Coding with canonical functions

Paul B. Larson*
Miami University
Oxford, Ohio USA

Saharon Shelah†
Hebrew University of Jerusalem
Rutgers University

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Abstract

A function f from ω_1 to the ordinals is called a canonical function for an ordinal α if f represents α in any generic ultrapower induced by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. We introduce here a method for coding sets of ordinals using canonical functions from ω_1 to ω_1. Combining this approach with arguments from [3], we show that for each cardinal κ there is a forcing construction preserving cardinalities and cofinalities forcing that every subset of κ is in the inner model $L(\mathcal{P}(\omega_1))$.

1 Introduction

Results in set theory over the last forty years show that the existence of certain large cardinals implies that there there are subsets of ω_1 which are not elements of the inner model $L(\mathcal{P}(\omega))$ (for instance, those coding an ω_1-sequence of distinct subsets of ω; see Chapter 6 of [1]). A natural question, asked of us by several researchers, is whether a similar phenomenon happens at higher cardinals. Here we show that this is not the case. In particular, we show, assuming the Continuum Hypothesis, that for any infinite cardinal κ there is an (ω, ∞)-distributive, $\aleph_2$-c.c. partial order forcing that $\mathcal{P}(\kappa) \subseteq L(\mathcal{P}(\omega_1))$. We employ a coding mechanism using canonical functions (from ω_1 to ω_1) to code subsets of the given cardinal κ. We introduce this mechanism in Section 2 and give the proof in the case $\kappa = \omega_2$ (which is simpler) at the end of the section. The result for arbitrarity κ is proved in Section 3. This proof involves combining the coding technique introduced here with previous work of the second author from [3], although in our context the argument is considerably simpler.

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2 Coding sets of ordinals by subsets of ω_1

In this section we introduce the forcing construction which is used in this paper to code sets of ordinals. Given functions f and g on ω_1, we let $\mathrm{eq}(f, g)$ denote the set of $\alpha < \omega_1$ for which $f(\alpha) = g(\alpha)$. Suppose that η is an ordinal and that

$$\bar{f} = \langle f_\alpha : \alpha < \eta \rangle$$

and

$$\bar{C} = \langle C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [\eta]^2 \rangle$$

are such that

- each f_α is a function from ω_1 to ω_1;
- $\bar{C}$ *witnesses that* $\bar{f}$ is a *mod*-NS_{ω_1}-*distinct sequence*, that is, each $C_{\{\alpha,\beta\}}$ is a club subset of ω_1 disjoint from $\mathrm{eq}(f_\alpha, f_\beta)$;

Let Y be a subset of η. We define a partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$ which adds a function from $\omega_1 \times \omega_1 \to 2$ coding Y via $\langle [f_\alpha]_{\mathrm{NS}_{\omega_1}} : \alpha < \omega_1 \rangle$, where $[f]_{\mathrm{NS}_{\omega_1}}$ (for f a function from ω_1 to ω_1) is the set of functions g from ω_1 to ω_1 for which $\mathrm{eq}(f, g)$ contains a club.

2.1 Definition. A condition in $\mathbb{P}_{\bar{f},\bar{C},Y}$ is a tuple $p = \langle u_p, i_p, h_p, \bar{E}_p \rangle$ such that

- $u_p \in [\eta]^{\aleph_0}$;
- $i_p \in \bigcap \{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u_p]^2\}$;
- letting
$$j_p = \sup\{f_\alpha(\xi) + 1 : (\alpha, \xi) \in u_p \times i_p\},$$
h_p is a function from $i_p \times j_p$ to 2;
- $\bar{E}_p$ is a sequence
$$\langle E_{p,\beta} : \beta \in u_p \rangle$$
such that each $E_{p,\beta}$ is a closed subset of i_p.

Given $p, q \in \mathbb{P}_{\bar{f},\bar{C},Y}$, $p \leq q$ (p is stronger than q) if

- $u_q \subseteq u_p$;
- $i_p \geq i_q$;
- for all $\beta \in u_q$, $E_{p,\beta} \cap i_q = E_{q,\beta}$;
- $h_q \subseteq h_p$;
- for all $\alpha \in u_q$ and all $\xi \in E_{p,\alpha} \setminus i_q$,
$$h_p(\xi, f_\alpha(\xi)) = 1 \Leftrightarrow \alpha \in Y.$$

Lemmas 2.2, 2.3 and 2.4 show that various sets are dense in $\mathbb{P}_{\bar{f},\bar{C},Y}$.

