Section 1 of the Appendix to Shelah's Proper and Improper Forcing (on weak diamond), retyped with minor modifications

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0.1 Definition. Weak Diamond (Φ) is the statement that for every function $F\colon {}^{<\omega_1}2 \to 2$ there exists an $h\colon \omega_1 \to 2$ such that for all $\eta\colon \omega_1 \to 2$ the set $\{\alpha < \omega_1 \mid F(\eta\restriction\alpha) = h(\alpha)\}$ is stationary.

Note that by replacing h with $1-h$, Φ is equivalent to statement that for every $F\colon {}^{<\omega_1}2 \to 2$ there exists an $h\colon \omega_1 \to 2$ such that for all $\eta\colon \omega_1 \to 2$ the set $\{\alpha < \omega_1 \mid F(\eta\restriction\alpha) \neq h(\alpha)\}$ is stationary.
Weak Diamond can be generalized to arbitrary cardinals as follows.

0.2 Definition. Given a cardinal κ and an ordinal λ of uncountable cofinality, Φ^{κ}_{λ} is the statement that for every $F\colon {}^{<\lambda}2 \to \kappa$ there exists an $h\colon \lambda \to \kappa$ such that for all $\eta\colon \lambda \to 2$ the set $\{\alpha < \lambda \mid F(\eta\restriction\alpha) = h(\alpha)\}$ is stationary.

Under this terminology, Φ is $\Phi^2_{\omega_1}$.

Notation. For any ordinal γ, for any $\alpha \leq \gamma$ and any sequence of nonzero ordinals $\bar{E} = \langle E(i) i < \gamma\rangle$ of length γ, let $D_\alpha(E)$ denote $\mathbf{\Pi}_{i<\alpha}E(i)$, and let $D(\bar{E})$ denote $\bigcup_{\alpha<\gamma}\mathbf{\Pi}_{i<\alpha}E(i)$. When $\bar{E}$ has the constant value γ, we write $D(\gamma)$ for $D(\bar{E})$ and $D_\alpha(\gamma)$ for $D_\alpha(\bar{E})$.

The negation of Φ can be generalized as follows.

0.3 Definition. Suppose that λ is an ordinal of uncountable cofinality, and let $\bar{\mu} = \langle \bar{\mu}(i) : i < \lambda\rangle$ and $\bar{\chi} = \langle \bar{\chi}(i) : i < \lambda\rangle$ be sequences of nonzero cardinals. Unif$(\lambda, \bar{\mu}, \bar{\chi})$ is the statement that there is a function F with domain $D(\bar{\mu})$ such that $F(\eta) \in \bar{\chi}(\alpha)$ for each $\eta \in D_\alpha(\bar{\mu})$ such that for every $h \in D_\lambda(\bar{\chi})$ there exists an $\eta \in D_\lambda(\bar{\mu})$ such that $\{\alpha < \lambda \mid F(\eta\restriction\alpha) = h(\alpha)\}$ contains a club subset of λ.

When $\bar{\mu}$ or $\bar{\chi}$ are constant, we write the constant value μ in place of $\bar{\mu}$ (and similarly for $\bar{\chi}$. So, for example, Unif$(\lambda, 2, 2)$ is the negation of Φ^2_λ.

Notation. When λ is understood, $\langle \mu(0), \mu(1)\rangle$ denotes the sequence of length λ whose first element is $\mu(0)$ and whose other elements are all $\mu(1)$. We let $D(\mu(0), \mu(1))$ and $D_\alpha(\mu(0), \mu(1))$ denote the corresponding versions of $D(\bar{E})$ and $D_\alpha(\bar{E})$.

0.4 Definition. Given $S \subseteq \lambda$, $\mathrm{Unif}(\lambda, S, \bar{\mu}, \bar{\chi})$ denotes the statement that there is a function F with domain $D(\bar{\mu})$ such that $F(\eta) \in \bar{\chi}(\alpha)$ for each $\alpha \in D_\alpha(\bar{\mu})$ such that for every $h \in D_\lambda(\bar{\chi})$ there exists an $\eta \in D_\lambda(\bar{\mu})$ such that

$$\{\alpha \in S \mid F(\eta{\restriction}\alpha) = h(\alpha)\}$$

contains a relative club subset of S.

