CONSISTENCY OF A STRONG UNIFORMIZATION PRINCIPLE

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Abstract. We prove the consistency of a strong uniformization principle for a subset of the Baire space of cardinality $\aleph_1$. As a consequence we get the consistency of a related group-theoretic principle.

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§ 0. Introduction

The abelian group theoretic property which was our original motivation is being a splitter: G is a splitter if $\mathrm{Ext}(G,G)=0$ (i.e. if whenever $G \subseteq H$ and $H/G \cong G$ then G is a direct summand of H). In Göbel-Shelah [3] this was investigated and it was claimed that no $\aleph_1$-free (abelian) group of cardinality $\aleph_1$ is a splitter, but this was replaced by a weaker version [2] following suspicions of Eklof. Those works deal with $R_{\mathbf{P}}$-modules where $\mathbf{P}$ is a proper subset of the set of primes and $R_{\mathbf{P}}$ is the sub-ring of $\mathbb{Q}$ generated by $\{1\} \cup \{1/p : p \text{ a prime} \in \mathbf{P}\}$, so an $R_{\mathbf{P}}$-module is a somewhat divisible abelian group. One problematic case was when $G \in \mathscr{K} = \{G :$ is $\aleph_1$-free and for some $G_0 \subseteq G$ we have $|G_0| = \aleph_0$ and G/G_0 is divisible$\}$.

This issue is not resolved here (see also [4]), but the question reminds us of the following problem : can there be a Whitehead group G of cardinality $\aleph_1$ such that for some countable $G_0 \subseteq G$, G/G_0 is divisible? This was shown to be consistent (noting that but both CH and $MA_{\aleph_1}$ contradict it) in [5] (see also [1]), using the consistency of

- $\boxtimes_1$ there exists an injective sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of elements of ${}^{\omega}\omega$ which has the 2-uniformization property, that is, such that if c_α $(\alpha < \omega_1)$ are elements of ${}^{\omega}2$ then for some $h : {}^{\omega>}\omega \to 2$, for every $\alpha < \omega_1$ and every sufficiently large $n < \omega$ we have $h(\eta_\alpha{\restriction}n) = c_\alpha(n)$.

Our intended application (see [4]) deals with coloring with $\aleph_1$ many colors (although after analysis only $\aleph_0$ many colors are used) and the parallel of $\boxtimes_1$ for this fails (see [5, 1.2(3)]), but as the kernel is large we can weaken the demand in another direction. This motivates us to formulate:

- $\boxtimes_2$ there exists an injective sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of elements of ${}^{\omega}\omega$ such that for every countable group $G = (G, +_G)$ and every sequence $\langle c_\alpha : \alpha < \omega_1 \rangle$ of elements of ${}^{\omega}G$ there exist functions $h\colon {}^{\omega>}\omega \to G$ and $\zeta\colon \omega_1 \to \omega_1$ such that for every $\alpha < \omega_1$ and $n < \omega$ we have
$$c_\alpha(n) = h(\eta_\alpha{\restriction}n) +_G h(\eta_{\zeta(\alpha)}{\restriction}n).$$

Note that we omit the restriction "for every large enough n" as we have the function ζ. This is, so far, immaterial. Unfortunately this is not enough for any result on Ext. This leads to the following relative but for it the proof does not work (contrary to a claim in an earlier version), see [4]:

- $\boxtimes_3$ for every infinite countable group $G = (G, +_G)$, we can find pairwise distinct $\eta_\alpha \in {}^{\omega}G$ for $\alpha < \omega_1$ such that: given $c_\alpha \in {}^{\omega}G$ for $\alpha < \omega_1$ we can find functions $h\colon {}^{\omega>}G \to G$ and $\zeta\colon \omega_1 \to \omega_1$ such that for any $\alpha < \omega_1$ and $n < \omega$ we have
$$c_\alpha(n) = h(\eta_\alpha{\restriction}(n+1)) +_G \eta_{\zeta(\alpha)}(n).$$

Our main result is the consistency of $\boxtimes_2$, which seems combinatorially interesting by itself; we first thought of using non-meagreness of $\{\eta_\alpha : \alpha < \omega_1\}$ but eventually continued the ideas from [6, §1].

Our algebraic questions (and proofs) are on abelian groups but in the principle $\boxtimes_2$ the groups are not necessarily commutative.

§ 1. Consistency of a uniformization principle for $\aleph_1$

Notation 1.1. For sequences η, ν, $\eta \trianglelefteq \nu$ means that η is an initial segment of ν, and $\eta \triangleleft \nu$ means that η is a proper initial segment of ν.

Notation 1.2. We let

(1) $\mathscr{F}_{\aleph_0}$ denote the set of pairs (h, ν) for which there exist a non-zero $n < \omega$ and a sequence $\eta \in {}^n\omega$ such that $\nu \in {}^n\omega$ and h is a function from
$$\{\rho : \rho \trianglelefteq \eta \vee \rho \triangleleft \nu\}$$
to ω (so (η, ν) can be reconstructed from $\mathrm{dom}(h)$);

(2) $\mathscr{F}_{*,\aleph_0}$ denote the set of functions from $\mathscr{F}_{\aleph_0}$ to ω.

The "s.i.u." defined in part (1) below is closely related to $\boxtimes_2$ from the introduction (see Theorem 2.1). Note that the main case below is $i_1^* = i_2^* = \aleph_1$.