Lemma 2.2. *For each $\gamma \in \eta$, the set of $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ with $\gamma \in u_p$ is dense in $\mathbb{P}_{\bar{f},\bar{C},Y}$.*

Proof. Fix a condition $\langle u, i, h, \bar{E}\rangle$ in $\mathbb{P}_{\bar{f},\bar{C},Y}$. If $\gamma \in u$ we are done, so suppose otherwise. Let

- $u' = u \cup \{\gamma\}$;
- $i' = \min \bigcap\{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u']^2\} \setminus i$;
- $\bar{F} = \langle F_\beta : \beta \in u'\rangle$ be such that $F_\beta = E_\beta$ for all $\beta \in u$, and $F_\gamma = \emptyset$;
- h' be any function from $i' \times \sup\{f_\alpha(\xi) + 1 : (\alpha, \xi) \in u' \times i'\}$ to 2 extending h.

Then $\langle u', i', h', \bar{F}\rangle$ is a condition in $\mathbb{P}_{\bar{f},\bar{C},Y}$ below $\langle u, i, h, \bar{E}\rangle$. □

Lemma 2.3. *For every $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ and every $\xi < \omega_1$, there exists a $q \leq p$ in $\mathbb{P}_{\bar{f},\bar{C},Y}$ with $i_q \geq \xi$ and*

$$i_p \in \bigcap\{E_{q,\beta} : \beta \in u_q\}.$$

Proof. Given p, define q by setting $u_q = u_p$,

$$i_q = \min(\bigcap\{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u_p]^2\}) \setminus (\xi \cup (i_p + 1)),$$

and $E_{q,\beta} = E_{p,\beta} \cup \{i_p\}$ for all $\beta \in u_p$.

It remains to extend h_p (whose domain is $i_p \times j_p$) to $h_q \colon i_q \times j_q \to 2$ such that, for all $\alpha \in u_p$, $h_q(i_p, f_\alpha(i_p)) = 1$ if and only if $\alpha \in Y$. Since $i_p \in C_{\{\alpha,\beta\}}$ for all $\{\alpha, \beta\} \in [u_p]^2$, there cannot be distinct $\alpha, \beta \in u_p$ such that $f_\alpha(i_p) = f_\beta(i_p)$. It follows that such an h_q exists. □

Lemmas 2.2 and 2.3 give the following.

Lemma 2.4. *The following sets are dense in $\mathbb{P}_{\bar{f},\bar{C},Y}$.*

1. *For each $\xi < \omega_1$, the set of $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ such that $i_p \geq \xi$.*
2. *For each $\xi < \omega_1$ and each $\alpha < \eta$, the set of $p \in \mathbb{P}_{\bar{f},\bar{C},Y}$ such that $\alpha \in u_p$ and $\sup(E_{p,\alpha}) \geq \xi$.*

Given a V-generic filter $G \subseteq \mathbb{P}_{\bar{f},\bar{C},Y}$, for each $\beta \in \eta$, we let

$$E_{G,\beta} = \bigcup\{E_{p,\beta} : p \in G, \beta \in u_p\}.$$

Lemma 2.4 shows that each $E_{G,\beta}$ ($\beta \in \eta$) is cofinal subsets of ω_1^V (which is $\omega_1^{V[G]}$, by Remark 2.5 below), and that

$$\mathrm{dom}(h_G) = \omega_1^V \times \sup\{f_\alpha(\xi) + 1 : \alpha < \eta, \xi < \omega_1\}$$

(which will be $\omega_1^V \times \omega_1^V$ in our applications). Each set $E_{G,\beta}$ is also closed. By Remark 2.5 and Lemma 2.11 below, forcing with $\mathbb{P}_{\bar{f},\bar{C},Y}$ preserves all cardinalities and cofinalities.

