MARTIN'S MAXIMUM AND DEFINABILITY IN $H(\aleph_2)$

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Abstract. In [6], we modified a coding device from [14] and the consistency proof of Martin's Maximum from [3] to show that from a supercompact limit of supercompact cardinals one could force Martin's Maximum to hold while the $\mathbb{P}_{max}$ axiom $(*)$ fails. Here we modify that argument to prove a stronger fact, that Martin's Maximum is consistent with the existence of a wellordering of the reals definable in $H(\aleph_2)$ without parameters, from the same large cardinal hypothesis. In doing so we give a much simpler proof of the original result.

Keywords: Forcing Axioms; Definability; Large Cardinals

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1. Introduction

The following maximal version of Martin's Axiom was introduced by Foreman, Magidor and Shelah in [3].

Definition 1.1. *Martin's Maximum (MM) is the statement that if $\mathbb{P}$ is a partial order such that forcing with $\mathbb{P}$ preserves stationary subsets of ω_1, and $\langle D_\alpha \mid \alpha < \omega_1\rangle$ is a collection of dense subsets of $\mathbb{P}$, then there is a filter $G \subset \mathbb{P}$ meeting each D_α.*

By convention, MM^+ is MM with the further requirement that if τ is a $\mathbb{P}$-name for a stationary set, then $\{\alpha < \omega_1 \mid \exists q \in G\ q \Vdash \alpha \in \tau\}$ is stationary, and MM^{++} is MM^+ but with $\aleph_1$ many names for stationary subsets of ω_1. We let $\mathrm{MM}^{+\omega}$ denote the version of MM^+ with countably many names.

The $\mathbb{P}_{max}$ axiom $(*)$ (Definition 5.1 of [14]) says that the Axiom of Determinacy (AD) holds in the inner model $L(\mathbb{R})$ and that $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$-generic extension

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of $L(\mathbb{R})$ (the partial order $\mathbb{P}_{max}$ is introduced in [14] and is not used in this paper, though Lemmas 4.4-4.6 use some of the $\mathbb{P}_{max}$ machinery). Woodin showed [14] that $\mathrm{MM}^{++}(\mathfrak{c})$ ($\mathrm{MM}^{++}(\kappa)$ is MM^{++} restricted to posets of cardinality κ or less) and $(*)$ are independent over ZFC, assuming the consistency of a strong form of determinacy (see Theorem 10.69 in [14]; that $(*)$ does not imply $\mathrm{MM}^{++}(\mathfrak{c})$ follows from arguments in [7] but was known previously by Woodin). In [6], we used some of the ideas from Woodin's proof that $\mathrm{MM}^{++}(\mathfrak{c})$ does not imply $(*)$ to show that $\mathrm{MM}^{+\omega}$ does not imply $(*)$. In this paper we modify the argument from [6] to produce a model of $\mathrm{MM}^{+\omega}$ in which there is a wellordering of the reals definable in $H(\aleph_2)$ without parameters. This answers a question asked privately by Todorcevic shortly after the results of [6] were announced. Our interest in the question was reawakened by recent work of Asperó ([1], for instance). We note that $(*)$ implies that there is no wellordering of the reals definable in $H(\aleph_2)$ without parameters (this follows immediately from the fact that the $\mathbb{P}_{max}$ extension is a homogeneous extension of a model of AD containing the reals), so $(*)$ fails in the model in this paper also.

We will use a variant of the set of reals $X^{\omega_1}_{(Code)}(\mathcal{S}, z)$ from [6], which is itself a variant of a set of reals from [14] (see Definition 10.22 of [14]). We will call our set of reals $X^2_{(Code)}(\mathcal{S})$. As with the other variants, under $(*)$ this set is equal to $\mathcal{P}(\omega)$ (we will not show this, though we will show that MM^{++} in conjunction with the existence of a Woodin cardinal below a measurable also implies that this set consists of all subsets of ω; modulo standard $\mathbb{P}_{max}$ arguments, the proof in the $\mathbb{P}_{max}$ context is the same). Our forcing construction is an iterated forcing which uses the construction from the consistency proof for MM from [3], adding forcings to make the set $X^2_{(Code)}(\mathcal{S})$ code the parameter $\mathcal{S}$. In the end $\mathcal{S}$ will be the only suitable parameter coding itself via $X^2_{(Code)}$, and will thus be definable in $H(\aleph_2)$. The parameter $\mathcal{S}$ is a partition of ω_1 into $\aleph_1$-many stationary sets, and there are numerous ways to define a wellordering of $\mathcal{P}(\omega_1)$ in $H(\aleph_2)$ from such a parameter under the assumption of MM (the axiom ψ_{AC} from [14] allows this, for instance).

We note that a ladder system on ω_1 (under PFA, [12]) and in fact any subset of ω_1 not constructible from a real (MM + "there exists a measurable cardinal", [9]) can also be used as parameters defining a wellordering of the reals definable in $H(\omega_2)$.

2. $X^2_{(Code)}$

If x is a set of ordinals, we let $ot(x)$ be the ordertype of x. The following definition is due to Woodin [14].

Definition 2.1. *Given $T \subset \omega_1$, $\tilde{T}$ is the set of $\beta \in [\omega_1, \omega_2)$ such that there exist a bijection $f : \omega_1 \to \beta$ and a club $C \subset \omega_1$ such that for all $\alpha \in C$, $ot(f[\alpha]) \in T$.*

If T is a subset of ω_1, then $\tilde{T}$ is the set of ordinals in the interval $[\omega_1, \omega_2)$ which are necessarily in the image of T by any embedding derived from forcing with the nonstationary ideal on ω_1 (which we denote by NS_{ω_1}). It is a standard fact ([7], for instance), and not too hard to show, that if A is a subset of ω_1 added generically by initial segments, then $\tilde{A} = \emptyset$, though this can be changed in further forcing extensions.

We denote by C^κ_β the set of the ordinals less than β of cofinality κ, where κ is a regular cardinal and β is an ordinal.

Definition 2.5 below is a variant of the one used in [6], which in turn is a variant of one used in [14] to show that $\mathrm{MM}^{++}(\mathfrak{c})$ does not imply $(*)$. The real z from the definition in [6] is always the empty real, so we drop it. We also add two more conditions to the definition, making it harder for reals to enter $X^2_{(Code)}(\mathcal{S})$.

Given an infinite $x \subset \omega$ we let $oe(x)$ denote the set

$$\{j < \omega \mid \text{ the } (j+3)\text{rd element of } x \text{ is even}\}$$

(here we mean that the 1st element of a set is its least element). The term $j+3$ in the definition ensures that for any pair $n, m \in \omega$,

$$\{oe(x) \mid x \in [\omega]^\omega \wedge \{n, m\} \subset x\} = \mathcal{P}(\omega)$$

(if we replaced $j+3$ with $j+1$, $\{0,1\} \subset x$ would imply that $0 \in oe(x)$ and $1 \notin oe(x)$, for infinite $x \subset \omega$). We note the following fact about oe, which we leave to the reader.

Lemma 2.2. *If x and y are subsets of ω, x is infinite and $oe(x)$ is infinite and co-infinite, then there exists an infinite $x' \subset x$ such that $oe(x') = y$.*

Let $t \colon \omega \to (2^{<\omega} \setminus \{\emptyset\})$ be the listing of the nonnull members of $2^{<\omega}$ that lists shorter sequences before longer ones, and which lists sequences of the same length in lexicographical order. We let $\mathbb{C}$ denote the set of $x \subset \omega$ such that for some $y \in 2^{\omega}$, $\{t(i) : i \in x\} = \{y \restriction n : 0 < n < \omega\}$ (we say in this case that x *codes* y *via* t, and we let t^* be the function on $\mathbb{C}$ taking $x \in \mathbb{C}$ to the real coded by x via t). Note that $\mathbb{C}$ is a Borel set, so membership in $\mathbb{C}$ is absolute between models of set theory containing ω_1. We note some additional facts about t and $\mathbb{C}$, left to the reader. The second part of the following lemma follows from the first.

Lemma 2.3. *Let y_0, y_1 and z be distinct subsets of ω.*

(1) If y_0 and y_1 are infinite and co-infinite, then

$$\lim_{n\to\infty} |t^{-1}(y_0 \restriction n) - t^{-1}(y_1 \restriction n)| = \infty;$$

(2) If $a \subset z$ is nonempty and finite, $oe(z) \in \mathbb{C}$ and $t^*(oe(z))$ is infinite and co-infinite, then $oe(z \setminus a) \notin \mathbb{C}$.

We let $li(x, \alpha)$ denote the least Silver indiscernible for $L[x]$ above α, when x is a real whose sharp exists and α is an ordinal. If A is a set of reals, the A-*uniform indiscernibles* are the ordinals which are indiscernibles for each member of A. We say *uniform indiscernibles* when A is the set of all reals, and M-*indiscernibles* for $\mathbb{R}^M$-indiscernibles when M is a model. By convention, u_2 denotes the second uniform indiscernible, which in our context, where the sharp of each real exists, is equal to $\sup\{li(x, \omega_1) : x \in \mathbb{R}\}$.

For any ordinal γ, a *cofinality function for* γ is a continuous, increasing cofinal function $f \colon cof(\gamma) \to \gamma$. The following abuse of terminology merits it own name.

Definition 2.4. *If γ_0 and γ_1 are ordinals of cofinality ω_1 and $A_0 \subset \gamma_0$ and $A_1 \subset \gamma_1$ are stationary in γ_0 and γ_1 respectively, then we say that A_0 and A_1 have* abused stationary intersection *if there exist cofinality functions f_0 for γ_0 and f_1 for γ_1 such that the set*

$$\{\eta < \omega_1 : f_0(\eta) \in A_0 \text{ and } f_1(\eta) \in A_1\}$$

is stationary.

The choice of f_0 and f_1 in Definition 2.4 is irrelevant (i.e., an equivalent notion is obtained by replacing "there exist" with "for all" and removing "such that").

The following definition is meant to be applied in the case where the sharp of every real exists.

Definition 2.5. *Suppose that $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ is a collection of pairwise disjoint stationary subsets of ω_1. We associate to $\mathcal{S}$ two subsets of $\mathcal{P}(\omega)$,*

$$X^2_{(Code)}(\mathcal{S}) = \cup\{X_\gamma : \gamma < \omega_2\} \qquad \text{and} \qquad Y^2_{(Code)}(\mathcal{S}) = \cup\{Y_\gamma : \gamma < \omega_2\},$$

where $\langle(\kappa_\gamma, X_\gamma, Y_\gamma) : \gamma < \omega_2\rangle$ is the sequence generated from $\mathcal{S}$ as follows.

(i) $Y_0 = \{\emptyset\}$, $X_0 = \emptyset$, *and* $\kappa_0 = li(\emptyset, \omega_1)$.

(ii) For all nonzero $\gamma < \omega_2$, κ_γ is the least Y_γ-uniform indiscernible η such that $k_\alpha < \eta$ for all $\alpha < \gamma$.

(iii) Suppose γ is not the successor of an ordinal of cofinality ω_1. Then

$$X_\gamma = \cup\{X_\alpha : \alpha < \gamma\} \qquad \text{and} \qquad Y_\gamma = \cup\{Y_\alpha : \alpha < \gamma\}.$$

(iv) Suppose γ has cofinality ω_1. For each $\alpha < \omega_1$, let

$$b_\alpha = oe(\{i < \omega \mid \tilde{S}_{\alpha i} \cap \kappa_\gamma \text{ is stationary}\})$$

if this is defined, and $\emptyset$ otherwise. Then $Y_{\gamma+1} = Y_\gamma \cup \{b_\alpha : \alpha < \omega_1\}$.