Definition 1.3. 1) We say that $(\bar{\eta}^1, \bar{\eta}^2)$ satisfies the $\aleph_0$-strong inside uniformization property ($\aleph_0$-s.i.u.) <u>when</u>:

(a) $\bar{\eta}^\ell = \langle \eta_i^\ell : i < i_\ell^* \rangle$ for $\ell \in \{1, 2\}$

(b) $\eta_i^\ell \in {}^\omega\omega \setminus \{\eta_j^\ell : j < i\}$ for $i < i_\ell^*$ and $\ell = 1, 2$

(c) for each sequence $\langle f_i : i < i_1^* \rangle \in {}^{i_1^*}(\mathscr{F}_{*,\aleph_0})$ we can find functions $h\colon {}^{\omega>}\omega \to \omega$ and $g\colon i_1^* \to i_2^*$ satisfying

 ($*$) for every $i < i_1^*$ and for every non-zero $n < \omega$ the function h obeys f_i at $((\eta_i^1 {\restriction} n), \eta_{g(i)}^2 {\restriction} n)$ which means that
$$h(\eta_{g(i)}^2 {\restriction} n) = f_i(h {\restriction} \{\rho : \rho \trianglelefteq \eta_i^1 {\restriction} n \quad \text{or} \quad \rho \triangleleft \eta_{g(i)}^2 {\restriction} n\}, \eta_{g(i)}^2 {\restriction} n).$$

2) We may replace $(\bar{\eta}^1, \bar{\eta}^2)$ by $\bar{\eta}$ if $\bar{\eta}^1 = \bar{\eta}^2 = \bar{\eta}$.
3) We say that λ has the $\aleph_0$-s.i.u.if some sequence $\bar{\eta} \in {}^\lambda({}^\omega\omega)$ has the $\aleph_0$-s.i.u..

Definition 1.4. A sequence $\bar{\eta}$ is universally $\aleph_0-$s.i.u. if

(a) $\bar{\eta} = \langle \eta_i : i < i^* \rangle$ where $\eta_i \in {}^\omega\omega \setminus \{\eta_j : j < i\}$ for $i < i^*$

(b) if $\bar{\eta}^1 = \langle \eta_i^1 : i < i^* \rangle$ and $\eta_i^1 \in {}^\omega\omega \setminus \{\eta_j^1 : j < i\}$ for $i < i^*$ then $(\bar{\eta}^1, \bar{\eta})$ has $\aleph_0-$s.i.u..

Our main result is the following.

Theorem 1.5. *There is a c.c.c. partial order of cardinality $2^{\aleph_1}$ forcing the existence of a universally $\aleph_0$-s.i.u. sequence of length ω_1.*

The proof is broken to a series of definitions and claims. We fix for this section a regular cardinal $\chi > 2^{2^{\aleph_1}}$, and let λ be $2^{\aleph_1}$.

Definition 1.6. For $\alpha \in [1, \lambda]$, let $\mathfrak{K}_\alpha$ be the family of
$$\mathfrak{q} = \langle (\mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \underset{\sim}{\bar{f}}_\beta, \bar{N}_\beta) : \beta < \alpha \rangle$$
such that

(a) $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ is a finite support iteration;

(*b*) $\mathbb{Q}_0$ is $\{p : p$ a finite function from ω_1 to ${}^{\omega>}\omega\}$, ordered by

$$p \leq q \quad \underline{\text{iff}}\, (\forall i \in \mathrm{Dom}(p))(i \in \mathrm{Dom}(q) \quad \& \quad p(i) \trianglelefteq q(i));$$

(*c*) $\underset{\sim}{\bar{f}}_0 = \bar{N}_0 = \emptyset$;

(*d*) if $\beta \in [1, \alpha)$ then

(*α*) $\underset{\sim}{\bar{f}}_\beta = \langle \underset{\sim}{f}_{\beta,j} : j < \omega_1 \rangle$ is a $\mathbb{P}_\beta$ - name for an ω_1-sequence of members of $\mathscr{F}^{\mathbf{V}[\mathbb{P}_\beta]}_{*,\aleph_0}$,

(*β*) $\underset{\sim}{\bar{\eta}}^1_\beta = \langle \underset{\sim}{\eta}^1_{\beta,j} : j < \omega_1 \rangle$ is a $\mathbb{P}_\beta$-name of pairwise distinct members of ${}^\omega\omega$

(*γ*) $\bar{N}_\beta$ is a $\subseteq$-increasing continuous sequence $\langle N_{\beta,i} : i < \omega_1 \rangle$ such that

- each $N_{\beta,i}$ is a countable elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$,
- $\mathfrak{q}\restriction\beta, \beta \in N_{\beta,0}$,
- $\bar{N}_\beta\restriction(i+1) \in N_{\beta,i+1}$ for each $i < \omega_1$;

(*e*) if $\beta \in (0, \alpha)$ (and $\omega_1^{\mathbf{V}}$ is uncountable in $\mathbf{V}^{\mathbb{P}_\beta}$; otherwise $\underset{\sim}{\mathbb{Q}}_\beta$ is the trivial forcing) then, in $\mathbf{V}^{\mathbb{P}_\beta}$, the conditions of $\underset{\sim}{\mathbb{Q}}_\beta$ are the triples $p = (h^p, w^p, g^p)$ such that (letting, for each $i < \omega_1$, $\underset{\sim}{\eta}_i$ denote the name for the ith element of ${}^\omega\omega$ added by $\mathbb{Q}_0$ and $\zeta_\beta(i)$ denote $N_{\beta,i} \cap \omega_1$):

(*α*) h^p is a function with domain some finite subset of ${}^{\omega>}\omega$ closed under initial segments and $\mathrm{Rang}(h^p) \subseteq \omega$,

(*β*) w^p is a finite subset of ω_1,

(*γ*) g^p is a function with domain w^p and each value $g^p(j)$ in the corresponding set $\{\zeta_\beta(\omega j + n) : 0 < n < \omega\}$,

(*δ*) if $n < \omega$ and $j \in w^p$ then $\underset{\sim}{\eta}^1_j\restriction n \in \mathrm{Dom}(h^p) \Leftrightarrow \underset{\sim}{\eta}_{g^p(j)}\restriction n \in \mathrm{Dom}(h^p)$,

(*ε*) if $j \in w^p$, then for some n we have

(i) $\underset{\sim}{\eta}^1_j\restriction n, \underset{\sim}{\eta}_{g^p(j)}\restriction n \in \mathrm{Dom}(h^p)$,

(ii) if $i \in w^p\backslash\{j\}$ then $\underset{\sim}{\eta}_{g^p(i)}\restriction(n+1) \neq \underset{\sim}{\eta}_{g^p(j)}\restriction(n+1)$,

(*ζ*) if $j \in w^p$, $0 < n < \omega$ and $\underset{\sim}{\eta}^1_j\restriction n \in \mathrm{Dom}(h^p)$, then h^p obeys $\underset{\sim}{f}_{\beta,j}$ at $(\underset{\sim}{\eta}^1_i\restriction n,\ \underset{\sim}{\eta}_{g^p(j)}\restriction n)$;

(*i*) if $\beta \in (0, \alpha)$ (and $\omega_1^{\mathbf{V}}$ is uncountable in $\mathbf{V}^{\mathbb{P}_\beta}$) then in $\mathbf{V}^{\mathbb{P}_\beta}$ the order of $\underset{\sim}{\mathbb{Q}}_\beta$ is : $p \leq q$ iff $h^p \subseteq h^q$ & $w^p \subseteq w^q$ & $g^p \subseteq g^q$.