2.5 Remark. The partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$ is σ-closed; moreover, every descending ω-sequence of conditions in $\mathbb{P}_{\bar{f},\bar{C},Y}$ has a greatest lower bound. To see this, suppose that $\langle p_n : n \in \omega\rangle$ is a sequence of conditions in $\mathbb{P}_{\bar{f},\bar{C},Y}$ such that $p_n \geq p_{n+1}$ for all $n \in \omega$. Let

- $u_q = \bigcup_{n\in\omega} u_{p_n}$;
- $i_q = \sup_{n\in\omega} i_{p_n}$;
- $E_{q,\beta} = \bigcup\{E_{p_n,\beta} : n \in \omega, \beta \in u_{p_n}\}$, for each $\beta \in u_q$;
- $h_q = \bigcup_{n\in\omega} h_{p_n}$.

Then q is the greatest lower bound for $\{p_n : n \in \omega\}$ in $\mathbb{P}_{\bar{f},\bar{C},Y}$.

It is a standard fact, and easy to see, that a countable support iteration of σ-closed partial orders is σ-closed. We formulate an abstract consequence of the preceding remark.

2.6 Remark. Suppose that P is a partial order and that $\underset{\sim}{Q}$ is a P-name for a partial order on a subset of the ground model. We say that $\underset{\sim}{Q}$ has *generic lower bounds* if whenever

- θ is a cardinal greater than $2^{|P*\underset{\sim}{Q}|}$;
- X is a countable elementary submodel of $H(\theta)$ with $P * \underset{\sim}{Q} \in X$;
- $G \subseteq (P * \underset{\sim}{Q}) \cap X$ is X-generic for $P * \underset{\sim}{Q}$;
- $p \in P$ is a lower bound for the restriction of G to P,

there exists an x such that $(p, \check{x})$ is a lower bound for G. The construction of the condition q in Remark 2.5 shows that whenever $\underset{\sim}{Q}$ is a P-name for a partial order of the form $\mathbb{P}_{\bar{f},\bar{C},Y}$, $\underset{\sim}{Q}$ has generic lower bounds.

Given a condition p in a forcing iteration, we let supp(p) denote the support of p (we also use sup(u) to mean the supremum of a set of ordinals u).

2.7 Remark. We say that a condition p in an forcing iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ is *fully realized* if, for each $\alpha \in \text{supp}(p)$, $p(\alpha)$ is $\check{x}$ (relative to P_α), for some set x. If $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ is a countable support iteration such that each $\underset{\sim}{Q}_\alpha$ has generic lower bounds, then (letting P_θ be the direct limit of this iteration) whenever

- χ is a cardinal greater than $2^{|P_\theta|}$;
- X is a countable elementary submodel of $H(\chi)$ with $P_\theta \in X$;
- $G \subseteq P_\theta \cap X$ is X-generic for $P * \underset{\sim}{Q}$;

there exists fully realized q which is a lower bound for G (this is easily proved by induction on the elements of $X \cap \theta$).

2.8 Remark. If $G \subseteq \mathbb{P}_{\bar{f},\bar{C},Y}$ is a V-generic filter, then, in $V[G]$, Y is the set of $\alpha < \eta$ such that $h_G(\xi, f_\alpha(\xi)) = 1$ for club many $\xi < \omega_1$. This implies that Y is in any inner model of ZF containing h_g which is correct about NS_{ω_1} and contains a sequence $\langle F_\alpha : \alpha < \eta\rangle$ such that each F_α is nonempty subset of the corresponding $[f_\alpha]_{\mathrm{NS}_{\omega_1}}$ (for instance, $L(\mathcal{P}(\omega_1))$, if $\eta \leq \omega_2$, and each f_α is a canonical function for α (see Theorem 2.13 and the paragraph before it)). Note that $\bar{f}$ and $\bar{C}$ do not need to be elements of the inner model.

2.9 Remark. In the definition of the partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$, the functions f_α are required only to differ pairwise on clubs, not necessarily to dominate one another. We use canonical functions (from ω_1 to ω_1) only because their NS_{ω_1}-classes are definable in $L(\mathcal{P}(\omega_1))$.

We turn now to showing that a countable support iteration of partial orders of the form $\mathbb{P}_{\bar{f},\bar{C},Y}$ is $\aleph_2$-c.c.. We start by noting a sufficient condition for compatibility.

