WRP*

Paul B. Larson

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Given a nonempty set X, a set $S \subseteq \mathcal{P}(X)$ is said to be *club* if there exists a function $F\colon X^{<\omega} \to X$ such that $S = \{Y \subseteq X \mid F[Y^{<\omega}] \subseteq Y\}$, and *stationary* if it intersects every club subset of $\mathcal{P}(X)$. A set S is said to be $\subseteq$-*cofinal* in a set T if for every $X \in T$ there exists a $Y \in S$ such that $X \subseteq Y$. Given a set X, an ideal $I \subseteq \mathcal{P}_{\aleph_1}(X)$ is *fine* if it contains every $\subseteq$-noncofinal set, i.e., if for all $x \in \mathcal{P}_{\aleph_1}(X)$, $\{y \in \mathcal{P}_{\aleph_1}(X) \mid x \not\subseteq y\} \in I$. The ideal I is *normal* if for all $f\colon X \to I$, $S_f = \{x \in \mathcal{P}_{\aleph_1}(X) \mid \exists a \in x \text{ s.t. } x \in f(a)\} \in I$, i.e., if I is closed under diagonal unions. We say that a set $S \subseteq \mathcal{P}_{\aleph_1}(X)$ *reflects* to a set $Y \subseteq X$ if $S \cap \mathcal{P}_{\aleph_1}(Y)$ is stationary in $\mathcal{P}_{\aleph_1}(Y)$.

In his book on $\mathbb{P}_{max}$ [2], Woodin defines the statements $\mathrm{WRP}^*_n(\omega_2)$. The natural generalization of these statement to an arbitrary regular cardinal $\kappa \geq \aleph_2$ is as follows.

0.1 Definition. Given a regular cardinal $\kappa \geq \aleph_2$ and an integer $n \in \omega$, $\mathrm{WRP}^*_n(\omega_2)$ is the statement that there exists a normal, fine ideal I on $\mathcal{P}_{\aleph_1}(\kappa)$ such that

- for all stationary $T \subseteq \omega_1$, $\{x \in \mathcal{P}_{\aleph_1}(\kappa) \mid x \cap \omega_1 \in T\} \notin I$;
- for all $S_1, \ldots, S_n \subseteq \mathcal{P}_{\aleph_1}(\kappa)$, if $S \notin I$, then the set of $X \subseteq \kappa$ such that
  - $\omega_1 \subseteq X$;
  - $|X| = \aleph_1$;
  - for all $i \in \{1, \ldots, n\}$, S_i reflects to X

  is stationary.

We write $\mathrm{WRP}^*(\kappa)$ for $\mathrm{WRP}^*_1(\kappa)$. Steel and Zoble [1] have shown that $WRP^*_2(\omega_2)$ + "NS_{ω_1} is saturated" + $2^{\aleph_0} \leq \aleph_2$ implies that the Axiom of Determinacy holds in $L(\mathbb{R})$.

Theorem 0.2. *Let $\kappa \geq \aleph_2$ be a regular cardinal. Then $WRP^*(\kappa)$ holds.*

Proof. Let I be the collection of $S \subseteq \mathcal{P}_{\aleph_1}(\kappa)$ for which the set of $X \in [\kappa]^{\aleph_1}$ for which $\omega_1 \subseteq X$ and S reflects to X is nonstationary. We claim that I satisfies the theorem. First of all, note that every $\subseteq$-noncofinal subset of $\mathcal{P}_{\aleph_1}(\kappa)$ is nonstationary, and every nonstationary subset of $\mathcal{P}_{\aleph_1}(\kappa)$ is in I. Note also that I is closed under countable unions and subsets. Clearly, I satisfies the first conclusion of the definition of $\mathrm{WRP}^*(\kappa)$. Let us check that I is closed under diagonal unions. Fix a function $f\colon \kappa \to I$, and suppose towards a contradiction that S_f reflects to a stationary set of $X \in [\kappa]^{\aleph_1}$ such that $\omega_1 \subseteq X$. Then we may fix such an X such that for all $\alpha \in X$, $f(\alpha)$ does not reflect to X. Let $\langle x_\beta : \beta < \omega_1 \rangle$ be a continuous, $\subseteq$-increasing sequence of countable sets with union X. By assumption, for a stationary set F of $\beta \in \omega_1$, there exists an $\alpha \in x_\beta$ such that $x_\beta \in f(\alpha)$. Pressing down, we have a stationary set F' for which a fixed α suffices, contradicting that assumption that $f(\alpha)$ does not reflect to γ. $\square$

References

[1] J. Steel, S. Zoble, *Determinacy from strong reflection*, preprint

[2] W.H. Woodin. *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal.* Logic and its Applications. de Gruyter, 1999.