Again, we can replace $\bar{\mu}$ with μ_0, μ_1, so Unif can take up to five arguments and as few as three.

0.5 Definition. $\text{Id-Unif}(\lambda, \bar{\mu}, \bar{\chi})$ is the set of $S \subseteq \lambda$ such that $\mathrm{Unif}(\lambda, S, \bar{\mu}, \bar{\chi})$ holds.

Note that $\text{Id-Unif}(\lambda, \bar{\mu}, \bar{\chi})$ trivially contains all nonstationary subsets of λ.
The following facts (Lemma 1.4) are straightforward.

Lemma 0.6.

1. *If $\bar{\chi}(i) = 1$ for club many i, then Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ holds.*
2. *The truth value of Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ is invariant under nonstationary changes to $\bar{\chi}$.*
3. *The truth value of Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ is invariant under nonstationary changes to S.*
4. *If Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ holds, it holds for any smaller $(\leq)$ χ-sequence and any larger $(\geq)$ μ-sequence.*
5. *Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ implies that $D_\lambda(\bar{\chi})$ has at most $|D_\lambda(\bar{\mu})|$ many equivalence classes under equivalence modulo the nonstationary ideal restricted to S.*
6. *Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ follows from the statement that for some $\beta < \lambda$, $D_\lambda(\bar{\chi})$ has at most $|D_\beta(\bar{\mu})|$ many equivalence classes under the nonstationary ideal restricted to S.*

The first major result of the appendix is the following (Shelah's Lemma 1.5), where for a sequence $\bar{\mu} = \langle \bar{\mu}(i) : i < \lambda \rangle$ and an $\alpha < \lambda$ we let $\bar{\mu}[\alpha]$ be the sequence of length λ whose ith element is $\bar{\mu}(\alpha + i)$.

Theorem 0.7. *Given λ, S, $\bar{\mu}$ and $\bar{\lambda}$ as above, and letting $\mu_0 = |D(\bar{\mu})|$ and $\mu_1 = \min_{\alpha<\lambda} |D(\bar{\mu}[\alpha])|$, Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ and Unif$(\lambda, S, \mu_0, \mu_1, \bar{\chi})$ are equivalent.*

Note that $\mu_0 \geq \mu_1$ in the statement of Theorem 0.7. The first lemma (Fact 1.5A) towards the proof of Theorem 0.7 involves a change of μ-sequence.

Lemma 0.8. *Suppose that $\bar{\mu}$ and $\bar{\nu}$ are sequences of nonzero cardinals of length λ, and suppose that there is a continuous, injective, order-preserving partial map $g\colon D(\bar{\mu}) \to D(\bar{\nu})$ such that for every $\eta \in D_\lambda(\bar{\mu})$ the set of $i < \lambda$ with $\eta{\restriction}i \in Dom(g)$ is club. Then Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ implies Unif$(\lambda, S, \bar{\nu}, \bar{\chi})$.*

Proof of Lemma 0.8. First note that we may assume that g is length-preserving as well. Given F witnessing $\mathrm{Unif}(\lambda, S, \bar{\mu}, \bar{\chi})$, define $F^*(\eta)$ to be $F(g^{-1}(\eta))$ for all η in the range of g, and let F^* be defined in any manner on other η's. Then F^* witnesses $\mathrm{Unif}(\lambda, S, \bar{\nu}, \bar{\chi})$. □

Proof of the forward direction of Theorem 0.7. Let $\alpha < \lambda$ be such that $\mu_1 = |D(\bar{\mu}[\alpha])|$. Then $\bar{\mu}(i) \leq \mu_1$ for all $i \in [\alpha, \lambda)$. Let $\mu_0' = |D_\alpha(\bar{\mu})|$, and let $\langle \nu_\xi : \xi < \mu_0' \rangle$ enumerate $D_\alpha(\bar{\mu})$. Then $\mu_0' \leq \mu_0$. Then there is a partial function $g \colon D(\bar{\mu}) \to D(\mu_0, \mu_1)$ as in the statement of Lemma 0.8 with domain the set of $\eta \in D(\bar{\mu})$ of length at least α, defined by $g(\nu_\xi{}^\frown\eta) = \langle \xi \rangle^\frown \eta$. □

For the reverse direction of Theorem 0.7, Shelah proves the following lemmas (the second and third of which make up Claim 1.6), which show that it suffices to suppose that $\langle \bar{\mu}(i) : 1 \leq i < \lambda \rangle$ is nondecreasing.