2.10 Remark. Let p and q be conditions in $\mathbb{P}_{\bar{f},\bar{C},Y}$ such that $h_p = h_q$ and $E_{p,\beta} = E_{q,\beta}$ for all $\beta \in u_p \cap u_q$. Then $i_p = i_q$ and $j_p = j_q$. Let

- u' be $u_p \cup u_q$;
- i' be $\min(\bigcap\{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [u']^2\} \setminus i_p$;
- j' be $\sup\{f_\alpha(\xi) + 1 : (\alpha, \xi) \in u' \times i'\}$;
- h' be any function from $i' \times j'$ to 2 such that $h_p \subseteq h'$.

Then the condition $\langle u', i', h', \bar{E}_p \cup \bar{E}_q\rangle$ is a lower bound for $\{p, q\}$.

We say that partial order $\mathbb{P}$ has satisfies the *regressive* $\aleph_2$*-chain condition (c.c.) on cofinality* $\aleph_1$ if for any sequence

$$\langle p_\alpha : \alpha < \omega_2\rangle$$

of conditions from $\mathbb{P}$, there exist a club $D \subseteq \omega_2$ and a regressive function r on the members of D of cofinality $\aleph_1$ such that for all $\gamma, \eta \in D$, if $r(\gamma) = r(\eta)$ then p_γ and p_η are compatible (this is a weakening of condition (c) from the first page of [2]).

Lemma 2.11. *Suppose that the Continuum Hypothesis holds. Let θ be an ordinal, and let $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ be a countable support forcing iteration such that each $\underset{\sim}{Q}_\alpha$ is a P_α-name for a partial order of the form $\mathbb{P}_{\bar{f},\bar{C},Y}$, where, in the P_α-extension,*

- *$\bar{f}$ is an η-sequence $\langle f_\alpha : \alpha < \eta\rangle$, for some ordinal η, of functions from ω_1 to ω_1;*

- *$\bar{C}$ witnesses that $\bar{f}$ is a mod-NS_{ω_1}-distinct sequence;*
- *Y is a subset of η.*

Let P_θ be the countable support limit of this iteration. Then P_θ satisfies the regressive $\aleph_2$-c.c. on cofinality $\aleph_1$.

Proof. Fix a sequence $\langle p_\alpha : \alpha < \omega_2\rangle$ consisting of distinct conditions in P_θ. It suffices to consider the case where each p_α is fully realized (so for each $\alpha < \omega_2$ and $\gamma < \theta$, we can let $u_{p_\alpha(\gamma)}$, etc., refer to the parts of $p_\alpha(\gamma)$). Let U be the union of all sets of the form $u_{p_\alpha(\gamma)}$ for $\alpha < \omega_2$ and $\gamma < \theta$. Let $\langle \xi_\delta : \delta < \zeta\rangle$ enumerate U, for some $\zeta \leq \omega_2$. Let κ be a regular cardinal with P_θ and $\langle p_\alpha : \alpha < \omega_2\rangle$ in $H(\kappa)$. Let D be a club subset of ω_2 such that for each $\gamma \in D$ of cofinality $\aleph_1$ there is an $X \prec H(\theta)$ of cardinality $\aleph_1$, closed under ω-sequences, with $\langle p_\alpha : \alpha < \omega_2\rangle \in X$ and $X \cap \omega_2 = \gamma$. For each $\gamma \in D$, then, there exists an $\alpha < \gamma$ such that, for all $\rho \in \mathrm{supp}(p_\gamma) \cap \gamma$,

- $\rho \in \mathrm{supp}(p_\alpha)$;
- $h_{p_\alpha(\rho)} = h_{p_\gamma(\rho)}$;
- $u_{p_\gamma(\rho)} \cap \{\xi_\delta : \delta < \min\{\gamma, \zeta\}\} \subseteq u_{p_\alpha(\rho)}$;
- $\bar{E}_{p_\alpha(\rho),\beta} = \bar{E}_{p_\gamma(\rho),\beta}$ for all $\beta \in u_{p_\gamma(\rho)} \cap \{\xi_\delta : \delta < \min\{\gamma, \zeta\}\}$.

Let $r(\gamma)$ be any such α. Remark 2.10 shows that this choice of r works. □

2.12 Remark. A generalized version of the partial order $\mathbb{P}_{\bar{f},\bar{C},Y}$ adds a coordinate to each condition containing a subset of i_p, and requires the coding in the last condition on the order to work only for members of this subset. This forcing then adds a stationary subset of ω_1 relative to which the coding works. We have chosen the restricted version so that the coding is absolute to outer models with the same ω_1 (not necessarily preserving stationary subsets of ω_1). The distinction is not important in this section, but it helps in the iterated forcing argument in Section 3.

