies that each $F_{n,p}$ is dense in $\mathbb{P}$.

It is now possible to apply $\mathrm{MA}(|a|)$ to obtain a filter $G \subseteq \mathbb{P}$ such that G has nonempty intersection with all of the dense sets above. Since G is a filter, if $(\alpha, r, \delta), (\beta, s, \eta) \in$

G, then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ as otherwise G contains incompatible elements. It thus follows that the weight sequence

$$w_a = \bigcup \{\alpha : (\exists r, \delta)((\alpha, r, \delta) \in G)\}$$

is well-defined.

Claim. $C_a \setminus \mathsf{HC}(w_a) \neq \emptyset$.

To prove this claim, choose a descending sequence $p_0 > p_1 > \dots$ of conditions in G by induction as follows:

- $p_0 \in G$ is arbitrary
- Given $p_0 > p_1 > \dots > p_n$, choose $p_{n+1} \in G \cap F_{n,p_n}$

Note that p_{n+1} will be an extension of p_n as otherwise p_{n+1} and p_n have no common extensions (by the definition of F_{n,p_n}), but this cannot happen since G is a filter.

For each n, suppose $p_n = (\alpha_n, r_n, \delta_n)$. It follows that there exists $x \in \ell^2$ with

$$x \in \bigcap_n \overline{U}_{r_n,\delta_n} = \bigcup_n U_{r_n,\delta_n} \subseteq \bigcap_n D_n = C_a$$

where the "$\subseteq$" is a consequence of the fact that $p_{n+1} \in F_{n,p_n}$ for each n. On the other hand, if $k \geq |\alpha_0|$ with n such that $k \in [|\alpha_{n-1}, |\alpha_n|)$ (such an n exists since $|\alpha_n| \geq n$). By the choice of p_n, it follows that

$$B^k_{\alpha_n^+}(x) \in \left\{z \in \ell^2 : \|z\|_2 < 1\right\}.$$

and hence

$$B^k_{w_a}(x) \in \left\{z \in \ell^2 : \|z\|_2 < 1\right\}$$

since α_n is an initial segment of w_a. In particular, $x \in C_a \setminus \mathsf{HC}(w_a)$ which proves the claim.

Claim. For each $b < a$, $y_b \in \mathsf{HC}(w_a)$.

To establish this claim, fix a basic open set $U_{q,\varepsilon}$ and let $\alpha \subseteq w_a$ and r, δ be such that

$$(\alpha, r, \delta) \in G \cap E_{b,q,\varepsilon}.$$

Hence, by the definition of $E_{b,q,\varepsilon}$, there exists $k \leq |\alpha|$ such that $B^k_{\alpha+}$ maps y_b nicely into $U_{q,\varepsilon}$. Since α is an initial segment of w_a and $k \leq |\alpha|$, it follows that B^k_w also maps y_b nicely into $U_{q,\varepsilon}$. As q, ε were arbitrary, this proves the claim.

To complete the construction, choose y_a to be any element of

$$\left(\bigcap_{b \leq a} \mathsf{HC}(w_b)\right) \cap C_a \cap \mathsf{NH}.$$

Note that this set is nonempty by Martin's Axiom as it is an $|a|$-size intersection of comeager sets. $\square$

Remark 5.10. Since all relevant partial orders in the proof above were countable, it would have been sufficient to make the weaker assumption that any $< \mathfrak{c}$-size intersection of comeager sets is nonempty. This statement, known as "$\mathsf{non}(\mathcal{M}) < \mathfrak{c}$" is an immediate consequence of Martin's Axiom.

References

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