Lemma 0.9. *If there exists $j < \lambda$ such that $\bar{\mu}(i) = 1$ for all $i \in [j, \lambda)$, then $\mu_0 = |D_j(\bar{\mu})|$, $\mu_1 = 1$, and the conclusion of Theorem 0.7 holds.*

Proof. By Lemma 0.6, parts (5) and (6). □

Lemma 0.10. *If $\langle \alpha_i : i < \lambda \rangle$ is an increasing, continuous sequence of elements of λ with $\alpha_0 = 0$, and for each $i < \lambda$, $\nu_i = |\mathbf{\Pi}_{\alpha_i \leq j < \alpha_{i+1}} \bar{\mu}(j)|$, then Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ and Unif$(\lambda, S, \bar{\nu}, \bar{\chi})$ are equivalent.*

Proof. Translate between F's using the natural bijection between $D(\bar{\mu})$ and $D(\bar{\nu})$. □

Lemma 0.11. *There exists $\langle \alpha_i : i < \lambda \rangle$, an increasing, continuous sequence of elements of λ with $\alpha_0 = 0$, such that, letting $\nu_i = |\mathbf{\Pi}_{\alpha_i \leq j < \alpha_{i+1}} \bar{\mu}(j)|$ for each $i < \lambda$, $\langle \nu_i : 1 \leq i < \lambda \rangle$ is nondecreasing.*

Proof. Let κ^* be the least cardinal κ such that $\{i < \lambda \mid \bar{\mu}(i) \geq \kappa\}$ is bounded below λ. Let $\alpha^* < \lambda$ be such that $\bar{\mu}(i) < \kappa^*$ for all $i \in [\alpha^*, \lambda)$. Let $\alpha_1 = \alpha^*$. There are three cases, depending on whether κ^* is a successor cardinal, has cofinality λ, or cofinality less than λ. □

Proof of the reverse direction of Theorem 0.7. By Lemmas 0.9, 0.10 and 0.11, we may assume that $\bar{\mu}(i) \leq \bar{\mu}(j)$ whenever $1 \leq i \leq j < \lambda$. By Lemma 0.8, we need only find a partial injective, continuous, order-preserving embedding g from $D(\mu_0, \mu_1)$ to $D(\bar{\mu})$ with the property that every $\eta \in D_\lambda(\mu_0, \mu_1)$ has club many initial segments in the domain of g. Let $\alpha^* \in [3, \lambda)$ be such that for all $\beta \in [\alpha^*, \lambda)$, $|D(\bar{\mu}[\beta])| = \mu_1$. The desired function g can be defined in a straightforward recursive manner once one sees that $D(\bar{\mu})$ contains an antichain of size μ_0 (consisting of elements of length at least α^*, and that there is an antichain of size μ_1 extending each element of $D(\bar{\mu})$ of length at least α^*. This construction is relatively straightforward, alternating codes and 0's until the coding is done, then punctuating with two 1's. In the first case, one starts with $\nu(0)$ and fills until α^* with all 0's. Since $\bar{\mu}$ is increasing, one can always save values for later. The second case is simpler. □

The following lemma (Shelah's Claim 1.7 (2)) says that fewer than λ many witnesses to Unif$(\lambda, S, \ldots)$ can be glued together to make a witness to Unif$(\lambda, S, \ldots)$ on the corresponding product. Shelah's Claim 1.7 (1) is a special case of Claim 1.7 (2).