We give an application of the material in this section, in the case $\eta = \omega_2$. Fix a sequence π_α $(\alpha < \omega_2)$ such that each π_α is a surjection from ω_1 to α, and for each $\alpha < \omega_2$, let $f_\alpha\colon \omega_1 \to \omega_1$ be such that $f_\alpha(\beta)$ is the ordertype of $\pi_\alpha[\beta]$, for each $\beta < \omega_1$ (such a function f_α is called a *canonical function* for α; it is not hard to see that it represents α in all generic ultrapowers formed by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$). Then $\langle [f_\alpha]_{\mathrm{NS}_{\omega_1}} : \alpha < \omega_2\rangle$ is in $L(\mathcal{P}(\omega_1))$. Let $\bar{f} = \langle f_\alpha : \alpha < \omega_2\rangle$ (we do not need $\bar{f}$ to be in $L(\mathcal{P}(\omega_1))$ for Theorem 2.13 below). For each $\{\alpha, \beta\} \in [\omega_2]^2$, let $C_{\{\alpha,\beta\}}$ be a club of countable limit ordinals such that $f_{\min\{\alpha,\beta\}}(\gamma) < f_{\max\{\alpha,\beta\}}(\gamma)$ for all $\gamma \in C_{\{\alpha,\beta\}}$. Fixing a regular $\theta > 2^{\aleph_2}$, there is a countable support forcing iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta\rangle$ such that each $\underset{\sim}{Q}_\alpha$ is a P_α-name for a partial order of the form $\mathbb{P}_{\bar{f},\bar{C},Y_\alpha}$, for $\bar{f}$ and $\bar{C}$ the fixed sets introduced in this paragraph, where by suitable bookkeeping the sets Y_α range through all subsets of ω_2 appearing in the final extension. Putting together the material in this section, then, we get the following.

Theorem 2.13. *If the Continuum Hypothesis holds, then there is a σ-closed, $\aleph_2$-c.c. partial order forcing that $\mathcal{P}(\omega_2) \subseteq L(\mathcal{P}(\omega_1))$.*

3 Coding $\mathcal{P}(\kappa)$ for larger κ

In this section we combine the argument of the previous section with the arguments of Section XVII §4 of [3] to produce a coding of the subsets of cardinals κ larger than ω_2. The object is to obtain the coding in the previous section along with the existence of a canonical function from ω_1 to ω_1 for each $\alpha < \kappa$.

3.1 Remark. The forcing construction from Section XVII §4 of [3] makes the constant function from ω_1 to $\{\omega_1\}$ into a canonical function. Here we need only that there are canonical functions for each $\alpha < \kappa$, so our job is considerably easier.

Given an ordinal γ, we say that *there exist canonical functions for each* $\alpha < \gamma$ if there exists a sequence of functions $f_\alpha \colon \omega_1 \to \omega_1$ ($\alpha < \gamma$) such that

- for all $\alpha < \beta < \gamma$, $\{\delta < \omega_1 : f_\alpha(\delta) < f_\beta(\delta)\}$ contains a club;
- for all $\alpha < \gamma$, all stationary $A \subseteq \omega_1$ and all $g \colon \omega_1 \to \omega_1$, if f_α *dominates* g *on* A (i.e. $g(\delta) < f_\alpha(\delta)$ for all $\delta \in A$), then there exists a $\beta < \alpha$ such that $A \cap \mathrm{eq}(g, f_\alpha)$ is stationary.

Each f_α is then called a canonical function for α.[1] If canonical functions exist for each $\alpha < \gamma$, then the sequence $\langle F_\alpha : \alpha < \gamma \rangle$ is in $L(\mathcal{P}(\omega_1))$, where each F_α is the set of canonical functions for α. The paragraph before Theorem 2.13 shows that canonical functions exist for each $\alpha < \omega_2$.

The rest of this section proves the following theorem.