Furthermore, suppose the following hold.

a) For all $\alpha < \omega_1$ and $i < \omega$, $\tilde{S}_{\alpha i} \cap \kappa_\gamma$ and $\tilde{S}_{\alpha(i+1)} \cap \kappa_\gamma$ are not both stationary.

b) Each $b_\alpha \in \mathbb{C}$.

c) $li(b_1, \omega_1) > \kappa_\gamma$.

d) The sequence $\langle li(b_\alpha, \omega_1) : \alpha < \omega_1 \rangle$ *is strictly increasing.*

e) For all $i, i' < \omega$*, for all* $\alpha, \alpha' < \omega_1$*, for all* $\beta \in C^{\omega_1}_\gamma$*, if* $\tilde{S}_{\alpha i} \cap \kappa_\gamma$ *is stationary in* κ_γ *and* $\tilde{S}_{\alpha' i'} \cap \kappa_\beta$ *is stationary in* κ_β*, then these sets have abused stationary intersection.*

Then $X_{\gamma+1} = X_\gamma \cup \{t^*(b_0)\}$*. Otherwise,* $X_{\gamma+1} = X_\gamma$*.*

When the set $\mathcal{S}$ is clear from context, we will simply refer to κ_γ, X_γ and Y_γ individually as needed, sometimes using κ^M_γ (for instance) for this set as computed in a given model M. When $\mathcal{S}$ is not clear, we write $\kappa_\gamma(\mathcal{S})$, etc. We will also refer to $\langle \kappa_\gamma : \gamma < \omega_2 \rangle$ as the *κ-sequence* of $\mathcal{S}$, $\langle Y_\gamma : \gamma < \omega_2 \rangle$ as the Y-sequence, and so on. The sequence $\langle b_\alpha : \alpha < \omega_1 \rangle$ will likewise be called the *b-sequence at stage* γ or sometimes the *$\mathcal{S}$-coding at stage* γ. We let $STAT(\mathcal{S}, \gamma)$ (for some $\gamma \leq \omega_2$) denote the collection of stationary sets of the form $\{\delta < \omega_1 \mid f(\delta) \in \tilde{S}_{\alpha i}\}$, for some $\beta \in C^{\omega_1}_\gamma$, $\alpha < \omega_1$, $i < \omega$ and some cofinality function f for κ_β.

Condition (ive) in Definition 2.5 ensures that if we shoot a club through any stationary set of the form $\tilde{S}_{\alpha i} \cap \kappa_\gamma$, or any union of such stationary sets, we don't don't destroy the stationarity of any earlier $\tilde{S}_{\alpha' i'} \cap \kappa_\beta$, and therefore don't change the b-sequence at any earlier stage. In practice, when we do shoot such a club, $STAT(\mathcal{S}, \gamma)$ will contain all the stationary subsets of ω_1 appearing in models from proper initial segments of our iteration so far (see Lemmas 4.7 and 4.8).

Since the definition of the $\mathcal{S}$-coding at a given stage is absolute between models which agree about stationary subsets of ω_1 (and whose ω_2 is large enough), if $M \subset N$ are models of ZFC such that $NS^M_{\omega_1} = NS^N_{\omega_1} \cap M$, then $X^2_{(Code)}(\mathcal{S})^M \subset X^2_{(Code)}(\mathcal{S})^N$ for any $\mathcal{S} \in M$. Furthermore, if $\langle M_\alpha : \alpha \leq \beta \rangle$ is an increasing sequence of models of ZFC which agree about stationary subsets of ω_1, and ${\omega_2}^{M_\beta} = sup\{\omega_2^{M_\alpha} : \alpha < \beta\}$, then $X^2_{(Code)}(\mathcal{S})^{M_\beta} = \bigcup\{X^2_{(Code)}(\mathcal{S})^{M_\alpha} : \alpha < \beta\}$.

We note two more useful facts about Definition 2.5. The first follows from condition (ivc) and the fact that $li(x, \omega_1) < \omega_2$ for any real x, and the second follows from conditions (ii) and (ivc).

Lemma 2.6. *Suppose that $M \subset N$ are models of ZFC such that $\mathbb{R}^M = \mathbb{R}^N$ and $NS^M_{\omega_1} = M \cap NS^N_{\omega_1}$, and let $\mathcal{S} \in M$ be a partition of ω_1^M into stationary sets indexed by $\omega_1^M \times \omega$. Then $X^2_{(Code)}(\mathcal{S})^M = X^2_{(Code)}(\mathcal{S})^N$.*

Lemma 2.7. *Let $\mathcal{S}$ be a partition of ω_1 into stationary sets indexed by $\omega_1 \times \omega$. Suppose that $\alpha < \omega_2$ is such that $\sup\{li(x, \omega_1) : x \in Y_\alpha\} = u_2$. Then for all $\beta \in [\alpha, \omega_2)$, $X_\beta = X_\alpha$.*

3. Standards

Definition 3.1. *Given a set X, $c \subset \mathcal{P}_{\aleph_1}(X)$ is* closed unbounded *(club) if there is a function $F\colon [X]^{<\omega} \to X$ such that c is the set of countable subsets of X closed under F. A set $a \subset \mathcal{P}_{\aleph_1}(X)$ is* stationary *if it intersects every club $c \subset \mathcal{P}_{\aleph_1}(X)$.*

Definition 3.2. *A partial order P is* semi-proper *if whenever p is a condition in P, $X \prec H((2^{|P|})^+)$ is countable and $p, P \in X$, there is a $q \leq p$ in P forcing that the realization of every P-name in X for a countable ordinal is in X.*

The following statement follows from MM and implies that every forcing which preserves stationary subsets of ω_1 is semi-proper [3].

Definition 3.3. *The Weak Reflection Principle (WRP) is the statement that for all cardinals $\lambda \geq \omega_2$ and for all stationary $Z \subset \mathcal{P}_{\aleph_1}(\lambda)$, there exists a $Y \in \mathcal{P}_{\aleph_2}(\lambda)$ containing ω_1 such that $Z \cap \mathcal{P}_{\aleph_1}(Y)$ is stationary in $\mathcal{P}_{\aleph_1}(Y)$.*

Given a cardinal γ and a set X, $Coll(\gamma, X)$ is the partial order consisting of partial functions from γ to X with domain of cardinality less than γ, ordered by inclusion. Given a cardinal γ and an ordinal η, $Coll(\gamma, {<}\eta)$ is the partial order consisting of functions p from $\gamma \times \eta$ to η with domain of cardinality less than γ and the stipulation that $p(\alpha, \beta) \in \beta$, ordered by inclusion.

We refer the reader to [5] for the definitions of the large cardinal concepts used in this paper (measurable, Woodin, (λ-)supercompact). The following well-known fact is due to Foreman, Magidor and Shelah and follows easily from the arguments in [3].

Lemma 3.4. *Assume that η is a supercompact cardinal. Then WRP holds after forcing with $Coll(\omega_1, <\eta)$.*

An ideal I on ω_1 is *saturated* if the Boolean algebra $\mathcal{P}(\omega_1)/I$ has no antichains of cardinality $\aleph_2$. Such an I *presaturated* if whenever $A \in I^+$ and $\mathcal{B}_i$ $(i < \omega)$ are maximal antichains in $\mathcal{P}(A)/I$ there exists an $A' \subset A$ in I^+ such that for each $i < \omega$, $|\{B \in \mathcal{B}_i : B \cap A' \in I^+\}| = \aleph_1$. Presaturation is weaker than saturation, and saturation of NS_{ω_1} follows from MM($\mathfrak{c}$) [3] (indeed, from the restriction of MM($\mathfrak{c}$) to posets which do not add reals). Presaturation also implies that $j(\omega_1) = \omega_2$, where j is any generic elementary embedding derived from forcing with $\mathcal{P}(\omega_1)/I$.

We will use the two following facts, due to Shelah and Woodin, respectively.

Theorem 3.5. *([13]) If δ is a Woodin cardinal, there there exists a semi-proper forcing of cardinality δ forcing that NS_{ω_1} is saturated.*

Theorem 3.6. *([14]) If NS_{ω_1} is saturated and there exists a measurable cardinal, then $u_2 = \omega_2$.*

As in the original consistency proof of Martin's Maximum, we will use the following result of Laver [11].

Theorem 3.7. *Let κ be a supercompact cardinal. Then there exists a function $L : \kappa \to V_\kappa$ such that for every set X and every cardinal λ there exists a λ-supercompact embedding $j : V \to M$ such that $j(L)(\kappa) = X$.*

We will be using iterations of semi-proper forcing with Revised Countable Support [13]. The following theorem from [13] (which we have rewritten in our own terminology) ensures that our forcing will be semi-proper.

Theorem 3.8. *Suppose that $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \delta\rangle$ is an RCS iteration, and that for every $\beta < \delta$, for arbitrarily large non-limit $\alpha < \beta + 1$, $P_{\beta+1}/P_\alpha$ is semi-proper, and for every $\alpha < \delta$, the empty condition in $P_{\alpha+n}$ (for some $n < \omega$) forces that $|P_\alpha| = \aleph_1$. Then P_δ is semi-proper.*

We will be taking advantage of a degree of freedom offered by Theorem 3.8: when β is a limit ordinal, the theorem does not require that Q_β preserve stationary subsets of ω_1 which are in the P_β-extension but in no earlier P_α-extension. Our situation is a special case. We will have WRP (and thus that partial orders preserving stationary subsets of ω_1 are semi-proper) in each nonlimit P_α-extension, so the semi-properness of each $P_{\beta+1}$ will follow from the fact that for each nonlimit $\alpha < \beta$, P_β/P_α is semi-proper and Q_β preserves stationary subsets of ω_1 from P_α.

4. Protection and Erasure

The first half of this section, Lemmas 4.1-4.6, deals with ways to put stationary sets into $STAT(\mathcal{S}, \omega_2)$ without adding reals to $X^2_{(Code)}(\mathcal{S})$. The basic lemma for doing this we call the Protection Lemma (Lemma 4.2 below). The proof of the Protection Lemma is based on ideas which have become standard. The lemma itself is a variation of Lemma 10.66 of [14]. For the arguments in Section 6 we will need more information about the forcings introduced here (hence Lemma 4.6).

We first note an end-extension property which is the basis for these arguments.

Lemma 4.1. *Suppose that $\langle \rho_\gamma : \gamma \leq \eta \rangle$ is a continuous increasing sequence of cardinals such that $\eta < \rho_0$ and ρ_γ is a measurable cardinal for each nonlimit γ. Let S_γ ($\gamma \leq \eta$) be stationary subsets of ω_1. Suppose that $\chi \geq (2^{\rho_\eta})^+$ is a regular cardinal, and let Y be a countable elementary substructure of $H(\chi)$ containing the sequence $\langle \rho_\gamma : \gamma \leq \eta \rangle$. Let ζ be any ordinal in the interval $[\eta, \rho_0)$. Then there exists a countable $Y' \prec H(\chi)$ containing Y such that*

- *$Y' \cap \rho_0$ end-extends $Y \cap \rho_0$;*
- *$Y' \cap \zeta = Y \cap \zeta$;*
- *$ot(Y' \cap \rho_\gamma) \in S_\gamma$ for each $\gamma \in Y \cap (\eta + 1)$.*

Proof. By induction on η. When η is a successor ordinal, the induction step follows immediately from a standard end-extension property of measurable cardinals (Lemma 1.1.21 of [8], say). Now suppose that η is a limit ordinal, and fix Y as in the statement of the lemma. Let Z be a countable elementary submodel of

$H((2^\chi)^+)$ with Y and $\langle S_\gamma : \gamma \leq \eta\rangle$ as elements and $Z \cap \omega_1 \in S_\eta$. Let $\langle \nu_i : i < \omega\rangle$ be an increasing cofinal sequence in $Z \cap \eta$ (with $\nu_0 = 0$) and let $\langle \alpha_i : i < \omega\rangle$ be an increasing cofinal sequence in $Z \cap \omega_1$. Let $Y_0 \in Z$ satisfy the lemma with respect to Y, ζ, $\langle S_\gamma : \gamma \leq 0\rangle$ and $\langle \rho_\gamma : \gamma \leq 0\rangle$.