Lemma 0.12. *Suppose that $\kappa < \lambda$, and Unif$(\lambda, S, \bar{\mu}_\xi, \bar{\chi}_\xi)$ holds for all $\xi < \kappa$. Define $\bar{\mu}(i)$ and $\bar{\chi}(i)$ for $i < \lambda$ by $\bar{\mu}(i) = |\mathbf{\Pi}_{\xi<\kappa}\bar{\mu}_\xi(i)|$ and $\bar{\chi}(i) = |\mathbf{\Pi}_{\xi<\kappa}\bar{\chi}_\xi(i)|$. Then Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ holds.*

Proof. Fix witnesses F_ξ $(\xi < \kappa)$ to Unif$(\lambda, S, \bar{\mu}_\xi, \bar{\chi}_\xi)$, and fix bijections $G^i : \bar{\mu}(i) \to \mathbf{\Pi}_{\xi<\kappa}\bar{\mu}_\xi(i)$ and $H^i : \mathbf{\Pi}_{\xi<\kappa}\bar{\mu}_\xi(i) \to \bar{\chi}(i)$. For each $i < \lambda$, $\xi < \kappa$ and $\alpha < \bar{\mu}(i)$, $G^i_\xi(\alpha)$ be the ξth coordinate of $G^i(\alpha)$, and let $H^i_\xi(\alpha)$ be the ξth coordinate of $H^i(\alpha)$. Given $\eta \in D_\delta(\bar{\mu})$ for some $\delta < \lambda$, let

$$F(\eta) = H^\delta(\langle F_\xi(\langle G^i_\xi(\eta(i)) : i < \delta\rangle) : \xi < \kappa\rangle).$$

Now given $h \in D_\lambda(\bar{\chi})$, let h_ξ $(\xi < \kappa)$ be such that $H^\delta(\langle h_\xi(\delta) : \xi < \kappa\rangle) = h(\delta)$ for all $\delta < \lambda$. Fix η_ξ $(\xi < \kappa)$ and clubs C_ξ $(\xi < \kappa)$ such that for each $\xi < \kappa$ and each $\delta \in C_\xi \cap S$, $F_\xi(\eta_\xi \restriction \delta) = h_\xi(\delta)$. Let $\eta \in D(\bar{\mu})$ be such that $G^i(\eta(i)) = \langle \eta_\xi(i) : \xi < \kappa\rangle$ for all $i < \lambda$. Then for every $\delta \in S \cap \bigcap_{\xi<\kappa} C_\xi$,

$$F(\eta\restriction\delta) = H^\delta(\langle F_\xi(\langle G^i_\xi(\eta(i)) : i < \delta\rangle) : \xi < \kappa\rangle) =$$

$$H^\delta(\langle F_\xi(\eta_\xi\restriction\delta) : \xi < \kappa\rangle) = H^\delta(\langle h_\xi(\delta) : \xi < \kappa\rangle) = h(\delta).$$

□

The following is Claim 1.7 (3).

Lemma 0.13. *Suppose that $\bar{\mu}$ is a nondecreasing sequence of infinite cardinals, Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$ holds, and $\bar{\rho}(i) \leq \bar{\chi}(i)^{|i|}$ for all $i < \lambda$. Then Unif$(\lambda, S, \bar{\mu}, \bar{\rho})$ holds.*

Proof. It suffices to suppose that $\bar{\rho}(i) = \bar{\xi}(i)^{|i|}$ for all $i < \lambda$, by Lemma 0.6 (4). Let F witness Unif$(\lambda, S, \bar{\mu}, \bar{\chi})$. For each $i < \lambda$, fix a bijection $H^i : \mathbf{\Pi}_{j<i}\bar{\xi}(i) \to \bar{\rho}(i)$. Fix increasing functions $k_\zeta : \lambda \to \lambda$ $(\zeta < \lambda)$ with disjoint ranges. For any $\eta \in D(\bar{\mu}) \cup D_\lambda(\bar{\mu})$ and any $\zeta < \lambda$, let $\eta[\zeta]$ be the sequence of values that η takes on the range of k_ζ.