Notation 1.7. Given a $\mathfrak{q}$ in $\mathfrak{K}_\alpha$ for some ordinal α, we let

$$\langle (\mathbb{P}^{\mathfrak{q}}_\beta, \underset{\sim}{\mathbb{Q}}^{\mathfrak{q}}_\beta, \underset{\sim}{\bar{f}}^{\mathfrak{q}}_\beta, \bar{N}^{\mathfrak{q}}_\beta) : \beta < \alpha^{\mathfrak{q}} \rangle$$

denote the components of $\mathfrak{q}$.

Notation 1.8. Given $\alpha \in [1, \lambda]$ and $\mathfrak{q}$ in $\mathfrak{K}_\alpha$, we let $\mathrm{Lim}(\mathfrak{q})$ denote $\mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is $\mathbb{P}_{\alpha-1} * \underset{\sim}{\mathbb{Q}}_{\alpha-1}$ if α is a successor ordinal and $\bigcup_{\beta<\alpha} \mathbb{P}_\beta$ otherwise. When $\mathfrak{q}$ is clear from context, we let

- $\zeta_\beta(i)$ (for $\beta \in (0, \alpha)$ and $i < \omega_1$) be $N_{\beta,i} \cap \omega_1$;
- $\underset{\sim}{\eta}_i$ (for $i < \omega$) be the $\mathbb{Q}_0$-name for the ith element of ${}^\omega\omega$ added by $\mathbb{Q}_0$ (i.e., the union of the sequences $p(i)$, for p in the $\mathbb{Q}_0$-generic filter);
- $\underset{\sim}{h}_\beta$ (for $\beta \in (0, \alpha)$) be the $\mathbb{P}_{\beta+1}$-name for $\cup\{h^{p(\beta)} : p \in \underset{\sim}{G}_{\mathbb{P}_\alpha}\}$ (in the case where $\omega_1^{\mathbf{V}}$ is uncountable in $\mathbf{V}^{\mathbb{P}_\beta}$).

The two following claims show that the partial orders $\mathbb{Q}_\alpha$ ($\alpha \in [1,\lambda)$) force instances of the universal $\aleph_0$-s.i.u.. The proof of Claim 1.9 is routine.

Claim 1.9. *If $\alpha \in [1,\lambda]$ and $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \underset{\sim}{\bar{f}}_\beta, \bar{N}_\beta : \beta < \alpha\rangle \in \mathfrak{K}_\alpha$ then $\Vdash_{\mathbb{Q}_0}$ "for all $i < j < \omega_1, \underset{\sim}{\eta}_i, \underset{\sim}{\eta}_j \in {}^\omega\omega$ and $\underset{\sim}{\eta}_i \neq \underset{\sim}{\eta}_j$".*

Claim 1.10. *If*

- $\alpha \in [1,\lambda]$,
- $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \underset{\sim}{\bar{f}}_\beta, \bar{N}_\beta : \beta < \alpha\rangle \in \mathfrak{K}_\alpha$,
- $\beta \in (0,\alpha)$,
- $\Vdash_{\mathbb{P}_\beta}$ "$\omega_1^{\mathbf{V}}$ *is uncountable*",

then

(1) $\Vdash_{\mathbb{P}_{\beta+1}}$ "*$\underset{\sim}{h}_\beta$ is a function from ${}^{\omega>}\omega$ to ω*",
(2) *in $\mathbf{V}^{\mathbb{P}_{\beta+1}}$ the function $\underset{\sim}{h}_\beta$ witnesses the universal* $\aleph_0$-s.i.u.*for $\langle \underset{\sim}{\eta}_i : i < \omega_1\rangle$ with respect to $\underset{\sim}{\bar{f}}_\beta$ and $\bar{\underset{\sim}{\eta}}^1_\beta$.*

Proof. Let $\mathbf{G}_\beta \subseteq \mathbb{P}_\beta$ be generic over $\mathbf{V}$, with $\mathbf{G}_0$ its restriction to $\underset{\sim}{\mathbb{Q}}_0$. For each $i < \omega_1$, let $\eta_i = \underset{\sim}{\eta}_i[\mathbf{G}_0]$, let $\eta^1_i = \underset{\sim}{\eta}^1_{\beta,i}[\mathbf{G}_\beta]$ and let $f_i = \underset{\sim}{f}_{\beta,i}[\mathbf{G}_\beta]$. For $i \in w^p$, let $\eta^2_i = \eta_{g^p(i)}$.

We prove the first part first. Trivially $\underset{\sim}{h}_\beta$ is forced to be a partial function from ${}^{\omega>}\omega$ to ω. Let $\nu \in {}^{\omega>}\omega$; we shall prove that $\Vdash_{\underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]} \nu \in \mathrm{Dom}(\underset{\sim}{h}_\beta)$. Fix $p \in \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]$. We need to find a condition q satisfying $p \leq q$ in $\underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]$ such that $\nu \in \mathrm{Dom}(h^q)$. If $\nu \in \mathrm{Dom}(h^p)$ we are done, so we suppose otherwise. Let $n^* \geq \ell g(\nu)$ be such that $n^* > \sup\{\ell g(\rho) : \rho \in \mathrm{Dom}(h^p)\}$. By extending ν if necessary we may assume that $\ell g(\nu) = n^*$.