Successively choose Y_i $(0 < i < \omega)$ in Z such that each Y_{i+1} satisfies the lemma for Y_i, ρ_{ν_i} (in the role of ζ), $\langle S_\gamma : \nu_i < \gamma \leq \nu_{i+1}\rangle$ and $\langle \rho_\gamma : \nu_i < \gamma \leq \nu_{i+1}\rangle$ and satisfies $ot(Y_{i+1} \cap \rho_{\nu_{i+1}}) \geq \alpha_i$ (which we can do by replacing S_{ν_i+1} with $S_{\nu_i+1} \setminus \alpha_i$, for instance). Then since the sets $Y_i \cap \rho_{\nu_i}$ end-extend one another, the union of these sets has ordertype $Z \cap \omega_1$, so the union of $\{Y_i : i < \omega\}$ satisfies the lemma for Y, $\langle S_\gamma : \gamma \leq \eta\rangle$, and $\langle \rho_\gamma : \gamma \leq \eta\rangle$. $\square$

Lemma 4.2. *(Protection Lemma) Suppose that $\mathcal{S}$,$\mathcal{T}$ are collections of stationary subsets of ω_1, and that the members of $\mathcal{T}$ are pairwise disjoint. Let π be a function from $\mathcal{T}$ to $\mathcal{S}$, and let $\langle \rho_\gamma : \gamma \leq \omega_1\rangle$ be a continuous, increasing sequence of cardinals such that ρ_γ is a measurable cardinal for each nonlimit γ.*

Then there is a (ω, ∞)-distributive semi-proper forcing in whose extension, for each $T \in \mathcal{T}$ and each $\gamma \in T$, $\rho_\gamma \in \tilde{S}$ if $S = \pi(T)$.

Proof. Let P be the set of countable, continuous, $\subset$-increasing sequences $\langle x_\nu : \nu \leq \mu\rangle \subset [\rho_{\omega_1}]^{\aleph_0}$ such that for each $\nu \leq \mu$, and each $\gamma \in x_\nu \cap \omega_1$, if there exists a $T \in \mathcal{T}$ with $\gamma \in T$, then $ot(x_\nu \cap \rho_\gamma) \in \pi(T)$. Forcing with this partial order has the desired effect by standard arguments, almost straight from the definition of $\tilde{S}$.

Lemma 4.1 gives that P is semi-proper and (ω, ∞)-distributive, as follows. Fix a condition p in a countable elementary submodel Y of $H((2^{\rho_{\omega_1}})^+)$ having P and $\langle \rho_\gamma : \gamma \leq \omega_1\rangle$ as members. By Lemma 4.1, there exists a countable $Y' \prec H((2^{\rho_{\omega_1}})^+)$ containing Y with $Y' \cap \omega_1 = Y \cap \omega_1$ and, for each $\gamma \in Y' \cap \omega_1$, $ot(Y' \cap \rho_\gamma) \in \pi(T)$ whenever $T \in \mathcal{T}$ and $\gamma \in T$. Then if $\bar{p}$ is any descending ω-sequence of conditions in $P \cap Y'$ extending p, such that $\bar{p}$ meets every dense subset of P in Y', then $(\bigcup \bar{p})^\frown\langle Y' \cap \rho_{\omega_1}\rangle$ is a (Y, P)-semi-generic condition extending p. $\square$

Suppose that $\langle E_\alpha : \alpha < 2^{\omega_1}\rangle$ is a listing of $NS^+_{\omega_1}$ and that g is a V-generic filter for $Coll(\omega_1, 2^{\omega_1})$ (note that this forcing makes CH hold). Then the sets $\{\gamma \in E_\alpha : g(\gamma) = \alpha\}$ $(\alpha < (2^{\omega_1})^V)$ are pairwise disjoint and stationary. If κ is a cardinal greater than $2^{\aleph_1}$, then, in the $Coll(\omega_1, {<}\kappa)$-extension there exists a partition $\mathcal{T}$ of ω_1 into stationary sets such that every stationary subset of the ground model contains a member of $\mathcal{T}$ modulo NS_{ω_1}. In such an extension, given an uncountable collection $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ of stationary subsets of ω_1, a listing $\langle x_\alpha : \alpha < \omega_1\rangle$ of $\mathcal{P}(\omega)$ and reals $\langle y_\alpha : \alpha < \omega_1\rangle$ such that each $oe(y_\alpha) = x_\alpha$, we let $\mathbb{PF}(\mathcal{S})$ be the forcing as in the Protection Lemma, with $\mathcal{T}$ as a partition as in the previous sentence, $\pi\colon \mathcal{T} \to \mathcal{S}$ any injection whose range is the set of $S_{\alpha i}$ such that $i \in y_\alpha$, and $\langle \rho_\gamma : \gamma \leq \omega_1\rangle$ the closure of the first ω_1 many measurable cardinals. Since neither of $Coll(\omega_1, {<}\kappa)$ and $\mathbb{PF}(\mathcal{S})$ add reals or destroy stationary subsets of ω_1, they do not add reals to $X^2_{(Code)}(\mathcal{S})$ or change $X^2_{(Code)}(\mathcal{S})$ or $Y^2_{(Code)}(\mathcal{S})$ below ω_2^V. Furthermore, in this extension the ordinal $\kappa_{\rho_{\omega_1}}(\mathcal{S})$ is equal to ρ_{ω_1}, each x_α is in $Y^2_{(Code)}(\mathcal{S})$ and, for every stationary $E \subset \omega_1$ in the ground model, there exists $(\alpha, i) \in \omega_1 \times \omega$ such that, for any cofinality function f for ρ_{ω_1}, the set of $\alpha \in \omega_1$ such that $f(\alpha) \in \tilde{S}_{\alpha i}$ is a stationary set contained modulo NS_{ω_1} in E.

Our forcing iterations will use the partial order $Coll(\omega_1, {<}\mu) * \mathbb{PF}(\mathcal{S})$ at every successor step, where μ is the least supercompact cardinal. This and condition (ive) of Definition 2.5 will allow us to undo accidentally coded reals by destroying the stationarity of certain subsets of ω_1 while preserving the stationarity of subsets of ω_1 added by initial segments of the iteration. It will also help us put reals into $X^2_{(Code)}$.

The Protection Lemma allows us to code stationary subsets of ω_1 into sets of the form $\tilde{S}_{\alpha i} \cap \kappa_\gamma$. In Section 5, we will need to code stationary sets of ordertype ω_2 into sets of this form. The following modified form of the Protection Lemma does this.

Lemma 4.3. *Suppose that κ is a strongly inaccessible cardinal and $\langle \rho_\alpha : \alpha \leq \kappa\rangle$ is a continuous increasing sequence of cardinals with supremum κ such that ρ_α is* *measurable for each nonlimit α. Let $\mathcal{S}$ be a collection of stationary subsets of ω_1. Then there is a semi-proper (ω,∞)-distributive partial order forcing that $\rho_\kappa = \omega_2$ and that $\tilde{S} \cap C^\omega_{\rho_\kappa}$ is stationary for each S in $\mathcal{S}$.*

Proof. This is largely the same as the proof of the Protection Lemma. Partition C^ω_κ into stationary sets E_S ($S \in \mathcal{S}$). A condition in our partial order P consists of a domain F - a countable subset of $\kappa \setminus \rho_0$ - and, for each $\eta \in F$ a countable, continuous, $\subset$-increasing sequence $\langle x^\eta_\nu : \nu \leq \mu\rangle \subset [\eta]^{\aleph_0}$ such that for each $\nu \leq \mu$, and each $\gamma \in x^\eta_\nu \cap C^\omega_\kappa$ with $\rho_\gamma < \eta$, $ot(x^\eta_\nu \cap \rho_\gamma) \in S$, where $S \in \mathcal{S}$ is such that $\gamma \in E_S$. Stronger conditions in this forcing extend the domain and extend the sequences for each member of the domain of the weaker condition. Forcing with this partial order has the desired effect.

Lemma 4.1 gives that P is semi-proper and (ω,∞)-distributive. Fix a condition p in a countable elementary submodel Y of $H((2^\kappa)^+)$ having P, $\mathcal{S}$ and $\{\rho_\alpha : \alpha < \kappa\}$ as members. By Lemma 4.1, there exists a countable $Y' \prec H((2^\kappa)^+)$ containing Y with $Y' \cap \omega_1 = Y \cap \omega_1$ and for each $\gamma \in Y' \cap \kappa$, $ot(Y' \cap \rho_\gamma) \in S$ if $S \in \mathcal{S}$ and $\gamma \in E_S$. Let $\langle p_i : i < \omega\rangle$ be a descending sequence of conditions in $P \cap Y'$ (with $p_0 = p$) meeting every dense subset of P in Y'. Then the condition consisting of domain $Y' \cap \kappa$ and, for each $\eta \in Y' \cap \kappa$, the sequence $p'(\eta)^\frown\langle Y' \cap \eta\rangle$ (where $p'(\eta)$ is the union of the sequences for η in the p_i's) is a (Y, P)-semi-generic condition extending p. $\square$

Lemma 4.5 below is relatively standard. We will not apply this lemma directly, but we will use the construction in the proof of the lemma to derive an extra property of forcings of the type used in the Protection Lemma (in Lemma 4.6). Similar arguments appear in [6, 7, 14] (where one can also find the definitions of *iteration* and *iterable*). First we will give a quick proof of another standard fact, which is the key to the proof of Lemma 4.5.

Lemma 4.4. *Suppose that M is a countable transitive model of ZFC in which NS_{ω_1} is presaturated and $u_2 = \omega_2$. Let $\langle (M_\alpha, I_\alpha), G_\beta : \beta < \gamma, \alpha \leq \gamma\rangle$ and $j_{\alpha\beta} \colon M_\alpha \to M_\beta$*

$(\alpha \leq \beta \leq \gamma)$ make up an iteration of $(M, NS^M_{\omega_1})$. Then $j_{0\gamma}[\omega_2^M]$ is cofinal in $\omega_2^{M_\gamma}$ and the critical sequence of $j_{0,\gamma}$ is the set of all the M-uniform indiscernibles ξ in the interval $[\omega_1^M, \omega_1^{M_\gamma})$.