Theorem 3.2. *Suppose that the Continuum Hypothesis holds, and that κ is an infinite cardinal. There exists an (ω, ∞)-distributive, $\aleph_2$-c.c. countable support iterated forcing construction forcing the following statements.*

- *There exist canonical functions for each $\alpha < \kappa$.*
- $\mathcal{P}(\kappa) \subseteq L(\mathcal{P}(\omega_1))$.

The previous section establishes the theorem in the case that $\kappa \leq \omega_2$, so we assume otherwise here. The second conclusion of the theorem will be obtained by forcing that for each $Y \subseteq \kappa$ there exists an $h \colon \omega_1 \times \omega_1 \to \omega_1$ such that for each $\alpha < \kappa$, $\alpha \in Y$ if and only if $h(\xi, f_\alpha(\xi)) = 1$ for club many $\xi < \omega_1$, for functions f_α ($\alpha < \kappa$) witnessing the first conclusion.

[1] Standard arguments, working by induction on α, show that each f_α represents α in any generic ultrapower formed by forcing with $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. The usual definition of canonical function allows functions which map into the ordinals, as opposed to only the countable ordinals. Since we will not need such functions, we modify the definition for this paper. It would interesting if the current argument could be modified to work with canonical functions in this generalized sense, while having $2^{\aleph_1} < \kappa$.

We consider countable support forcing iterations $\bar{P} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \theta \rangle$, for some ordinal θ, satisfying the following conditions, where for notational convenience we let G_α denote a generic filter for P_α.

1. For $\alpha < \kappa$, $\underset{\sim}{Q}_\alpha$ is a P_α-name for the partial order which adds a function from ω_1 to ω_1, by countable initial segments. We let $\dot{f}_\alpha$ be the $P_{\alpha+1}$-name for this function, and write f_α for $\dot{f}_{\alpha,G_{\alpha+1}}$ when this causes no confusion.
2. For $\gamma \in [\kappa, \kappa \cdot \kappa)$, letting $\alpha, \beta \in \kappa$ be such that $\gamma = \kappa \cdot (1+\alpha) + \beta$, if $\alpha \geq \beta$ then $\underset{\sim}{Q}_\gamma$ is a P_γ-name for the trivial partial order (which we take to be the partial order on $\{\emptyset\}$), otherwise it is a P_γ-name for the partial order which adds by countable initial segments a club set of $\xi \in \omega_1$ for which $f_\alpha(\xi) < f_\beta(\xi)$. Again, we let $\dot{C}_{\{\alpha,\beta\}}$ be the $P_{\gamma+1}$-name for this club, and we write $C_{\{\alpha,\beta\}}$ for $\dot{C}_{\{\alpha,\beta\},G_{\gamma+1}}$ when convenient.
3. For $\alpha \in [\kappa \cdot \kappa, \theta)$ of the form $\gamma + 2n$, for γ a limit ordinal and $n \in \omega$, $\underset{\sim}{Q}_\alpha$ is a P_α-name such that, for some $\delta < \kappa$ and some P_α-names $\dot{g}_\alpha$ and $\dot{A}_\alpha$,
   - $\dot{A}_\alpha$ is a P_α-name for a subset of ω_1;
   - $\dot{g}_\alpha$ is a P_α-name for a function from ω_1 to ω_1 which is dominated by f_δ on $\dot{A}_{\alpha,G_\alpha}$;
   - if, in $V[G_\alpha]$, $\dot{A}_{\alpha,G_\alpha}$ is stationary and there does not exist $\beta < \delta$ such that $\dot{A}_{\alpha,G_\alpha} \cap \mathrm{eq}(f_\beta, \dot{g}_{\alpha,G_\alpha})$ is stationary, then $\underset{\sim}{Q}_{\alpha,G_\alpha}$ is the partial order which adds a club subset of ω_1 disjoint from $\dot{A}_{\alpha,G_\delta}$, by countable initial segments;
   - if, in $V[G_\alpha]$, $\dot{A}_{\alpha,G_\alpha}$ is not stationary or there does exist such a β, then $\underset{\sim}{Q}_{\alpha,G_\alpha}$ is the trivial partial order.
4. For $\alpha \in [\kappa \cdot \kappa, \theta)$ of the form $\gamma + 2n + 1$, for γ a limit ordinal and $n \in \omega$, $\underset{\sim}{Q}_\alpha$ is a P_α-name for the partial order $\mathbb{P}_{\bar{f},\bar{C},\dot{Y}_\alpha}$, where
   - $\bar{f} = \langle f_\alpha : \alpha < \kappa \rangle$;
   - $\bar{C} = \{C_{\{\alpha,\beta\}} : \{\alpha, \beta\} \in [\kappa]^2\}$;
   - $\dot{Y}_\alpha$ is a P_α-name for a subset of κ.