Our condition q will have $w^q = w^p$ and $g^q = g^p$. It remains to define h^q, which will extend h^p. We let $\mathrm{Dom}(h^q) = \{\rho : \rho \trianglelefteq \nu$ or $\rho \in \mathrm{Dom}(h^p)$ or $(\exists j \in w)(\exists \ell \in \{1,2\})(\rho \trianglelefteq \eta^\ell_j \& \ell g(\rho) \leq n^*)\}$. If $\rho \in \mathrm{Dom}(h^q) \setminus \mathrm{Dom}(h^p)$ and ρ is not of the form $\eta_{g^p(j)} \restriction m$ for some $j \in w^p$ and $m \leq n^*$, then we let $h^q(\rho) = 0$. For the remaining sequences ρ, we define $h^q(\rho)$ by recursion on j, and for each j by m, letting

$$h^q(\eta_{g^p(j)} \restriction m) = f_j(h^q{\restriction}\{\rho' : (\rho' \triangleleft \eta_{g^p(j)} \restriction m) \vee (\rho' \trianglelefteq \eta_j{\restriction}m)\}, \eta_{g^p(j)} \restriction m).$$

By part (e)(ϵ)(ii) of Definition 1.6 there are no conflicts in doing this. This completes the proof of the first part of the claim.

We now prove the second part. By the definition of the order on $\underset{\sim}{\mathbb{Q}}_\beta$, and Claim 1.10, it suffices to prove that, in $\mathbf{V}^{\mathbb{P}_\beta}$ for every $i < \omega_1$ the set of $p \in \underset{\sim}{\mathbb{Q}}_\beta$ with $i \in w^p$ is a dense subset of $\underset{\sim}{\mathbb{Q}}_\beta$. Fix $i < \omega_1$ and $p \in \underset{\sim}{\mathbb{Q}}_\beta[\mathbf{G}_\beta]$.

By genericity,

$$\Vdash_{\mathbb{Q}_0} \text{“}{}^{\omega>}\omega = \{\eta_j{\restriction}n : j \in \{N_{\beta,\omega\varepsilon+k} \cap \omega_1 : k \in (0,\omega)\} \text{ and } n < \omega\}\text{”}.$$

It follows that we can find $j \in \{N_{\beta,\omega i+k} \cap \omega_1 : k \in (0,\omega)\}$ such that $\{\rho : \rho \triangleleft \eta_j$ and $\ell g(\rho) > 0\}$ is disjoint from $\mathrm{Dom}(h^p) \cup \{\rho : \rho \triangleleft \eta^2_k,\ k \in \mathrm{Dom}(g^p)\}$ (it is enough to choose a suitable value for $\eta_j(0)$).

Choose $n^* > 0$ such that $\rho \in \mathrm{Dom}(h^p) \Rightarrow \ell g(\rho) < n^*$. As in the proof of the first part we can find h^*, a function from $\mathrm{Dom}(h^p) \cup \{\rho : \rho \trianglelefteq \eta^1_i{\restriction}n^*$ or $\rho \trianglelefteq \eta^\ell_k{\restriction}n^*$ for some $\ell \in \{1,2\}$ and $k \in \mathrm{Dom}(g^p)\}$ to ω such that $h^p \subseteq h^*$ and h^* obeys f_k at $(\eta^1_k{\restriction}m, \eta_{g^p(k)}{\restriction}m)$ for all $k \in w^p$ and $m \in [1,n^*]$. Next we choose $h^{**} \supseteq h^*$ with domain $\mathrm{Dom}(h^*) \cup \{\eta_j{\restriction}m : m \leq n^*\}$, as in the proof of Claim 1.10 so that h^{**} obeys f_i at $(\eta^1_i{\restriction}m, \eta_j{\restriction}m)$ for all $m \in [1,n^*]$.

Lastly, we let $g^q = g^p \cup \{\langle i,j\rangle\}$, $w^q = w^p \cup \{i\}$ and h^q as h^{**}. Easily $p \leq q$ and $i \in w^q$ so we are done. $\square_{1.10}$

We make one additional observation about the successor stages of our iterations (Claim 1.12 below).

Definition 1.11. 1) Let $\mathbb{Q}_*$ be defined by

- (A) $p \in \mathbb{Q}_*$ iff $p = (h,w,g) = (h^p,w^p,g^p)$ satisfies
  - (α) h^p is a function from some finite subset of ${}^{\omega>}\omega$ closed under initial segments and $\mathrm{Rang}(h^p) \subseteq \omega$
  - (β) w^p is a finite subset of ω_1
  - (γ) g^p is an increasing function from w^p to ω_1, $\alpha < g^p(\alpha)$
- (B) $\mathbb{Q}_*$ is ordered by $p \leq q \Leftrightarrow h^p \subseteq h^q$ & $w^p \subseteq w^q$ & $g^p \subseteq g^q$.

The following claim is straightforward.

Claim 1.12. *For each $\beta \leq \lambda$, $\Vdash_{\mathbb{P}_\beta} \underset{\sim}{\mathbb{Q}}_\beta \subseteq \mathbf{V}$. Furthermore, in $V^{\mathbb{P}_\beta}$, for all $p,q \in \underset{\sim}{\mathbb{Q}}_\beta$ we have $p \leq_{\underset{\sim}{\mathbb{Q}}_\beta} q \Leftrightarrow p \leq_{\mathbb{Q}_*} q$".*

We now move to an analysis of the initial segments of our iterations.

Definition 1.13. Let $\mathfrak{K}^+_\alpha$ be the set of $\mathfrak{q} \in \mathfrak{K}_\alpha$ such that for every $\beta < \alpha$, the forcing notion $\mathbb{P}^{\mathfrak{q}}_\beta$ satisfies the c.c.c.

Claim 1.14. *For proving Theorem 1.5 it suffices to prove that for all $\alpha < \lambda$ and all $\mathfrak{q} \in \mathfrak{K}^+_\alpha$ the forcing notion $\mathbb{P}^{\mathfrak{q}}_\alpha$ satisfies the c.c.c.*

Proof. By bookkeeping, as $\lambda^{\aleph_1} = \lambda$ there is $\mathfrak{q} \in \mathfrak{K}_\lambda$ such that

$(*)$ if $\beta < \lambda$ and $\underset{\sim}{\bar{f}}$ is a $\mathbb{P}_\beta$ - name of a member of ${}^{\omega_1}(\mathscr{F}_{*,\aleph_0})$ and $\underset{\sim}{\bar{\eta}}^1$ is a $\mathbb{P}_\beta$-name of a member of ${}^{\omega_1}({}^\omega\omega)$, then for some $\gamma \in [\beta,\lambda)$ we have $\Vdash_{\mathbb{P}_\gamma}$ "$\underset{\sim}{\bar{f}}{}^{\mathfrak{q}}_\gamma = \underset{\sim}{\bar{f}}$ and $\bar{\eta}^1_\gamma = \underset{\sim}{\bar{\eta}}^1$".