Proof. This is easily proved by induction on γ, using the fact that for each $\alpha < \gamma$, $j(\omega_2^{M_\alpha})$ is the least M-uniform indiscernible above $\omega_2^{M_\alpha}$. This in turn follows from the fact that no $f \colon \omega_1^{M_\alpha} \to \omega_2^{M_\alpha}$ could represent the least M-uniform indiscernible above $\omega_2^{M_\alpha}$, since the range of each such function is bounded by the least indiscernible above $\omega_1^{M_\alpha}$ of some real in M. $\square$

Lemma 4.5. *Suppose that NS_{ω_1} is presaturated, $u_2 = \omega_2$ and β is an ordinal. Suppose that $\bar{\xi} = \langle \xi_\alpha : \alpha \leq \beta \rangle$ is an increasing sequence of uniform indiscernibles, and $\bar{S} = \langle S_\alpha : \alpha \leq \beta \rangle$ is a sequence of stationary subsets of ω_1. Suppose that there exists a measurable cardinal greater than ξ_β. Then the set of countable $x \subset \xi_\beta$ such that $ot(x \cap \xi_\alpha) \in S_\alpha$ for all $\alpha \in (\beta + 1) \cap x$ is stationary.*

Proof. Let $\kappa > \xi_\beta$ be measurable and let $\theta > 2^\kappa$ be regular. Fix a function $F \colon [\xi_\beta]^{<\omega} \to \xi_\beta$. Let T be the tree of attempts to build a countable $x \subset \xi_\beta$ closed under F and an increasing sequence of ordinals $\langle \delta_\alpha : \alpha \in (\beta + 1) \cap x \rangle$ such that each $\delta_\alpha \in S_\alpha$ and such that $ot(x \cap \xi_\alpha) = \delta_\alpha$ for all $\alpha \leq \beta$ (so T is a tree of height ω using some fixed set of bijections between ω and each countable ordinal). Let X be a countable elementary submodel of $H(\theta)$ with κ, F, $\bar{\xi}$ and $\bar{S}$ as elements. Let M be the transitive collapse of X, and for each $\alpha \in (\beta + 1) \cap X$, let ξ^*_α and S^*_α be the images of ξ_α and S_α respectively under this collapse. Let T^* be the image of T. By standard arguments ([14], see also [10, 2]) using the presence of the measurable cardinal κ, $(M, NS^M_{\omega_1})$ is iterable. Furthermore, each ξ^*_α is an M-uniform indiscernible, and every M-uniform indiscernible is on the critical sequence of every sufficiently long iteration of $(M, NS^M_{\omega_1})$, by Lemma 4.4. Let $j \colon (M, NS^M_{\omega_1}) \to (N, NS^N_{\omega_1})$ be an iteration of $(M, NS^M_{\omega_1})$ such that for each $\alpha \in (\beta + 1) \cap X$, $\xi^*_\alpha \in j(S^*_\alpha)$. Then $j(T^*)$ has an infinite branch corresponding to the sequence $\langle \xi^*_\alpha : \alpha \in (\beta + 1) \cap X \rangle$ and the set $j[\xi^*_\beta]$. This set is not necessarily in N, but since N is wellfounded, there must be a branch through $j(T^*)$ in N, which by elementarity means that there is a branch through T. $\square$

Putting together the proofs of Lemma 4.5 and the Protection Lemma we have Lemma 4.6 below. The proof uses a forcing to add a partition of ω_1 indexed by $\omega_1 \times \omega$ by initial segments. Conditions in this forcing are partitions $\langle c_{\alpha i} : (\alpha, i) \in \zeta \times \omega \rangle$ of a countable ordinal χ, for some countable ordinal ζ. We say that

$$\langle c_{\alpha i} : (\alpha, i) \in \zeta \times \omega \rangle \leq \langle d_{\alpha i} : (\alpha, i) \in \xi \times \omega \rangle$$

if $\zeta \geq \xi$ and $c_{\alpha i} \cap \theta = d_{\alpha i}$ for each $(\alpha, i) \in \xi \times \omega$, where $\langle d_{\alpha i} : (\alpha, i) \in \xi \times \omega \rangle$ is a partition of θ. The forcing P_0 in the statement of Lemma 4.6 is the product of this forcing with itself.

Lemma 4.6. *Suppose that NS_{ω_1} is presaturated and $u_2 = \omega_2$, and let*

$$\langle \kappa_\alpha : \alpha \leq \omega_1 \cdot 2 \rangle$$

be a continuous increasing sequence of cardinals such that κ_α is a measurable cardinal for each nonlimit α. Let $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ and $\mathcal{T} = \langle T_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ be partitions of ω_1 into stationary sets.

Let P_0 be the forcing which adds partitions of ω_1 $\langle A_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ and $\langle B_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ by initial segments.

Let P_1 be the forcing in the P_0-extension consisting of all countable continuous, increasing sequences $\langle x_\gamma : \gamma \leq \delta \rangle \subset [\kappa_{\omega_1 \cdot 2}]^{\aleph_0}$ such that for each $\gamma \leq \delta$, each $\eta \in x_\gamma \cap \omega_1$, each $\alpha < \omega_1$ and each $i < \omega$, $ot(x_\gamma \cap \kappa_\eta) \in T_{\alpha i}$ if $\eta \in A_{\alpha i}$ and $ot(x_\gamma \cap \kappa_{\omega_1 + 1 + \eta}) \in S_{\alpha i}$ if $\eta \in B_{\alpha i}$ (ordered by extension).

Then $P_0 * P_1$ is (ω, ∞)-distributive, preserves stationary subsets of ω_1 and forces that for every V-indiscernible ξ not in $\{\omega_1\} \cup \{\kappa_\alpha : \alpha \leq 2 \cdot \omega_1\}$, every stationary $K \subset \omega_1$ in the ground model, all $\alpha, \beta < \omega_1$ and all $i, j < \omega$, the set of countable $x \subset \kappa_{\omega_1 \cdot 2}$ such that

$$x \cap \omega_1 \in A_{\alpha i} \cap B_{\beta j} \text{ and } ot(x \cap \xi) \notin K$$

is stationary.

Proof. Let λ be a regular cardinal greater than $2^{\kappa_{\omega_1 \cdot 2}}$. Given a condition $(\bar{a}, \bar{b})$ in P_0, say that a set X is $(\bar{a}, \bar{b})$-*good* if it is a countable elementary submodel of $H(\lambda)$ with $(\bar{a}, \bar{b})$ as an element and there exist (X, P_0)-generic

$$(\langle a_{\alpha i} : (\alpha, i) \in (X \cap \omega_1) \times \omega \rangle, \langle b_{\alpha i} : (\alpha, i) \in (X \cap \omega_1) \times \omega \rangle)$$

extending $(\bar{a}, \bar{b})$ such that for each $\eta \in X \cap \omega_1$, each $\alpha < \omega_1$ and each $i < \omega$, $ot(X \cap \kappa_\eta) \in T_{\alpha i}$ if $\eta \in a_{\alpha i}$ and $ot(X \cap \kappa_{\omega_1 + 1 + \eta}) \in S_{\alpha i}$ if $\eta \in b_{\alpha i}$.

Let E and K be stationary subsets of ω_1. Fix a V-indiscernible ξ not in the set $\{\omega_1\} \cup \{\kappa_\alpha : \alpha \leq 2 \cdot \omega_1\}$, countable ordinals α, β and integers $i, j < \omega$. It suffices to show that for every condition $(\bar{a}, \bar{b}) \in P_0$ the following set is stationary: the set of $(\bar{a}, \bar{b})$-good X such that

$$X \cap \omega_1 \in a_{\alpha i} \cap b_{\beta j} \cap E \text{ and } ot(X \cap \xi) \notin K,$$

for some $(\langle a_{\gamma k} : (\gamma, k) \in (X \cap \omega_1) \times \omega \rangle, \langle b_{\gamma k} : (\gamma, k) \in (X \cap \omega_1) \times \omega \rangle)$ extending $(\bar{a}, \bar{b})$ and witnessing that X is $(\bar{a}, \bar{b})$-good.

To see that this set is stationary, fix a function $F \colon [H(\lambda)]^{<\omega} \to H(\lambda)$. Let T be the tree of attempts to build an $(\bar{a}, \bar{b})$-good X closed under F and a witness

$$(\langle a_{\gamma k} : (\gamma, k) \in (X \cap \omega_1) \times \omega \rangle, \langle b_{\gamma k} : (\gamma, k) \in (X \cap \omega_1) \times \omega \rangle)$$

extending $(\bar{a}, \bar{b})$ such that $X \cap \omega_1 \in a_{\alpha i} \cap b_{\beta j} \cap E$ and $ot(X \cap \xi) \notin K$.

Let $\theta > 2^\lambda$ be regular and let $Y \prec H(\theta)$ be countable with λ, P_0, $(\bar{a}, \bar{b})$, $\mathcal{S}$, $\mathcal{U}$, α, β, E, K and F in Y. Let $(\langle a_{\gamma k} : (\gamma, k) \in (Y \cap \omega_1) \times \omega \rangle, \langle b_{\gamma k} : (\gamma, k) \in (Y \cap \omega_1) \times \omega \rangle)$ be a (Y, P_0)-generic condition in P_0 extending $(\bar{a}, \bar{b})$, with $Y \cap \omega_1 \in a_{\alpha i} \cap b_{\beta j}$.

Let M be the transitive collapse of Y, and let $h \colon Y \to M$ be the collapsing function. As in the proof of Lemma 4.5, $(M, NS^M_{\omega_1})$ is iterable, and every M-uniform indiscernible is on the critical sequence of every sufficiently long iteration of (M, NS_{ω_1}).

For each $\eta \in Y \cap \omega_1$, $h(\kappa_\eta)$ and $h(\kappa_{\omega_1 + 1 + \eta})$ are both M-indiscernibles, as is $h(\xi)$.

Let $j \colon (M, NS^M_{\omega_1}) \to (N, NS^N_{\omega_1})$ be an iteration of $(M, NS^M_{\omega_1})$ such that

- $h(\omega_1) \in j(h(E))$;
- for each $\delta, \eta \in \omega_1 \cap Y$ and each $i < \omega$, $h(\kappa_\eta) \in j(h(T_{\delta i}))$ if $\eta \in a_{\delta i}$ and $h(\kappa_{\omega_1+1+\eta}) \in j(h(S_{\delta i}))$ if $\eta \in b_{\delta i}$;
- $h(\xi) \notin j(h(K))$.

Then $j(h(T))$ has an infinite branch corresponding to $(\langle a_{\gamma k} : (\gamma, k) \in (Y \cap \omega_1) \times \omega\rangle, \langle b_{\gamma k} : (\gamma, k) \in (Y \cap \omega_1) \times \omega\rangle)$ and $j[h[Y \cap V_\lambda]]$. This branch is not necessarily in N, but since N is wellfounded, there must be a branch through $j(h(T))$ in N, which by elementarity means that there is a branch through T. $\square$

The second half of this section shows how we can undo codings (into $X^2_{(Code)}(\mathcal{S})$) arising at limit stages or imposed by the partial orders given by the Laver function. The Protection Lemma is used as preparation for the Erasure Lemma below to ensure that we can undo these codings while preserving stationary subsets of ω_1 from initial stages of the iteration.

Let $\mathcal{U} = \langle U_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ be a collection of stationary, costationary subsets of ω_1, and fix $\kappa \in C^{\omega_1}_{\omega_2}$, $B \subset \omega_1 \times \omega$ and a cofinality function f for κ. The *Recoding Forcing* $\mathbb{REC}(\mathcal{U}, B, f)$ is the forcing to shoot a club through

$$\omega_1 \setminus \triangledown\{\gamma < \omega_1 \mid \exists(\alpha, i) \in \omega_1 \times \omega \, f(\gamma) \in \tilde{U}_{\alpha i} \wedge (\alpha, i) \in B\}$$

by initial segments (with a bijection from $\omega_1 \times \omega$ to ω_1 as a suppressed parameter). Since $\tilde{S}_0 \cap \tilde{S}_1$ is empty whenever S_0 and S_1 are disjoint subsets of ω_1, this forcing preserves the stationarity of any subset of ω_1 having stationary intersection with any set of the form $\{\gamma < \omega_1 \mid f(\gamma) \in \tilde{U}_{\alpha i}\}$ for some $(\alpha, i) \notin B$.