Let C^* be the set of $\delta < \lambda$ such that the range of each k_ζ $(\zeta < \delta)$ has ordertype δ below δ. Then C^* is a club. Given $\delta \in C^*$, let $E(\delta)$ be the set of $\eta \in D_\delta(\bar{\mu})$ such that for each $\zeta < \delta$, $\eta[\zeta] \in D_\delta(\bar{\mu})$. For each $\eta \in E(\delta)$, let

$$F^*(\eta) = H^\delta(\langle F(\eta[\zeta]) : \zeta < \delta\rangle).$$

Then given $h \in D_\lambda(\bar{\rho})$, for each $\zeta < \lambda$, let h_ζ be the function with domain (ζ, λ) such that for each $\delta < \lambda$, $H^\delta(\langle h_\zeta(\delta) : \zeta < \delta\rangle) = h(\delta)$. Then for each $\zeta < \lambda$ there exist an $\eta_\zeta \in D_\lambda(\bar{\mu}$ and a club C_ζ such that for all $\delta \in C_\zeta$, $F(\eta_\zeta\restriction\delta) = h_\zeta(\delta)$. Let $\eta \in D_\lambda(\bar{\mu})$ be such that the sequence of values η gives on the range of each

k_ζ is equal to η_ζ. Then if δ is in C^* and in the diagonal intersection of the C_ζ's, then

$$F^*(\eta\restriction\delta) = H^\delta(\langle F((\eta\restriction\delta)[\zeta]) : \zeta < \delta\rangle) =$$
$$H^\delta(\langle F(\eta_\zeta\restriction\delta) : \zeta < \delta\rangle) = H^\delta(\langle h_\zeta(\delta) : \zeta < \delta\rangle) = h(\delta).$$

□

In his Conclusion 1.8, Shelah mentions that if Unif$(\lambda, \mu(0), 2, \chi)$ holds, and $1 < \kappa < \lambda$ and $\mu(0)^\kappa = \mu(0)$, then by Lemma 0.12 we have Unif$(\lambda, \mu(0), 2^\kappa, \chi^\kappa)$, which by our Lemma 0.7 applied twice is equivalent to Unif$(\lambda, \mu(0), 2, \chi^\kappa)$, since the corresponding μ_0 and μ_1 are the same in each case.

In his Lemma 1.9, (1), Shelah mentions that Id-Unif$(\lambda, \bar{\mu}, \bar{\chi})$ is either all of $\mathcal{P}(\lambda)$ or an ideal. To see this, given F_0 and F_1 which work for S_0 and S_1, use F_0 on sequences whose lengths are in S_0, and F_1 for sequences whose lengths are in S_1.

The first half of Shelah's Remark 1.9A notes that Id-Unif$(\lambda, \bar{\mu}, \bar{\chi})$ is equivalent to Id-Unif$(\lambda, \mu_0, \mu_1, \bar{\chi})$, by our Lemma 0.7. The second half of the remark uses later material, so we will save it for later (after Lemmas 0.19-0.23).

The second half of Shelah's Lemma 1.9 is the following.

Lemma 0.14. *If $\bar{\mu}$ is nondecreasing then Id-Unif$(\lambda, \bar{\mu}, \bar{\chi})$ is closed under diagonal unions.*

Proof. Let S_i $(i \in \lambda)$ be elements of Id-Unif$(\lambda, \bar{\mu}, \bar{\chi})$ as witnessed by functions F_i $(i < \lambda)$. Let S be the diagonal union of $\langle S_i : i < \lambda\rangle$, and let $f \colon S \to \lambda$ be a regressive function such that $\delta \in S_{f(\delta)}$ for each $\delta \in S$. Let $\langle k_\zeta : \zeta < \lambda\rangle$ and C^* be as in the proof of Lemma 0.13. For each $\delta \in C^*$ and $\eta \in D_\delta(\bar{\mu})$, let $F(\eta) = F_{f(\delta)}(\eta[f(\delta)])$, where $\eta[\zeta]$ is defined as in Lemma 0.13. Now given $h \in D(\bar{\chi})$, for each $\zeta < \lambda$, let $\eta_\zeta \in D_\lambda(\bar{\mu})$ and club $C_\zeta \subseteq \lambda$ be such that $F_\zeta(\eta_\zeta\restriction\delta) = h(\delta)$ for all $\delta \in C_\zeta \cap S_\zeta$. Let $\eta \in D_\lambda(\bar{\mu})$ be such that $\eta[\zeta] = \eta_\zeta$ for all $\zeta < \lambda$. Then for all $\delta \in S \cap C^* \cap \triangle_{\zeta<\lambda} C_\zeta$,

$$F(\eta\restriction\delta) = F_{f(\delta)}((\eta\restriction\delta)[f(\delta)]) =$$
$$F_{f(\delta)}(\eta_{f(\delta)}\restriction\delta) = h(\delta).$$

□

Now we are finally up to Shelah's Theorem 1.10.