We let P_θ denote the countable support limit of $\bar{P}$, and we let P'_θ denote the set of fully realized conditions in P_θ (see Remark 2.7). We will show by induction on θ that for any iteration satisfying conditions (1)-(4) above, the following properties hold.

(a) P_θ is (ω, ∞)-distributive (that is, forcing with P_θ adds no new ω-sequences of ordinals);

(b) P'_θ is dense in P_θ;

Given that (b) holds, the argument that the corresponding P_θ satisfies the regressive $\aleph_2$-c.c. on cofinality $\aleph_1$ is a slight modification (using the assumption of the Continuum Hypothesis to deal with the stages in cases (1)-(3)) of the proof of Lemma 2.11. Given this, we may fix a cardinal θ such that $\theta^\kappa = \theta$, and an iteration satisfying conditions (1)-(4) along with the following two conditions.

5. For all $\beta \in [\kappa \cdot \kappa, \theta)$ and $\delta < \kappa$, if $\dot{g}$ and $\dot{A}$ are such that $\dot{A}$ is a P_β-name for a subset of ω_1 and $\dot{g}$ is a P_β-name for a function from ω_1 to ω_1 which is dominated by f_δ on $\dot{A}_{\alpha,G_\alpha}$, there are cofinally many $\alpha \in [\beta, \theta)$ (in case (3) above) such that $\dot{g}_\alpha$ and $\dot{A}_\alpha$ are the natural reinterpretations of $\dot{g}$ and $\dot{A}$ as P_α-names.

6. For every $\beta \in [\kappa \cdot \kappa, \theta)$, and every P_β-name $\dot{Y}$ for a subset of κ, there is an $\alpha \in [\beta, \theta)$ (in case (4)) such that $\dot{Y}_\alpha$ is $\dot{Y}$ (again, reinterpreted).

The proof of Theorem 3.2 will then be complete. We briefly review the argument that each f_δ is a canonical function for δ. The first part of the definition is satisfied by the definition of the first $\kappa \cdot \kappa$ stages of the iteration. For the second, suppose that $G \subseteq P_\theta$ is a V-generic filter, and that, in $V[G]$, A is a subset of ω_1 and g is a function from ω_1 to ω_1 dominated by f_δ on A. We may fix a $\gamma \in [\kappa \cdot \kappa, \theta)$ and P_γ-names $\dot{g}$ and $\dot{A}$ such that $\dot{g}_G = g$ and $\dot{A}_G = A$. Then the sets $A \cap \mathrm{eq}(g, f_\beta)$ exist in the P_γ-extension, for each $\beta < \delta$. If $\rho \in [\gamma, \theta)$ is a stage in case (3) with respect to $\dot{g}$ and $\dot{A}$, and either A is nonstationary or one of the sets $A \cap \mathrm{eq}(g, f_\beta)$ is stationary in $V[G_\rho]$, then $\underset{\sim}{Q}_{\rho,G_\rho}$ is the trivial partial order. Otherwise, the partial order $\underset{\sim}{Q}_{\rho,G_\rho}$ destroys the stationarity of A. The cofinality condition in case (5) is then essential, but our argument does not require us to preserve the stationarity of a witness to a challenge given by g and A, since all possible witnesses exist as soon as g and A do, and if they all fail then the stationarity of A is destroyed.

We now return to the proof, by induction on θ, that (a) and (b) hold. They hold trivially in the case $\theta = 0$, and the successor case is similarly straightforward. We now give the proof in the case where θ is a limit ordinal, assuming that (a) and (b) hold for all $\alpha < \theta$.