Lemma 4.7. *(Erasure Lemma) Suppose that $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ is a collection of pairwise disjoint stationary subsets of ω_1. Fix $\zeta < \omega_2$, and let A_i $(i < \omega)$ be stationary subsets of ω_1. Let $\mathcal{A}$ be the union of $\{A_i : i < \omega\}$ and $STAT(\mathcal{S}, \zeta)$.*

Then there is a forcing R preserving the stationarity of each member of $\mathcal{A}$ and forcing that $X^2_{(Code)}(\mathcal{S}) = \bigcup\{X_\beta : \beta < \zeta\}^V$.

Proof. Let $\gamma \geq \zeta$ be least such that $X_{\gamma+1} \neq \bigcup_{\beta<\zeta} X^V_\beta$ (if there is no such γ we are done). The partial order R consists of two steps, the first of which is $Coll(\omega_1, \mathbb{R})$, which adds a listing $\langle x_\alpha : \alpha < \omega_1 \rangle$ of all the reals of V. For each $\alpha < \omega_1$, let y_α be the set of $i < \omega$ such that $\tilde{S}_{\alpha i} \cap \kappa_\gamma$ is stationary. Let f be a cofinality function for κ_γ and let $\delta \in [2, \omega_1)$ be such that for each $i < \omega$, if there exists $(\xi, j) \in \omega_1 \times \omega$ such that $\{\nu \in A_i \mid f(\nu) \in \tilde{S}_{\xi j}\}$ is stationary, then there exists such a (ξ, j) with $\xi < \delta$.

By condition (ivb) of Definition 2.5, each $oe(y_\alpha)$ is infinite and co-infinite. By Lemma 2.2, then, for each $\alpha < \omega_1$ there exists an infinite $y'_\alpha \subset y_{\delta+1+\alpha}$ such that $oe(y'_\alpha) = x_\alpha$. By condition (iva), no y_α contains a consecutive pair of integers, so no y'_α does, either.

The second step of R is $\mathbb{REC}(\mathcal{S}, B, f)$, where

$$B = (\{\delta\} \times \omega) \cup \{(\delta + 1 + \alpha, i) \in \omega_1 \times \omega \mid \alpha < \omega_1 \wedge i \in y_{\delta+1+\alpha} \setminus y'_\alpha\}.$$

Since the real b_1 at stage γ is nonempty, there exists a $k < \omega$ such that $\tilde{S}_{1k} \cap \kappa_\gamma$ is stationary, and by condition (ive) of Definition 2.5, $\tilde{S}_{1k} \cap \kappa_\gamma$ has abused stationary intersection with each member of $STAT(\mathcal{S}, \gamma)$, which contains $STAT(\mathcal{S}, \zeta)$. Furthermore, $\tilde{S}_{1k} \cap \kappa_\gamma$ is disjoint from each $\tilde{S}_{\alpha i} \cap \kappa_\gamma$, $(\alpha, i) \in B$, which means that $\mathbb{REC}(\mathcal{S}, B, f)$ preserves the stationarity of each member of $STAT(\mathcal{S}, \zeta)$. Lastly, the choice of δ means that the stationarity of each A_i is preserved.

This forcing also makes $b_\delta = \emptyset$ at stage γ, which since $li(b_1, \omega_1) > li(\emptyset, \omega_1)$ means that $X_{\gamma+1} = \bigcup_{\beta<\gamma} X_\beta$ in the R-extension, by condition (ivd) of Definition 2.5. It also makes $b_{\delta+1+\alpha} = x_\alpha$ at stage γ, for each $\alpha < \omega_1$. Furthermore, R does not add reals, so the reals of the R-extension are contained in the $Y_{\gamma+1}$ of the R-extension, which by Lemma 2.7 means that no reals enter $X^2_{(Code)}(\mathcal{S})$ at any stage after γ, either. $\square$

There are many natural ways to increase the collection of stationary sets preserved in the Erasure Lemma, which can be carried directly over to stronger versions of $\mathrm{MM}^{+\omega}$ in the main theorem. We cannot, however, preserve the stationarity of all the members of any given set of cardinality $\aleph_1$, however (for instance, all the stationary sets of the form $\tilde{S}_{\alpha i} \cap \kappa_\gamma$).

Let $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ be a collection of pairwise disjoint stationary subsets of ω_1. Suppose that $M[G]$ is a forcing extension of a model M by an iterated forcing $P = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \eta \rangle$ of limit length. For each $\alpha < \eta$, let G_α be the restriction of G to P_α. We say that $M[G]$ is $(P, \mathcal{S})$-*safe* if for each stationary, costationary set $T \subset \omega_1$ appearing in some $M[G_\alpha]$ $(\alpha < \eta)$, there exists a $T' \in STAT(\mathcal{S}, \omega_2)^{M[G]}$ such that $T' \setminus T \in NS_{\omega_1}^{M[G]}$.

Since each successor step of our iterations has the form $Coll(\omega_1, {<}\mu) * \mathbb{PF}(\mathcal{S})$, the extensions by each initial segment of our iterations of limit length will be safe. The notion of safety is applied in the following corollary of Lemma 4.7, which shows that we can erase any accidental coding at limit stages of our iteration, even after an additional semi-proper forcing which may affect the coding.

Corollary 4.8. *Suppose that $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ is a collection of pairwise disjoint stationary subsets of ω_1. Suppose that $M[G]$ is $(P, \mathcal{S})$-safe for some iterated forcing $P = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \eta \rangle$ in M of limit length. Let Q be a forcing in $M[G]$ preserving stationary subsets of ω_1, and let $H \subset Q$ be $M[G]$-generic. Let ζ be an ordinal in the interval $[\sup\{\omega_2^{M[G_\alpha]} : \alpha < \eta\}, \omega_2^{M[G][H]})$, where for each $\alpha < \eta$, G_α is the restriction of G to P_α.*

Then in $M[G][H]$ there is a forcing R preserving stationary subsets of ω_1 from the models of the form $M[G_\alpha]$, $\alpha < \eta$, such that if I is $M[G][H]$-generic for R, then $X^2_{(Code)}(\mathcal{S})^{M[G][H][I]} = \bigcup_{\alpha < \zeta} X_\alpha^{M[G][H]}$. Furthermore, R can be chosen so that the members of any fixed countable set of stationary subsets of ω_1 in $M[G][H]$ are stationary in $M[G][H][I]$.

5. Coding reals

In this section we show that it is possible to force reals into $X^2_{(Code)}(\mathcal{S})$ while preserving stationary subsets of ω_1.

Lemma 5.1. *Suppose that there exists a Woodin cardinal below a measurable cardinal, and let $\mathcal{S}$ be a collection of pairwise disjoint stationary subsets of ω_1 indexed by $\omega_1 \times \omega$. Let z be a subset of ω. Then there is a partial order preserving stationary subsets of ω_1 from V such that if $V[G]$ is a generic extension of V by this partial order, then $X^2_{(Code)}(\mathcal{S})^{V[G]} = X^2_{(Code)}(\mathcal{S})^V \cup \{z\}$.*

Proof. Let $\mathcal{S} = \langle S_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$. First force with $Coll(\omega_1, 2^{\omega_1}) * \mathbb{PF}(\mathcal{S})$. Then force as in Lemma 4.3, making $\tilde{S}_{\alpha i} \cap C^{\omega}_{\omega_2}$ stationary for each $(\alpha, i) \in \omega_1 \times \omega$. Call this extension V^*, and let ρ denote $\omega_2^{V^*}$. Force now with $Coll(\omega_1, \rho)$, and note that each $\tilde{S}_{\alpha i} \cap C^{\omega}_{\rho}$ now has abused stationary intersection with each stationary subset of ω_1 in V^* (this will ensure that item (ive) of Definition 2.5 is satisfied at stage ρ after the following forcing, and will also ensure that the forcing $\mathbb{REC}(\mathcal{S}, B, f)$ below does not alter $X^2_{(Code)}(\mathcal{S})$ or $Y^2_{(Code)}(\mathcal{S})$ below ρ). None of these partial orders adds reals, and so $X^2_{(Code)}(S)$ remains the same. Furthermore, all reals are in $\bigcup\{Y_\gamma : \gamma < \rho\}$ in this extension.

Now use the Woodin cardinal and Shelah's semi-proper forcing to make NS_{ω_1} saturated. Since there is a measurable cardinal in this extension, $u_2 > \rho$ there (any forcing preserving stationary subsets of ω_1 and making $u_2 > \rho$ would suffice here). Fix in this extension a sequence of reals $\langle c_\alpha : \alpha < \omega_1 \rangle$ such that $t^*(oe(c_0)) = z$, no c_α contains a consecutive pair of elements of ω, each $oe(c_\alpha) \in \mathbb{C}$,

$$li(oe(c_1), \omega_1) > \max\{li(oe(c_0), \omega_1), \rho\}$$

and $\langle li(oe(c_\alpha), \omega_1) : \alpha < \omega_1 \rangle$ is increasing. Now force with $\mathbb{REC}(\mathcal{S}, B, f)$ (as defined before Lemma 4.7), where $B = \{(\alpha, i) \in \omega_1 \times \omega \mid i \notin c_\alpha\}$. Since ρ is the ρth uniform indiscernible of V, $\kappa_\rho = \rho$ in this extension, and it is straightforward to check that in this extension z enters $X^2_{(Code)}(\mathcal{S})$ at stage ρ.

Finally, we can force as in the Erasure Lemma to ensure that no reals enter $X^2_{(Code)}(\mathcal{S})$ after stage ρ. Since the first step of our forcing was $Coll(\omega, 2^{\omega_1}) * \mathbb{PF}(\mathcal{S})$, the forcing from the Erasure Lemma preserves the stationarity of each stationary subset of ω_1 from V. $\square$

Corollary 5.2. *Suppose that MM^{++} holds and that there exists a Woodin cardinal below a measurable cardinal. Let $\mathcal{S}$ be a collection of pairwise disjoint subsets of ω_1 indexed by $\omega_1 \times \omega$. Then $X^2_{(Code)}(\mathcal{S}) = \mathcal{P}(\omega)$.*

The Protection and Erasure lemmas, plus Corollary 5.2, give a much simpler proof of the main result from [6] (modulo the fact that the conclusion of Corollary 5.2 follows from $(*)$, which follows from Lemma 5.1 and standard $\mathbb{P}_{max}$ arguments). The argument can be easily modified to produce models of $\mathrm{MM}^{+\omega}$ in which $X^2_{(Code)}(\mathcal{S})$ is any desired set of reals. In the proof of our main theorem, we let $X^2_{(Code)}(\mathcal{S})$ be a set of reals coding $\mathcal{S}$.