Theorem 0.15. *If λ is regular, $2^{<\lambda} < 2^\lambda$ and there is no collection of sets $S_i \in [\mu]^\lambda$ $(i < 2^\lambda)$ with pairwise finite intersection, then Unif$(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ fails.*

Shelah notes that the existence of such a collection of S_i's would imply that $\mu^{\aleph_0} \geq 2^\lambda$, by picking a countable subset of each S_i. He notes that the nonexistence of such a collection implies that $\mu < 2^\lambda$. He also notes that his Theorem 1.10 has the following corollary.

Corollary 0.16. *If for some $\theta < \lambda$, $2^\theta = 2^{<\lambda} < 2^\lambda$, then Unif$(\lambda, 2^\theta, 2)$ fails.*

Proof of Corollary 0.16. By Theorem 0.15 we get that $\mathrm{Unif}(\lambda, 2^\theta, 2^\theta, 2^\theta)$ fails. This is equal to $\mathrm{Unif}(\lambda, 2^\theta, 2^\theta, 2)$ by Lemmas 0.6 (2) and 0.12. $\square$

Proof of Theorem 0.15. Let F witness $\mathrm{Unif}(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$. Let Mod be the set of sequences

$$\langle \alpha, C_0, g_0, C_1, g_1, \ldots C_\beta, g_\beta, \ldots \rangle_{\beta < \beta(0)}$$

where α and $\beta(0)$ are elements of λ, each g_β is a function from $\alpha \setminus \{0\}$ to ${}^{<\lambda}2$, and each C_β is a closed subset of α. Then Mod has cardinality $2^{<\lambda}$, so we can fix a bijection H from Mod to ${}^{<\lambda}2$. For each function $f \in {}^\lambda 2$ and each $\beta < \lambda$, define $h_{f,\beta} \colon \lambda \to {}^{<\lambda}2$, $g_{f,\beta} \in D_\lambda(\mu, 2^{<\lambda})$ and club $C_{f,\beta} \subseteq \lambda$ as follows.

- $h_{f,0} = g_{f,0} = f$ and $C_{f,0} = \lambda \setminus \{0\}$;
- for $\gamma > 0$:
  - $h_{f,\gamma}(i) =$

    $$H(\langle \alpha, C_{f,0} \cap \alpha, g_{f,0} \restriction (\alpha \setminus \{0\}, \ldots, C_{f,\beta} \cap \alpha, g_{f,\beta} \restriction (\alpha \setminus \{0\}), \ldots \rangle_{\beta < \gamma},$$

    where $\alpha = \alpha(i, f, \gamma) = \min \bigcap_{\beta < \gamma} C_{f,\beta} \setminus (i+1)$;
  - $g_{f,\gamma}$ is such that some club C is subset of

    $$\{\delta < \lambda \mid F(g_{f,\gamma} \restriction \delta) = h_{f,\gamma}(\delta)\},$$

    and $C_{f,\gamma}$ is the set of limit points of $C \cap \bigcap_{\beta < \gamma} C_{f,\beta}$.

The key claim is the following: if f_1 and f_2 are distinct functions from λ to 2, and $f_1(0) = f_2(0)$, then the set of $\gamma < \lambda$ such that

$$g_{f_1,\gamma} \restriction \min C_{f_1,\gamma} = g_{f_2,\gamma} \restriction \min C_{f_2,\gamma}$$

is finite.