Fix a regular cardinal χ greater than $2^{|P_\theta|}$. Let p be a condition in P_θ and let τ be a P_θ-name for an ω-sequence of ordinals. Let N be a countable elementary submodel of $H(\chi)$ with p and τ in N. Let G be an N-generic filter for P_θ with $p \in G$. For each $\alpha \in N \cap (\theta + 1)$, let G_α be $\{p \restriction \alpha : p \in G\}$. Let q be the function on θ which takes the value $1_{\underset{\sim}{Q}_\alpha}$ for all $\alpha \in \theta \setminus N$ (which is $\check{\emptyset}$ is all cases except for (4), and $\check{x}$ for $x = \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$ in (4)), and which is defined as follows on $N \cap \theta$.

- For each $\alpha \in N \cap \kappa$, $q(\alpha)$ is $\check{x}$, where x is the function determined by the α-th coordinates of the elements of G, extended by taking the value o.t.$(N \cap \alpha)$ at $N \cap \omega_1$.
- For each $\alpha \in N \cap [\kappa, \kappa \cdot \kappa)$ in the trivial case, $q(\alpha)$ is $\check{\emptyset}$.

- For each $\alpha \in N \cap [\kappa, \kappa \cdot \kappa)$ in the nontrivial case, $q(\alpha)$ is $\check{x}$, where x is the club subset of $N \cap \omega_1$ determined by the α-th coordinates of the elements of G, extended by adding the ordinal $N \cap \omega_1$.
- For each $\alpha \in N$ in case (3) for which $\underset{\sim}{Q}_\alpha$ is forced by some condition in G_α to be trivial, $q(\alpha) = \check{\emptyset}$.
- For each $\alpha \in N$ in case (3) for which $\underset{\sim}{Q}_\alpha$ is forced by some condition in G_α to be nontrivial, $q(\alpha)$ is $\check{x}$, where x is the club subset of $N \cap \omega_1$ determined by the α-th coordinates of the elements of G, extended by adding the ordinal $N \cap \omega_1$.
- For each $\alpha \in N$ in case (4), $q(\alpha)$ is the limit of the conditions in the α-coordinates of the elements of $G \cap P'_\theta$, as described in Remark 2.5.

It suffices now to see that q is in P_θ, and below each element of G. We show by induction on $\alpha \in N \cap (\theta + 1)$ that $q \restriction \alpha$ is in P_α and that $q \restriction \alpha$ is below each element of G_α. This is straightforward in the cases where $\alpha = 0$ or α is a limit ordinal. The successor cases of the form $\alpha + 1$, where α is in cases (1) or (4), or $\underset{\sim}{Q}_\alpha$ is forced by some condition in G_α to be the trivial partial order, are also straightforward. For α in the nontrivial subcase of case (2), the fact that the sequence of values $\langle q(\beta)(N \cap \omega_1) : \beta \in N \cap \kappa \rangle$ is increasing implies that $N \cap \omega_1$ is forced by $q \restriction \alpha$ to be in the desired set.

Finally, we consider $\alpha \in N$ in case (3) for which $\underset{\sim}{Q}_\alpha$ is forced by some condition p_0 in $G \cap P_\alpha$ to be nontrivial. We want to see that $q \restriction \alpha$ forces that $N \cap \omega_1$ is not in $\dot{A}_\alpha$. To see this, suppose to the contrary that $r \leq q \restriction \alpha$ is in P_α and forces the opposite. We may fix $\delta \in N \cap \alpha$ such that r forces that $\dot{g}_\alpha$ is dominated on $\dot{A}_\alpha$ by $\dot{f}_\delta$. Strengthening r, we may assume that it forces a value ξ to $\dot{g}_\alpha(N \cap \omega_1)$. Since $q \restriction \alpha$ forces that $f_\delta(N \cap \omega_1) = \text{o.t.}(N \cap \delta)$, ξ is less than $N \cap \delta$, so ξ is $\text{o.t.}(N \cap \gamma)$ for some $\gamma \in N \cap \delta$. Then there is a P_α-name $\dot{C}$ in N for a subset of ω_1 which p_0 forces to be a club disjoint from the set of $\zeta \in \dot{A}_\alpha$ for which $f_\gamma(\zeta) = \dot{g}_\alpha(\zeta)$. Then r forces that $N \cap \omega_1$ is in $\dot{C}$, but also that $\dot{f}_\gamma(\zeta) = \dot{g}_\alpha(\zeta)$ and that $N \cap \omega_1$ is in $\dot{A}_\alpha$, giving a contradiction.

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