Theorem 5.3. *Suppose that λ is a supercompact limit of supercompact cardinals. Then there is a semi-proper partial order of cardinality λ forcing that Martin's Maximum$^{+\omega}$ holds and forcing that there exists a partition $\mathcal{S}$ of ω_1 into stationary sets (indexed by $\omega_1 \times \omega$) such that $X^2_{(Code)}(\mathcal{S}) = \emptyset$. Furthermore, in this extension MM^{++} holds for all partial orders that do not add reals to $X^2_{(Code)}(\mathcal{S})$.*

Proof. Force to add a partition $\mathcal{S} = \langle S_{\alpha i} : \alpha < \omega_1, i < \omega\rangle$ of ω_1 by initial segments. Then each member of $\mathcal{S}$ is stationary and each $\tilde{S}_{\alpha i} = \emptyset$, so $X^2_{(Code)}(\mathcal{S}) = \emptyset$. Let $L\colon \lambda \to V_\lambda$ be as in Lemma 3.7.

Let $\bar{P} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \lambda\rangle$ be a semi-proper forcing iteration satisfying the following conditions, where G_α denotes the restriction of the generic filter to the first α stages of the iteration.

- Q_0 is the trivial forcing and each forcing $Q_{\alpha+1}$ has the form $Coll(\omega_1, {<}\mu) * \mathbb{PF}(\mathcal{S})$, where μ is the least supercompact cardinal (these forcings are (ω, ∞)-distributive and preserve stationary subsets of ω_1, so they do not add reals to $X^2_{(Code)}(\mathcal{S})$).
- If α is a limit ordinal and $L(\alpha)$ is a pair (τ, σ), where τ is a P_α-name for a semi-proper forcing and σ is a $P_\alpha * \tau$-name for a countable collection $\mathcal{A}$ of stationary subsets of ω_1, then the first step of Q_α is τ_{G_α}. If $X^2_{(Code)}(\mathcal{S}) = \emptyset$ after forcing with τ_{G_α}, then Q_α is just this forcing. Otherwise, Q_α is τ_{G_α}

followed by a forcing as in the Erasure Lemma, preserving the stationarity of each member of $\mathcal{A}$ and each stationary subset of ω_1 in each $V[G_\beta]$ $(\beta < \alpha)$ while making $X^2_{(Code)}(\mathcal{S})^{V[G_{\alpha+1}]} = \emptyset$.

- If α is a limit ordinal and the previous case does not apply, and $X^2_{(Code)}(\mathcal{S})$ is not empty in $V[G_\alpha]$, then Q_α is a forcing as in the Erasure Lemma, preserving the stationarity of each stationary subset of ω_1 in each $V[G_\beta]$ $(\beta < \alpha)$ while making $X^2_{(Code)}(\mathcal{S})^{V[G_{\alpha+1}]} = \emptyset$.

Then in the $\bar{P}$-extension, Martin's Maximum$^{+\omega}$ holds as in [3] and $X^2_{(Code)}(\mathcal{S}) = \emptyset$. Furthermore, MM^{++} holds for all partial orders that do not add a real to $X^2_{(Code)}(\mathcal{S})$. □

We note the following standard (except for the last, which follows from the argument above and the consistency proof of MM from [3]), nonoptimal facts [3, 11, 4], where for any cardinal γ, $\beth_\omega(\gamma)$ denotes the supremum of 2^γ, 2^{2^γ}, $2^{2^{2^\gamma}}$, etc.

- If λ is a supercompact cardinal, then stationarily many cardinals κ below κ are $\beth^+_\omega(\kappa)$-supercompact.
- If κ is $\beth^+_\omega(\kappa)$-supercompact, then there is a semi-proper forcing of cardinality κ forcing MM$^{++}(2^{\mathfrak{c}})$.
- If κ is $\beth^+_\omega(\kappa)$-supercompact and a limit of supercompact cardinals, and $\mathcal{S}$ is a partition of ω_1 into stationary sets indexed by $\omega_1 \times \omega$, then there is a semi-proper partial order of cardinality κ not adding reals to $X^2_{(Code)}(\mathcal{S})$ and forcing MM$^{++}(2^{\mathfrak{c}})$ for all partial orders not adding reals to $X^2_{(Code)}(\mathcal{S})$.

Mixing in forcings of the form $Coll(\omega_1, <\mu) * \mathbb{PF}(\mathcal{U})$ for supercompact cardinals μ below κ and all partitions $\mathcal{U}$ of ω_1 into stationary sets indexed by $\omega_1 \times \omega$ (again, these forcings don't add reals to $X^2_{(Code)}(\mathcal{S})$) we have the following.

Theorem 5.4. *Let $\mathcal{S}$ be a partition of ω_1 into stationary sets, indexed by $\omega_1 \times \omega$. Suppose that κ is a $\beth^+_\omega(\kappa)$-supercompact cardinal and a limit of supercompact cardinals. Then there is a semi-proper partial order of cardinality κ that does not add reals to $X^2_{(Code)}(\mathcal{S})$ and forces that Martin's Maximum$^{+\omega}(2^{\mathfrak{c}})$ holds and MM$^{++}(2^{\mathfrak{c}})$*

holds for all partial orders that do not add reals to $X^2_{(Code)}(\mathcal{S})$. Furthermore, this forcing forces for all partitions $\mathcal{U}$ of ω_1 indexed by $\omega_1 \times \omega$ that the following statements hold.

- $Y^2_{(Code)}(\mathcal{U}) = \mathcal{P}(\omega)$;
- *every stationary subset of ω_1 contains a member of $STAT(\mathcal{U}, \omega_2)$ modulo NS_{ω_1}.*

6. Distinguishing partitions

Our remaining goal is to show that by modifying the iteration in Theorem 5.3, we can make some partition $\mathcal{S}$ of ω_1 into stationary sets have a property (definable in $H(\aleph_2)$) not shared by any other partition. This plus the fact that Martin's Maximum holds in this extension will imply that there is a wellordering of $\mathcal{P}(\omega_1)$ in this extension definable in $H(\aleph_2)$ without parameters. We will use $\mathcal{U}$ to denote partitions that we wish to distinguish from $\mathcal{S}$. Lemma 6.1 gives conditions under which any given real can be forced into $X^2_{(Code)}(\mathcal{U})$, presuming that u_2 is greater than the κ^* of the lemma, and that the sets $\tilde{F} \cap \kappa^*$ $(F \in \mathcal{F})$ in the lemma have abused stationary intersection with the sets required in item (ive) of Definition 2.5.

Lemma 6.1. *Suppose that $\mathcal{U} = \langle U_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ is a partition of ω_1 into stationary sets. Suppose that κ^* is a member of the κ-sequence of $\mathcal{U}$ of cofinality ω_1, and let f be a cofinality function for κ^*. Suppose that d_α $(\alpha < \omega_1)$ are subsets of ω. Suppose that $\mathcal{F}$ is a collection of pairwise disjoint stationary subsets of ω_1 such that for each member F of $\mathcal{F}$, F is contained in some $U_{\alpha i}$ and $\tilde{F} \cap \kappa^*$ is stationary. Suppose that for each $\alpha < \omega_1$, $U_{\alpha i}$ contains a member of $\mathcal{F}$ for infinitely many even integers i and for infinitely many odd integers i.*

Suppose that F_0 and F_1 are elements of $\mathcal{F}$ such that, letting (α_0, i_0) and (α_1, i_1) be the unique members of $\omega_1 \times \omega$ such that $F_0 \subset U_{\alpha_0 i_0}$ and $F_1 \subset U_{\alpha_1 i_1}$, at least one of the following holds:

- $\alpha_0 \neq \alpha_1$,
- $|i_0 - i_1| \neq 1$.

Then there is an uncountable $\mathcal{F}' \subset \mathcal{F}$ containing $\{F_0, F_1\}$ such that the partial order to shoot a club through $\bigtriangledown\{\{\gamma < \omega_1 \mid f(\gamma) \in \tilde{F}\} : F \in \mathcal{F}'\}$ (under any fixed enumeration of $\mathcal{F}'$) forces that for each $\alpha < \omega_1$,

$$\{i < \omega | \tilde{U}_{\alpha i} \cap \kappa^* \text{ is stationary}\}$$

contains no consecutive pair of elements of ω, and

$$oe(\{i < \omega | \tilde{U}_{\alpha i} \cap \kappa^* \text{ is stationary}\}) = d_\alpha.$$

Furthermore, this forcing is (ω, ∞)-distributive, preserves stationarity of any subset of ω_1 having abused stationary intersection with any of the sets $\tilde{F} \cap \kappa^*$ ($F \in \mathcal{F}'$) and forces that $\tilde{E} \cap \kappa^*$ is nonstationary for any E disjoint from every member of $\mathcal{F}$.

Lemma 6.2 below presents a preparation forcing which in the right circumstances allows one to force a real into $X^2_{(Code)}(\mathcal{U})$ via Lemma 6.1 without adding a real to $X^2_{(Code)}(\mathcal{S})$, for suitable $\mathcal{U}$ and $\mathcal{S}$.

Lemma 6.2. *Suppose that there exists a cardinal κ below a Woodin cardinal below a measurable cardinal such that κ is $\beth^+_\omega(\kappa)$-supercompact and a limit of supercompact cardinals. Let $\mathcal{S}$ and $\mathcal{U}$ be partitions of ω_1 into stationary sets, indexed by $\omega_1 \times \omega$. Then there exists a partial order P preserving stationary subsets of ω_1 such that, letting κ^* be the supremum of the first ω_1 many measurable cardinals above κ and letting $\mathcal{F}$ be the collection of stationary sets of the form $S_{\alpha i} \cap U_{\beta j}$,*

- *P forces each of the following statements:*
  - *for every $F \in \mathcal{F}$, $\tilde{F} \cap \kappa^*$ has abused stationary intersection with every member of $STAT(\mathcal{S}, \kappa^*) \cup STAT(\mathcal{U}, \kappa^*)$;*
  - *every stationary subset of ω_1 in the ground model contains modulo NS_{ω_1} a member of $STAT(\mathcal{S}, \kappa^*)$ and a member of $STAT(\mathcal{U}, \kappa^*)$;*
  - *$Y_{\kappa^*}(\mathcal{S}) \cap Y_{\kappa^*}(\mathcal{U})$ contains all the reals of the ground model;*
  - *$u_2 > \kappa^*$;*
- *P adds no reals to $X^2_{(Code)}(\mathcal{S})$;*

- *if $\dot{Q}$ is a P-name for a partial order which shoots a club through a set of the form $\bigtriangledown\{\{\gamma < \omega_1 \mid f(\gamma) \in \tilde{F}\} : F \in \mathcal{F}'\}$ for some uncountable $\mathcal{F}' \subset \mathcal{F}$ and some cofinality function f for κ^*, and this forcing does not add a real to $X^2_{(Code)}(\mathcal{S})$ at stage κ^*, then there is a $P * \dot{Q}$-name $\dot{R}$ for a partial order preserving the stationarity of the members of $STAT(\mathcal{U}, \kappa^* + 1)$ and $STAT(\mathcal{S}, \kappa^*)$ such that $X^2_{(Code)}(\mathcal{S})$ in the $P * \dot{Q} * \dot{R}$-extension is the same as $X^2_{(Code)}(\mathcal{S})$ in the ground model.*

Proof. The partial order P consists of three steps. First force as in Lemma 5.4 with a partial order of cardinality κ (while preserving stationary subsets of ω_1) to make $\mathrm{MM}^{++}(2^{\mathfrak{c}})$ hold for all forcings not adding reals to $X^2_{(Code)}(\mathcal{S})$, without adding any reals to $X^2_{(Code)}(\mathcal{S})$. Call this extension $V[G]$. In $V[G]$, NS_{ω_1} is saturated (since the antichain-sealing forcing adds no reals, see Lemma 2.6 and the remarks after Lemma 3.4), $u_2 = \omega_2$, and

$$Y^2_{(Code)}(\mathcal{S}) = Y^2_{(Code)}(\mathcal{U}) = \mathcal{P}(\omega).$$

Furthermore, every stationary subset of ω_1 in the ground model contains modulo NS_{ω_1} a member of $STAT(\mathcal{S}, \omega_2)^{V[G]}$ and a member of $STAT(\mathcal{U}, \omega_2)^{V[G]}$.