Then one gets by induction for all $\alpha \in [1,\lambda]$, $\mathrm{Lim}(\mathfrak{q}{\restriction}\alpha)$ satisfies the c.c.c., noting that the c.c.c. is preserved by finite support iterations. $\square_{1.14}$

For the rest of the section we fix $\alpha \in [1,\lambda)$ and $\mathfrak{q} \in \mathfrak{K}^+_\alpha$.

By the definition of finite support iterations, for each $\beta \leq \lambda$, $\mathbb{P}_\beta$ is the set of finite functions p with domain contained in β such that for each $\gamma \in \mathrm{Dom}(p)$, $p(\gamma)$ is a $\mathbb{P}_\gamma$-name of a member of $\underset{\sim}{\mathbb{Q}}_\gamma$. We define some dense subsets of $\mathbb{P}_\alpha$.

Definition 1.15. Fix $\beta \leq \alpha$.

(1) We let D^0_β be the set of $p \in \mathbb{P}_\beta$ such that
  - (a) $0 \in \mathrm{Dom}(p)$;
  - (b) for each $\gamma \in \mathrm{Dom}(p)$, there exists a set $x \in \mathbf{V}$ such that $p(\gamma) = \check{x}$";
  - (c) for all $\gamma \in \mathrm{Dom}(p)\setminus\{0\}$ and $i \in w^{p(\gamma)}$, if $j = g^p(i)$ then $j \in \mathrm{Dom}(p(0))$, and, letting n^* be the length of the largest initial segment of $p(0)(j)$ in $\mathrm{Dom}(h^{p(\gamma)})$,
    - (i) for some $\nu \in {}^{(n^*)}2 \cap \mathrm{Dom}(h^{p(\gamma)})$, $(p \restriction \gamma) \Vdash (\underset{\sim}{\eta}^1_{\gamma,i} \restriction n^*) = \check{\nu}$,
    - (ii) $n^* < \ell g(p(0)(j))$,
    - (iii) $p(0)(j){\restriction}(n+1)$ is not equal to $p(0)(g^{p(\gamma)}(k)){\restriction}(n+1)$, for any $k \in w^{p(\gamma)}$ with $g^{p(\gamma)}(k) < j$.

(2) We let D^1_β be the set of finite functions with domain $\subseteq \beta$ such that

   (a) $0 \in \mathrm{Dom}(p)$;
   (b) for $\gamma \in \mathrm{Dom}(p)$ we have :
      (α) if $\gamma = 0$ then $p(\gamma) \in \mathbb{Q}_0$,
      (β) if $\gamma > 0$, then
      - $p(\gamma)$ is $(h^{p(\gamma)}, w^{p(\gamma)}, g^{p(\gamma)}) \in \mathbb{Q}_*$,
      - $\mathrm{Rang}(g^{p(\gamma)}) \subseteq \mathrm{Dom}(p(0))$.

(3) The order $\leq_{D^1_\beta}$ on D^1_β is: $p \leq_{D^1_\beta} q$ iff

   (a) $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$,
   (b) $\mathbb{Q}_0 \models$ "$p(0) \leq q(0)$",
   (c) $\forall \gamma \in \mathrm{Dom}(p) \setminus \{0\}$, $\mathbb{Q}_* \models$ "$p(\gamma) \leq q(\gamma)$".

(4) We let $D^{0,*}_\beta$ be the set of $p \in D^1_\beta$ such that for all $\gamma \in \mathrm{Dom}(p) \setminus \{0\}$ and $i \in w^p$, if $j = g^{p(\gamma)}(i)$ then then $j \in \mathrm{Dom}(p(0))$, and, letting $n^* = \ell g(p(0)(j))$,
   (a) $p(0)(j) \in \mathrm{Dom}(h^{p(\gamma)})$;
   (b) there is $q \in D^0_\gamma \cap N_{\gamma+1,i+1}$ satisfying $q \leq_{D^1_\beta} p{\restriction}\gamma$ such that
      (i) for some $\nu \in {}^{(n^*)}2 \cap \mathrm{Dom}(h^{p(\gamma)})$, $q \Vdash (\underset{\sim}{\eta}^1_{\gamma,i} \restriction n^*) = \check{\nu}$,
      (ii) q forces that $h^{p(\gamma)}$ obeys $\underset{\sim}{f}_{\gamma,i}$ at $(\nu{\restriction}m, p(0)(j){\restriction}m)$, for all $m \in (0, n]$.

(5) For $p \in D^{0,*}_\beta$ and $n < \omega$ we let $p^{\langle n\rangle}$ be the following function:

   (a) $\mathrm{Dom}(p^{\langle n\rangle}) = \mathrm{Dom}(p)$,
   (b) $\forall \gamma \in \mathrm{Dom}(p) \setminus \{0\}$, $p^{\langle n\rangle}(\gamma) = p(\gamma)$,
   (c) $\mathrm{Dom}(p^{\langle n\rangle}(0)) = \mathrm{Dom}(p(0))$,
   (d) $i \in \mathrm{Dom}(p(0)) \Rightarrow (p^{\langle n\rangle}(0))(i) = (p(0)(i))^\frown\langle n + \mathrm{otp}(i \cap \mathrm{Dom}(p(0)))\rangle$.

(6) Given $\beta \leq \alpha$, $p \in D^1_\beta$ and a countable elementary submodel N of

$$(\mathscr{H}(\chi), \in, <^*_\chi),$$

we let $p \restriction N$ denote the element q of D^1_β such that:

   (a) $\mathrm{Dom}(q) = \mathrm{Dom}(p) \cap N$
   (b) $q(0) = p(0) \restriction (N \cap \omega_1)$
   (c) for all $\gamma \in \mathrm{Dom}(q) \setminus \{0\}$, $q(\gamma) = (h^{q(\gamma)}, w^{q(\gamma)}, g^{w(\gamma)})$ is defined by:
      (α) $h^{q(\gamma)} = h^{p(\gamma)}$
      (β) $w^{q(\gamma)} = \{i \in w^{p(\gamma)} : g^{p(\gamma)}(i) \in N\}$
      (γ) $g^{q(\gamma)} = g^{p(\gamma)}{\restriction}w^{q(\gamma)}$.

Remark 1.16. (1) Each member of D^0_β has a clear description but the satisfaction of "$p \in D^0_\beta$" is complicated; it depends on the bookkeeping involved in the definition of $\mathfrak{q}$.

(2) The set D^0_β can be viewed as a subset of D^1_β (it is not literally a subset but we ignore this distinction in what follows, and above). Unlike with D^0_β, membership in D^1_β is simply defined.

(3) The set of $D^{0,*}_\beta$ consists of $p \in D^1_\beta$ which are in some sense close to being in D^0_β, needing only to be strengthened in coordinate 0 (i.e., $\mathbb{Q}_0$). Clause (d) is crucial; having such $q \in N_{\gamma+1,i+1}$ will hold densely often.

Claim 1.17 lays out some of the basic properties of the terms defined in Definition 1.15.

Claim 1.17. *Fix $\beta \leq \alpha$.*
0) For all $\gamma < \beta$,

- $D^0_\gamma = \{p \in D^0_\beta : \mathrm{Dom}(p) \subseteq \gamma\} = \{p{\restriction}\gamma : p \in D^0_\beta\}$;
- $D^1_\gamma = \{p \in D^1_\beta : \mathrm{Dom}(p) \subseteq \gamma\} = \{p{\restriction}\gamma : p \in D^1_\beta\}$;
- $D^{0,*}_\gamma = \{p \in D^{0,*}_\beta : \mathrm{Dom}(p) \subseteq \gamma\} = \{p{\restriction}\gamma : p \in D^{0,*}_\beta\}$;
- $\leq_{D^1_\gamma} = \leq_{D^1_\beta} {\restriction} D^1_\gamma$.

1) D^0_β is a dense subset of $\mathbb{P}_\beta$.
2) If $p \in D^1_\beta$, $v \subseteq \mathrm{Dom}(p)$ and $0 \in v$ then $p{\restriction}v \in D^1_\beta$.
3) If $\beta \leq \alpha$, $p \in D^{0,*}_\beta$ and $i < \omega_1$ then $p \upharpoonright N_{\beta,i} \in D^{0,*}_\beta$ and $D^1_\beta \models$ "$p \upharpoonright N_{\beta,i} \leq p$".
4) If $p, q \in D^0_\beta$ then $p \leq_{\mathbb{P}_\beta} q$ iff $p \leq_{D^1_\beta} q$.
5) $\leq_{D^1_\beta}$ is a partial order on D^1_β.

Proof. Parts (0), (2), (4) and (5) follow immediately from the definitions, and part (1) is routine.

For part (3), let $p' = p \upharpoonright N_{\beta,i}$. Clauses (4a) of Definition 1.15(4) should be clear, so the main issue is clause (4b). So assume that $\gamma_1 \in \mathrm{Dom}(p')\backslash\{0\}$ and $h^{p'(\gamma_1)}(i_1) = j_1$, hence $\gamma_1 \in N_{\beta,i} \cap \beta$ and $i_1, j_1 \in N_{\beta,i} \cap \omega_1$. Now as p satisfies clause (4b) there is q as there; in particular, $q \in D^0_{\gamma_1} \cap N_{\gamma_1+1,i_1+1}$. But $\gamma_1 \in \mathrm{Dom}(p') \subseteq N_{\beta,i}$ and $i_1 \in N_{\beta,i}$ (as $g^{p'(\gamma_1)}(i_1) = j_i$) and $\langle N_{\gamma_1,\varepsilon} : \varepsilon < \omega_1\rangle$ is in $N_{\beta,i}$ hence $N_{\gamma_1+1,i_1+1} \in N_{\beta,i}$ recalling Definition 1.6(e), so easily $q \leq_{D^1_\beta} p{\restriction}\gamma$ implies $q \leq_{D^1_\beta} p' \upharpoonright \gamma$. $\square_{1.17}$

Extending a $p \in D^{0,*}_\beta$ to an element of D^0_β (for some $\beta \leq \alpha$) requires only extending the members of $p(0)$ to make them distinct. Claim 1.18 records one way of doing this.

Claim 1.18. *Suppose that $\beta \leq \alpha$ and $p \in D^{0,*}_\beta$. For all but finitely many $n \in \omega$, $p \leq_{D^1_\beta} p^{\langle n\rangle} \in D^0_\beta$.*

Definition 1.19. Conditions $p_1, p_2 \in D^{0,*}_\beta$ are a Δ-*system pair* when:

(a) if $0 \in \mathrm{Dom}(p_i) \cap \mathrm{Dom}(p_2)$ then for all $i \in \mathrm{dom}(p_1) \cap \mathrm{dom}(p_2(0))$, $p_1(0)(i) = p_2(0)(i)$;
(b) $\mathrm{dom}(p_1(0)) \cap \mathrm{dom}(p_2(0))$ is an initial segment of both $\mathrm{dom}(p_1(0))$ and $\mathrm{dom}(p_2(0))$;
(c) for all $\gamma \in \mathrm{Dom}(p_1) \cap \mathrm{Dom}(p_2)\backslash\{0\}$,
  (α) $h^{p_1(\gamma)} = h^{p_2(\gamma)}$,
  (β) $w^{p_1} \cap w^{p_2}$ is an initial segment of both w^{p_1} and w^{p_2},
  (γ) for all $i \in w^{p_1} \cap w^{p_2}$, $g^{p_1(\gamma)}(i) = g^{p_2(\gamma)}(i)$.

Remark 1.20. If $\beta \leq \alpha$ and p_1, p_2 in D^1_β are compatible, then they have a least upper bound in D^1_β, which we call $p_1 + p_2$. A If p_1, p_2 are a Δ-system pair then they are compatible.

Claim 1.21 is used in the proof of Crucial Claim 1.23.