Letting $\langle \kappa_\alpha : \alpha \leq \omega_1 \cdot 2\rangle$ be the closed increasing sequence generated by the first $\omega_1 \cdot 2$ many measurable cardinals (so κ_{ω_1} is the κ^* of the statement of the lemma), force now as in Lemma 4.6, with $\mathcal{S}$ as itself and $\mathcal{F}$ as $\langle T_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ (suitably reindexed). Recall that the first step of this forcing adds generic partitions of ω_1 $\langle A_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ and $\langle B_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega\rangle$ by initial segments; the genericity of each $A_{\alpha i}$ implies that $\tilde{F} \cap \kappa_{\omega_1}$ has abused stationary intersection with every member of $STAT(\mathcal{S}, \omega_2^{V[G]}) \cup STAT(\mathcal{U}, \omega_2^{V[G]})$, for each $F \in \mathcal{F}$. Call this extension $V[G][H]$. Note that since the forcing from Lemma 4.6 does not add reals and preserves stationary subsets of ω_1, $X^2_{(Code)}(\mathcal{S})^{V[G][H]} = X^2_{(Code)}(\mathcal{S})^V$. Note also that for every stationary $K \subset \omega_1$ in $V[G]$, the only $V[G]$-indiscernibles in $\tilde{K}$ are in the set $\{\omega_1\} \cup \{\kappa_\alpha : \alpha \leq \omega_1 \cdot 2\}$. It follows that κ_{ω_1} is the only $\rho \in C^{\omega_1}_{\kappa_{\omega_1 \cdot 2}}$ is greater than $\omega_2^{V[G]}$ for which there exists $(\alpha, i) \in \omega_1 \times \omega$ such that either of $\tilde{S}_{\alpha i} \cap \rho$

and $\tilde{U}_{\alpha i} \cap \rho$ is stationary. We will see that these last two facts are preserved by the remaining forcings in this proof, which will preserve the fact that no real enters either $X^2_{(Code)}(\mathcal{U})$ or $X^2_{(Code)}(\mathcal{S})$ at any stage in the interval $(\omega_2^{V[G]}, \kappa_{\omega_1 \cdot 2})$ other than κ_{ω_1}. Note also then that $STAT(\mathcal{S}, \kappa_{\omega_1}) = STAT(\mathcal{S}, \omega_2^{V[G]})$ and $STAT(\mathcal{U}, \kappa_{\omega_1}) = STAT(\mathcal{U}, \omega_2^{V[G]})$

Finally, force with any partial order preserving stationary subsets of ω_1 making $u_2 > \kappa_{\omega_1}$. This is the last step of P. Call this extension $V[G][H][I]$. Since this last forcing preserves stationary subsets of ω_1, κ_{ω_1} and $\kappa_{\omega_1 \cdot 2}$ remain on the κ-sequences of both $\mathcal{S}$ and $\mathcal{U}$ in $V[G][H][I]$, and, for each $\alpha < \kappa_{\omega_1 \cdot 2} + 1$, $X_\alpha(\mathcal{S})^{V[G][H]} = X_\alpha(\mathcal{S})^{V[G][H][I]}$.

Now suppose that we force over $V[G][H][I]$ with a forcing Q of the form $\triangledown\{\{\gamma < \omega_1 \mid f(\gamma) \in \tilde{F}\} : F \in \mathcal{F}'\}$ for some uncountable $\mathcal{F}' \subset \mathcal{F}$ and some cofinality function f for κ_{ω_1}, and that this forcing does not add a real to $X^2_{(Code)}(\mathcal{S})$ at stage κ_{ω_1}. This partial order shoots a club through a subset of ω_1 containing some $A_{\alpha i}$ modulo NS_{ω_1}. Now force as in the Erasure Lemma (the forcing $\dot{R}$ of this lemma) to make no real enter $X^2_{(Code)}(\mathcal{S})$ at stage $\kappa_{\omega_1 \cdot 2}$ or later. This partial order wellorders the reals in ordertype ω_1 and then shoots a club through a subset of ω_1 containing some $B_{\beta j}$ modulo NS_{ω_1}. Since $B_{\beta j}$ has stationary intersection with each $A_{\gamma k}$, this second forcing preserves the stationarity of each $A_{\gamma k}$ left stationary by forcing with the realization of Q, and thus preserves the stationarity of each member of $STAT(\mathcal{U}, \kappa_{\omega_1} + 1)$.

By Lemma 4.6, no $V[G]$-indiscernible in $\kappa_{\omega_1 \cdot 2} \setminus \{\kappa_\alpha : \alpha < \omega_1 \cdot 2\}$ is forced by $Q * \dot{R}$ into $\tilde{K}$ for any stationary $K \subset \omega_1$ in $V[G]$. To see this, fix a condition in $Q * \dot{R}$ deciding the values of (α, i) and (β, j) as in the previous paragraph, fix such an indiscernible ξ and a name τ for a function supposedly witnessing that ξ is forced into $\tilde{K}$, and choose a countable elementary submodel X of a suitable $H(\theta)$ with this condition and τ in X, $X \cap \omega_1 \in A_{\alpha i} \cap B_{\beta j}$ and $ot(X \cap \xi) \notin K$ (the set of such X is stationary in $V[G][H]$ by Lemma 4.6 and genericity, and remains stationary in $V[G][H][I]$ since ξ has cardinality $\aleph_1$ in $V[G][H]$). Then any X-generic filter for

$Q * \dot{R}$ naturally defines a $Q * \dot{R}$ condition witnessing that τ is not a name for a function witnessing that $\xi \in \tilde{K}$.

It follows that in the $Q * \dot{R}$ extension, the b-sequences for both $\mathcal{S}$ and $\mathcal{U}$ still consist of the empty real at every stage in the interval $(\omega_2^{V[G]}, \kappa_{\omega_1 \cdot 2})$ other than κ_{ω_1}. In particular, no real enters $X^2_{(Code)}(S)$ at any stage γ in the interval $(\kappa_{\omega_1}, \kappa_{\omega_1 \cdot 2})$ either in this final extension. □

Fixing a coding of elements of $H(\omega_1)$ by reals, for each partition $\mathcal{U} = \langle U_{\alpha i} : (\alpha, i) \in \omega_1 \times \omega \rangle$ of ω_1 into stationary sets, let $E_{\mathcal{U}}$ be the set of $x \subset \omega$ such that x codes in this fixed coding a set of the form $\langle U_{\alpha i} \cap \beta : \alpha < \beta \rangle$, for some $\beta < \omega_1$. Given subsets A, B of ω_1, we say $A =^* B$ if $A \triangle B \in NS_{\omega_1}$ and $A \subset^* B$ if $A \setminus B \in NS_{\omega_1}$.

Lemma 6.3 below allows one to distinguish between $\mathcal{S}$ and $\mathcal{U}$ in the forcing extension in the main theorem.

Lemma 6.3. *Suppose that there exists a cardinal λ below a Woodin cardinal below a measurable cardinal such that λ is $\beth^+_\omega(\lambda)$-supercompact and a limit of supercompact cardinals.*

Let $\mathcal{S} = \langle S_{\alpha i} : \alpha < \omega_1, i < \omega \rangle$ and $\mathcal{U} = \langle U_{\alpha i} : \alpha < \omega_1, i < \omega \rangle$ be two partitions of ω_1 into stationary sets. Then there is a partial order preserving stationary subsets of ω_1 which forces no real outside of $E_{\mathcal{S}}$ into $X^2_{(Code)}(\mathcal{S})$ but forces a real outside of $E_{\mathcal{U}}$ into $X^2_{(Code)}(\mathcal{U})$.

Proof. We break into two cases, the second of which has four subcases. In the first case, where $\mathcal{U}$ is essentially the same as $\mathcal{S}$, we can force a real in $E_{\mathcal{S}} \setminus E_{\mathcal{U}}$ into both $X^2_{(Code)}(\mathcal{S})$ and $X^2_{(Code)}(\mathcal{U})$. In the second case, where $\mathcal{U}$ and $\mathcal{S}$ are significantly different, we can force a real not in $E_{\mathcal{U}}$ into $X^2_{(Code)}(\mathcal{U})$ without adding any real to $X^2_{(Code)}(\mathcal{S})$.

The first (close) case is when $S_{\alpha i} =^* U_{\alpha i}$ for all $(\alpha, i) \in \omega_1 \times \omega$. In this case we first force as in Theorem 5.4. Then since $u_2 = \omega_2$ and NS_{ω_1} is saturated in this extension, for each pair $S_{\alpha i}, U_{\alpha i}$ there is a real in $Y^2_{(Code)}(\mathcal{S}) \cap Y^2_{(Code)}(\mathcal{U})$ whose indiscernibles are disjoint from $S_{\alpha i} \triangle U_{\alpha i}$ (see Theorem 3.16 of [14]) and thus from

$\tilde{S}_{\alpha i} \triangle \tilde{U}_{\alpha i}$. Then each set of the form $\tilde{S}_{\alpha i} \triangle \tilde{U}_{\alpha i}$ is disjoint from the κ-sequences of both $\mathcal{S}$ and $\mathcal{U}$ above some ordinal below the ω_2 of this extension. It follows that in all forcing extensions preserving stationary subsets of ω_1, the $\mathcal{S}$-codings and $\mathcal{U}$-codings are the same at all stages after the ω_2 of this extension.

Let $\beta < \omega_1$ be large enough so that $\langle S_{\alpha i} \cap \beta : (\alpha, i) \in \beta \times \omega\rangle$ and $\langle U_{\alpha i} \cap \beta : (\alpha, i) \in \beta \times \omega\rangle$ are distinct, and let z be a real coding $\langle S_{\alpha i} \cap \beta : (\alpha, i) \in \beta \times \omega\rangle$ in our fixed coding. Then forcing as in Lemma 5.1 adds z and only z to both $X^2_{(Code)}(\mathcal{S})$ and $X^2_{(Code)}(\mathcal{U})$.

The second case is when $S_{\alpha i} \triangle U_{\alpha i}$ is stationary for some $(\alpha i) \in \omega_1 \times \omega$ (we do not, however, fix such a pair (α, i)). In this case, we first force with P as in Lemma 6.2, and fix κ^* as in the statement of the lemma. Note that κ^* is on the κ-sequences of both $\mathcal{S}$ and $\mathcal{U}$ in this extension, and, for every stationary set F of the form $S_{\alpha i} \cap U_{\beta j}$, $\tilde{F}$ has abused stationary intersection with every member of $STAT(\mathcal{S}, \kappa^*)$ and $STAT(\mathcal{U}, \kappa^*)$, and that every stationary subset of ω_1 in the ground model contains modulo NS_{ω_1} a member of each of these sets.

The second case has four subcases, which will see are exhaustive. Fix a sequence $\langle d_\alpha : \alpha \in \omega_1\rangle$ such that

- each d_α is an infinite subset of ω;
- no d_α contains consecutive elements of ω;
- each $oe(d_\alpha) \in \mathbb{C}$;
- $\langle li(oe(d_\alpha), \omega_1) : \alpha < \omega_1\rangle$ is increasing;
- $li(oe(d_1), \omega_1) > \kappa^*$;
- each $t^*(oe(d_\alpha))$ is infinite and co-infinite;
- $t^*(oe(d_0)) \in \mathcal{P}(\omega) \setminus E_{\mathcal{U}}$.