Claim 1.21. *Suppose that*

- $\beta_* \leq \beta \leq \alpha$,
- $i < \omega_1$,
- $q, r \in D^{0,*}_{\beta_*}$,
- $r \in N_{\beta,i}$,
- $r \geq q \restriction N_{\beta,i}$.

Then r and q are compatible, and $r + q$ is in $D^{0,*}_{\beta_*}$.

Proof. For each $\gamma \in \mathrm{Dom}(q)$, $N_{\beta,i} \cap \omega_1 = N_{\gamma,(N_{\beta,i}\cap\omega_1)} \cap \omega_1$ is a limit ordinal, so for all $i \in N_{\beta,i} \cap \mathrm{Dom}(g^{q(\gamma)})$, $g^{q(\gamma)}(i) \in N_{\beta,i} \cap \omega_1$. Given this, the compatibility of r and q is straightforward. $\square$

Definition 1.22. We say p is (β, δ)-good <u>when</u>:

(i) $p \in D^0_{\beta+1}$

(ii) if $\beta \in \mathrm{Dom}(p) \setminus \{0\}, g^{p(\beta)}(i) = j$ and $\delta < j$ then for some n the demands in Definition 1.15(4) of $D^{0,*}_{\beta}$ hold.

Crucial Claim 1.23. *If $p \in D^0_\beta$ then for some $q \in D^{0,*}_\beta$ we have $p \leq q$.*

Proof. By induction on β. For $\beta \in \{0, 1\}$ this is trivial, and limit steps follow from the fact that our iteration is by finite support. So fix β_* for which the claim holds, and let $\beta = \beta_* + 1$. We prove by induction on limit $\delta < \omega_1$ that

$\boxplus_\delta$ if $p \in D^0_\beta$ is (β_*, δ)-good then for some $q \in D^{0,*}_\beta$ we have

- $p \leq q$,
- $q(\beta_*) = p(\beta_*)$,
- $q(0)\restriction[\delta, \omega_1) = p(0)\restriction[\delta, \omega_1)$.

This is enough because trivially every $p \in D^0_\beta$ is (β_*, δ)-good for all sufficiently large δ.

If $\delta = \omega$ then we apply the induction hypothesis for β_* to obtain a $q_0 \in D^{0,*}_{\beta_*}$ above $p \restriction \beta_*$ with $q_0(0) \restriction [\delta, \omega_1) = p(0) \restriction [\delta, \omega_1)$. Then $q_0 \cup (\beta_*, p(\beta_*))$ is as desired, as $\zeta_\gamma(i) > \omega + \omega$ for all γ, i.

Fix then a countable limit ordinal δ such that $\boxplus_{\delta'}$ holds for all limit $\delta' < \delta$, and fix a (β_*, δ)-good $p \in D^0_\beta$. If there is no $i \in \mathrm{Dom}(g^{p(\beta_*)})$ with $g^{p(\beta_*)}(i) = \delta$ then p is (β_*, δ')-good for some limit $\delta' < \delta$ and we are done, so suppose otherwise. Let p_0 be p with i removed from $w^{p(\beta_*)}$ (and thus $\mathrm{Dom}(g^{p(\beta_*)})$). Then p_0 is (β_*, δ')-good for some limit $\delta' < \delta$, so there exists a q_0 as in $\boxplus_{\delta'}$ relative to p_0. By Claim 1.17(1) there is $p_1 \in D^0_\beta$ above q_0, and again we may assume that $p_1(\beta_*) = q_0(\beta_*)$. As $\beta_* < \alpha$, $\mathbb{P}_{\beta_*}$ satisfies the c.c.c., there exists an $r_0 \in \mathbb{P}_{\beta_*} \cap N_{\beta,i+1}$ above $q_0 \restriction N_{\beta,i+1}$ deciding enough of $\underaccent{\tilde}{f}_{\beta_*,i}$ and $\underaccent{\tilde}{\eta}^1_{\beta_*,i}$, in agreement with p_1, to satisfy Definition 1.15(4) with respect to β_* and i (we may also do this in such a way that $r_0(0) = p_1(0) \restriction N_{\beta,i+1} \cap \omega_1$). We can strengthen r_0 inside $N_{\beta,i+1}$ to a condition $r_1 \in D^0_{\beta_*}$ and then again to a condition $r_2 \in D^{0,*}_{\beta_*}$. Now let $q = q_0 + r_2$, which is in $D^{0,*}_{\beta_*}$, by Claim 1.21. Then $q \cup (\beta_*, p(\beta_*))$ is as desired. $\square_{1.23}$

Conclusion 1.24. *$\mathbb{P}_\alpha$ satisfies the c.c.c.*

Proof. Let $p_\varepsilon \in \mathbb{P}_\alpha$ for $\varepsilon < \omega_1$, without loss of generality $p_\varepsilon \in D^0_\alpha$ (see 1.17(1)).

Applying Crucial Claim 1.23, choose q_ε $(\varepsilon < \omega_1)$ such that $p_\varepsilon \leq q_\varepsilon \in D^{0,*}_\alpha$ holds for each ε. Use the Δ-system lemma to fix $\varepsilon < \zeta$ such that (q_ε, q_ζ) form a Δ-system pair, as in Definition 1.19, so they have a common upper bound $q \in D^{0,*}_\alpha$.

By Claim 1.18 there is a $p \in D^0_\alpha$ such that $q \leq p$ has $p_\varepsilon \leq q_\varepsilon \leq q \leq p, p_\zeta \leq q_\zeta \leq q \leq p$, and so by Claim 1.17(5), $\mathbb{P}_\alpha \models$ "$p_\varepsilon \leq p \wedge p_\zeta \leq p$", so we are done. $\square_{1.24}$

§ 2. Conclusion

In this section we show that an $\aleph_0$-s.i.u. sequence witnesses the principle $\boxtimes_2$ from the introduction. We prove this in slightly greater generality, modifying Definition 1.3 by replacing ${}^\omega\omega$ with ${}^\omega\mu$ and making the obvious changes. For any set X, we let $\mathscr{F}_X = \{(h, \nu_1)$: for some $n, \nu_0 \in {}^nX, \nu_1 \in {}^{n+1}X$ we have h is a function from $\{\rho : \rho \triangleleft \nu_0 \vee \rho \triangleleft \nu_1\}$ to $X\}$ and define $\mathscr{F}_{*,X}$ and the X-s.i.u. analogously.