It suffices to see that at this point there is a suitable choice of F_0, F_1 and $\mathcal{F}$ as in Lemma 6.1 which adds $t^*(oe(d_0))$ to $X^2_{(Code)}(\mathcal{U})$ at stage κ^* without adding any real to $X^2_{(Code)}(\mathcal{S})$ at stage κ^*. Suitable here means that $\tilde{F}_0 \cap \kappa^*$ and $\tilde{F}_1 \cap \kappa^*$ both being stationary, and $\tilde{E} \cap \kappa^*$ being nonstationary for each $E \subset \omega_1$ having nonstationary intersection with each member of $\mathcal{F}$, implies that no real enters $X^2_{(Code)}(\mathcal{S})$ at stage

κ^*, while for each $\alpha < \omega_1$, $U_{\alpha i}$ will have stationary intersection with some member of $\mathcal{F}$ for infinitely even i and infinitely many i odd, which means that it is still possible to force any desired real into $X^2_{(Code)}(\mathcal{U})$ (in particular, $t^*(oe(d_0))$). Then by Lemma 6.2 we can further force to ensure that no reals enter $X^2_{(Code)}(\mathcal{S})$ at a stage after κ^*, either. We will do just this in the four subcases below, with $\mathcal{F}$ as a subset of the collection of stationary sets of the form $S_{\alpha i} \cap U_{\beta j}$, for $\alpha, \beta < \omega_1$ and $i, j < \omega$.

In the first subcase, suppose that there exist $\alpha, \beta, \gamma < \omega_1$, $i, j, k < \omega$ such that $S_{\alpha i} \cap U_{\beta j}$ and $S_{\alpha(i+1)} \cap U_{\gamma k}$ are both stationary and one of the following cases holds:

- $\beta \neq \gamma$,
- $|j - k| \neq 1$.

Then let $F_0 = S_{\alpha i} \cap U_{\beta j}$, $F_1 = S_{\alpha(i+1)} \cap U_{\gamma k}$ and $\mathcal{F}$ be all stationary sets of the form $S_{\delta n} \cap U_{\rho m}$, for $m, n \in \omega$, $\delta, \rho \in \omega_1$.

For the rest of the proof we assume that the first subcase does not apply, and so there exists a function $g\colon \omega_1 \to \omega_1$ such that whenever $S_{\alpha i} \cap U_{\beta j}$ and $S_{\alpha i} \cap U_{\gamma k}$ are stationary, $\beta = \gamma = g(\alpha)$ and $|k - j| \in \{0, 2\}$.

In the second subcase, suppose that there exists $\alpha^* < \omega_1$ such that one of the two following statements holds.

- $\bigcup\{S_{\alpha^* i} : i < \omega \text{ even}\}$ fails to contain (modulo NS_{ω_1}) all but finitely many of the sets $U_{g(\alpha^*)j}$ for j odd and fails to contain (modulo NS_{ω_1}) all but finitely many of the sets $U_{g(\alpha^*)j}$ for j even.
- $\bigcup\{S_{\alpha^* i} : i < \omega \text{ odd}\}$ fails to contain (modulo NS_{ω_1}) all but finitely many of the sets $U_{g(\alpha^*)j}$ for j odd and fails to contain (modulo NS_{ω_1}) all but finitely many of the sets $U_{g(\alpha^*)j}$ for j even.

In the first (subsub)case, let $\mathcal{F}$ be all stationary sets of the form $S_{\delta n} \cap U_{\rho m}$ where $n, m \in \omega$, $\delta, \rho \in \omega_1$ such that either n is not even or $\delta \neq \alpha^*$, and let F_0 and F_1 both be any one element of $\mathcal{F}$. In the second case, let $\mathcal{F}$ be all stationary sets of the form $S_{\delta n} \cap U_{\rho m}$ where $n, m \in \omega$, $\delta, \rho \in \omega_1$ such that either n is not odd or $\delta \neq \alpha^*$, and let F_0 and F_1 both be any one element of $\mathcal{F}$.

Assuming that the first two subcases do not apply, g is injective.

In the third subcase, suppose that the first two subcases do not apply, and two distinct set $S_{\alpha i}$ and $S_{\alpha k}$ both intersect some $U_{g(\alpha)j}$ stationarily. Since the first subcase does not apply, $S_{\alpha i} \setminus U_{g(\alpha)j}$ intersects at most one set of the form $U_{g(\alpha)n}$ stationarily (fix n if it exists; n must be either $j-2$ or $j+2$, though we will not use this fact). Let $F_0 = S_{\alpha i} \cap U_{g(\alpha)j}$, $F_1 = S_{\alpha k} \cap U_{g(\alpha)j}$ and $\mathcal{F}$ be all stationary sets of the form $S_{\delta p} \cap U_{\rho m}$ (excluding $S_{\alpha i} \cap U_{g(\alpha)n}$ if n as above exists). If the forcing corresponding to each $\mathcal{F}' \subset \mathcal{F}$ as in Lemma 6.1 adds a real to $X^2_{(Code)}(\mathcal{S})$ at stage κ^*, then fix one such $\mathcal{F}'$ and force instead with $\mathcal{F}' \setminus \{S_{\alpha i} \cap U_{g(\alpha)j}\}$. This codes the same real into $X^2_{(Code)}(\mathcal{U})$ at stage κ^*, but it cannot be that both forcings add a real to $X^2_{(Code)}(\mathcal{S})$, since the set $oe(\{m < \omega \mid \tilde{S}_{\alpha m} \cap \kappa^* \text{ is stationary}\})$ cannot be in $\mathbb{C}$ in both cases, by Lemma 2.3.

If the first three cases do not apply, then for each $\alpha < \omega_1$ there is a permutation $e_\alpha \colon \omega \to \omega$ such that for each $i < \omega$, $S_{\alpha i} =^* U_{g(\alpha)e_\alpha(i)}$. It follows easily from the fact that $|e_\alpha(i) - e_\alpha(i+1)| = 1$ for all $\alpha < \omega_1$, $i < \omega$ that each e_α is the identity function. If g is not the identity function, $\langle li(d_{g^{-1}(\beta)}, \omega_1) : \beta < \omega_1 \rangle$ is not increasing, so we can let $\mathcal{F}$ be all stationary sets of the form $S_{\alpha i} \cap U_{\beta j}$ and let F_0 and F_1 both be any one of these sets. If g is the identity then we are in the first case of the proof. $\square$

7. The main theorem

In Theorem 7.1 we modify the iteration from the proof of Theorem 5.3 to make $X^2_{(Code)}(\mathcal{S}) = E_{\mathcal{S}}$, which makes $\mathcal{S}$ definable in the $H(\aleph_2)$ of the extension without parameters. The third condition on the iteration is probably subsumed by the second one, but we include it to make certain.

Undoubtedly there are many other ways to make the parameter $\mathcal{S}$ definable.

Theorem 7.1. *Assume that there exists a supercompact limit of supercompact cardinals. Then there is a semi-proper partial order forcing Martin's Maximum$^{+\omega}$ and forcing the existence of a partition $\mathcal{S}$ of ω_1 into uncountably many stationary sets* *such that $\mathcal{S}$ is definable in $H(\aleph_2)$ without parameters. Furthermore, in this extension, Martin's Maximum^{++} holds for partial orders that don't add reals not in $E_{\mathcal{S}}$ to $X^2_{(Code)}(\mathcal{S})$,*

Proof. Let λ be a supercompact limit of supercompact cardinals. Force to add a partition $\mathcal{S} = \langle S_{\alpha i} : \alpha < \omega_1, i < \omega\rangle$ be a partition of ω_1 of ω_1 by initial segments. Then each member of $\mathcal{S}$ is stationary and each $\tilde{S}_{\alpha i} = \emptyset$, so $X^2_{(Code)}(\mathcal{S}) = \emptyset$.

Let $L\colon \lambda \to V_\lambda$ be as in Lemma 3.7.

Let $\bar{P} = \langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \lambda\rangle$ be a semi-proper forcing iteration satisfying the following conditions, where G_α denotes the restriction of the generic filter to the first α stages of the iteration.

- Q_0 is the trivial forcing and each forcing $Q_{\alpha+1}$ has the form $Coll(\omega_1, {<}\mu) * \mathbb{PF}(\mathcal{S})$.
- If α is a limit ordinal and $L(\alpha)$ is a triple $(0, \tau, \sigma)$, where τ is a P_α-name for a semi-proper forcing and σ is a $P_\alpha * \tau$-name for a countable collection $\mathcal{A}$ of stationary subsets of ω_1, then the first step of Q_α is τ_{G_α}. If $X^2_{(Code)}(\mathcal{S}) \subset E_{\mathcal{S}}$ after forcing with τ_{G_α}, then Q_α is just this forcing. Otherwise, Q_α is τ_{G_α} followed by a forcing as in the Erasure Lemma, preserving the stationarity of each member of of $\mathcal{A}$ and each stationary subset of ω_1 in each $V[G_\beta]$ ($\beta < \alpha$) while making $X^2_{(Code)}(\mathcal{S})^{V[G_{\alpha+1}]} = \bigcup\{X^2_{(Code)}(\mathcal{S})^{V[G_\beta]} : \beta < \alpha\}$.
- If α is a limit ordinal and $L(\alpha)$ is a triple $(1, \tau, \sigma)$, where τ is a P_α-name for a partition of ω_1 into stationary sets (indexed by $\omega_1 \times \omega$) and σ is P_α-name for a real, then if there exists a forcing preserving stationary subsets of ω_1 and forcing σ_{G_α} into $X^2_{(Code)}(\tau_{G_\alpha})$ in whose extension $X^2_{(Code)}(\mathcal{S}) \subset E_{\mathcal{S}}$, then Q_α is such a forcing.
- If α is a limit ordinal and the previous two cases do not apply, and $X^2_{(Code)}(\mathcal{S})$ is not a subset of $E_{\mathcal{S}}$ in $V[G_\alpha]$, then Q_α is a forcing as in the Erasure Lemma, preserving the stationarity of each stationary subset of ω_1 in each $V[G_\beta]$ ($\beta < \alpha$) while making $X^2_{(Code)}(\mathcal{S})^{V[G_{\alpha+1}]} = \bigcup\{X^2_{(Code)}(\mathcal{S})^{V[G_\beta]} : \beta < \alpha\}$.

Then in the $\bar{P}$-extension, Martin's Maximum$^{+\omega}$ holds, and Martin's Maximum^{++} holds for partial orders that don't add reals not in $E_{\mathcal{S}}$ to $X^2_{(Code)}(\mathcal{S})$, by the original consistency proof for Martin's Maximum in [3]. By construction, $X^2_{(Code)}(\mathcal{S}) \subset E_{\mathcal{S}}$ in this extension, and $X^2_{(Code)}(\mathcal{S}) \supset E_{\mathcal{S}}$ by Lemma 5.1. By Lemma 6.3, in this extension $\mathcal{S}$ is the unique partition $\mathcal{U}$ of ω_1 into stationary sets indexed by $\omega_1 \times \omega$ such that $X^2_{(Code)}(\mathcal{U}) = E_{\mathcal{U}}$. □

We note that the following question is still open.

Question 7.2. *Does Martin's Maximum^{++} imply $(*)$?*

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