Theorem 2.1. *Let λ_1 and λ_2 be ordinals, and let μ be a cardinal. Suppose that*

(a) *$\eta^\ell_\alpha \in {}^\omega\mu$ for $\alpha < \lambda_\ell$ and $\bar\eta^\ell = \langle \eta^\ell_\alpha : \alpha < \lambda_\ell \rangle$ for $\ell = 1, 2$,*

(b) *$(\bar\eta^1, \bar\eta^2)$ has the μ-s.i.u.,*

(c) *G is a group of cardinality μ.*

Then

$\boxtimes^2_{\bar\eta,G}$ *given $c_\alpha \in {}^\omega G$ $(\alpha < \lambda_1)$ we can find functions $h : {}^{\omega>}\mu \to G$ and $\zeta : \lambda_1 \to \lambda_2$* such that

$$c_\alpha(n) = h(\eta^1_\alpha\restriction n) \cdot_G h(\eta^2_{\zeta(\alpha)}\restriction n)$$

for all $\alpha < \lambda_1$ and $n \in (0, \omega)$.

Proof. For notational simplicity, we suppose that μ is the set of elements of G. Given $c_\alpha \in {}^\omega\mu$ $(\alpha < \lambda_1)$ we define functions f_α $(\alpha < \lambda_1)$ as follows. If $n < \omega$, $\nu \in {}^n\mu$ and h is a function from

$$\{\rho : \rho \triangleleft \eta^1_\alpha\restriction(n+1) \quad \text{or } \rho \triangleleft \nu\}$$

to μ, we let $f_\alpha(h)$ be the unique $x \in \mu$ such that

$$c_\alpha(n) = h(\eta^1_\alpha\restriction n) \cdot_G x.$$

Since $(\bar\eta^1, \bar\eta^2)$ has the μ-s.i.u. there exist $h : {}^{\omega>}\mu \to \mu$ and $\zeta : \lambda_1 \to \lambda_2$ such that:

$(*)$ for all $\alpha < \lambda_1$ and every non-zero $n < \omega$, h obeys f_i at n, i.e.,

$$h(\eta^2_{\zeta(\alpha)}\restriction n) = f_\alpha(h\restriction\{\rho \triangleleft \eta^1_\alpha\restriction(n+1) \quad \text{or} \quad \rho \triangleleft \eta^2_{\zeta(\alpha)}\restriction n\}).$$

It follows that for all $\alpha < \lambda_1$ and all $n \in (0, \omega)$,

$$c_\alpha(n) = h(\eta^1_\alpha\restriction n) \cdot_G h(\eta^2_{\zeta(\alpha)}\restriction n)$$

as required. $\square_{2.1}$

Corollary 2.2. *If $\aleph_1$ has the $\aleph_0$-s.i.u., then $\boxtimes_2$ holds.*

We briefly discuss further generalizations.

Definition 2.3. Let μ and κ be cardinals.

(1) We say that $(\bar\eta^1, \bar\eta^2)$ has the (μ,κ)-strong inside uniformization ($(\mu,\kappa)-$ s.i.u. in short) <u>when</u>

(a) $\bar\eta^\ell = \langle \eta^\ell_i : i < i^*_\ell\rangle$ for $\ell = 1, 2$

(b) $\eta^\ell_i \in {}^\kappa\mu \setminus \{\eta^\ell_j : j < i\}$ for $i < i^*_\ell$ and $\ell = 1, 2$

(c) for any sequence $\langle f_i : i < i^*_1\rangle \in {}^{i^*_1}(\mathscr{F}_{*,\aleph_0})$ we can find $h : {}^{\kappa>}\mu \to \omega$ and function $\zeta : i^*_1 \to i^*_2$ satisfying

(∗) for any sequence $i < i^*_1$ for every non-zero $\varepsilon < \kappa$ the function h obeys f_i at $((\eta^1_i{\restriction}\varepsilon), \eta^2_{\zeta(i)}{\restriction}\varepsilon)$ (but we may just say at ε if $(\eta^1_i, \eta^2_{\zeta(i)})$ is clear from the context), which means

$$h(\eta^2_{\zeta(i)}{\restriction}\varepsilon) = \ f_i(h{\restriction}\{\rho : \rho \trianglelefteq \eta^1_i{\restriction}\varepsilon \quad \text{or} \quad \rho \triangleleft \eta^2_{\zeta(i)}{\restriction}\varepsilon\}, \eta^2_{\zeta(i)}{\restriction}\varepsilon)$$

(2) We may replace $(\bar\eta^1, \bar\eta^2)$ by $\bar\eta$ if $\bar\eta^1 = \bar\eta^2 = \bar\eta$.

(3) We say λ has $\aleph_0-$ s.i.u. <u>if</u> for some sequence $\bar\eta \in^\lambda ({}^\kappa\mu)$ has $\aleph_0-$ s.i.u..

(4) We say that $\bar\eta$ is universally $(\mu,\kappa)-$ s.i.u. if

(a) $\bar\eta = \langle \eta_i : i < i^*\rangle$ where $\eta_i \in {}^\kappa\mu \setminus \{\eta_j : j < i\}$ for $i < i^*$

(b) if $\bar\eta^1 = \langle \eta^1_i : i < i^*\rangle$ and $\eta^1_i \in {}^\kappa\mu \setminus \{\eta^1_j : j < i\}$ for $i < i^*$ then $(\bar\eta^1, \bar\eta)$ has $(\mu,\kappa)-$ s.i.u..

The proof of the following result, a modification of the proof of Theorem 1.5, will appear elsewhere.

Theorem 2.4. *Assume V satisfies $\kappa = \kappa^{<\kappa} = \mu, \theta = \kappa^+ < \lambda = \lambda^\theta, 2^\kappa = \kappa^+ = 2^\kappa$. <u>Then</u> for some κ^+-c.c. $(<\kappa)$-complete forcing notion $\mathbb{P}$ of cardinality λ we have $\Vdash_{\mathbb{P}}$ "there is a universal $\kappa-$* s.i.u. *sequence $\bar\eta \in {}^\theta({}^\kappa\kappa)$".*

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