Given the claim, the remainder of the proof is as follows. For each $f \colon \lambda \to 2$, let A_f be the set of sequences $\langle \gamma, g_{f,\gamma}(0), g_{f,\gamma} \restriction (\delta \setminus \{0\}), f(0) \rangle$, where $\gamma < \lambda$ and $\delta = \min C_{f,\gamma}$. For distinct f's, these sets have finite intersection, so there are 2^λ such A_f's. Each A_f is a subset of $\lambda \times \mu \times {}^{<\lambda}({}^{<\lambda}2) \times 2$ of cardinality λ. If $\mu \geq 2^{<\lambda}$ then we have a contradiction. Otherwise, 2^λ is less than or equal to $(2^{<\lambda})^{\aleph_0}$ (since the A_f's have distinct countable subsets), which is equal to $2^{<\lambda}$, giving another contradiction. This finishes the proof of the theorem, assuming the claim.

Finally, we prove the claim. To do this, suppose that f_1 and f_2 are functions from λ to 2 such that $f_1(0) = f_2(0)$, and let $\langle j_n : n < \omega \rangle$ be an increasing sequence of members of λ, with supremum j, such that

$$g_{f_1,j_n} \restriction \min C_{f_1,j_n} = g_{f_2,j_n} \restriction \min C_{f_2,j_n}$$

for all $n < \omega$. For each $\ell \in \{1, 2\}$, let $C^\ell = \bigcap_{n<\omega} C_{f_\ell,j_n}$ and let $\langle \gamma_i^\ell : i < \lambda \rangle$ be an increasing enumeration of C^ℓ. For notational convenience, let $\gamma_\lambda^\ell = \lambda$ for $\ell \in \{1, 2\}$.

We prove by induction on $i \leq \lambda$ that

a) $\gamma_i^1 = \gamma_i^2$;

b) for all $\zeta < j$,

$$g_{f_1,\zeta}\restriction(\gamma_i^1 \setminus \{0\}) = g_{f_2,\zeta}\restriction(\gamma_i^2 \setminus \{0\})$$

and

$$C_{f_1,\zeta} \cap \gamma_i^1 = C_{f_2,\zeta} \cap \gamma_i^2.$$

Letting $i = \lambda$ and $\zeta = 0$, this shows that $f_1 = f_2$, since $g_{f,0} = f$ for all $f \in {}^{\lambda}2$.

The induction has three cases: $i = 0$, i limit and i successor. For each $n \in \omega$ we let $\delta_n = \min Cf_1, j_n = \min C_{f_2,j_n}$.

Case $i = 0$. For each $\ell \in \{1, 2\}$, $\langle C_{f_\ell,j_n} : n < \omega\rangle$ is a $\subseteq$-decreasing sequence of clubs, so $\gamma_0^\ell = \sup\{\delta_n : n < \omega\}$. This shows (a). To see (b), fix $\zeta < j$ and $n \in \omega$ such that $\zeta < \jmath_n$ and if

$$g_{f_1,\zeta}\restriction(\gamma_0^1 \setminus \{0\}) \neq g_{f_2,\zeta}\restriction(\gamma_0^2 \setminus \{0\})$$

then

$$g_{f_1,\zeta}\restriction(\delta_n \setminus \{0\}) \neq g_{f_2,\zeta}\restriction(\delta_n \setminus \{0\})$$

and if

$$C_{f_1,\zeta} \cap \gamma_0^1 \neq C_{f_2,\zeta} \cap \gamma_0^2$$

then

$$C_{f_1,\zeta} \cap \delta_n \neq C_{f_2,\zeta} \cap \delta_n.$$

We have assumed that $g_{f_1,j_n}\restriction\delta_n = g_{f_2,j_n}\restriction\delta_n$. Also, for each $\ell \in \{1, 2\}$, $\delta_n \in C_{f_\ell,j_n}$, so

$$F(g_{f_\ell,j_n}\restriction\delta_n) = h_{f_\ell,j_n}(\delta_n) = H(\langle \alpha, \dots, C_{f_\ell,\beta} \cap \alpha, g_{f_\ell,\beta}\restriction(\alpha \setminus \{0\}), \dots\rangle_{\beta<j_n}),$$

where

$$\alpha = \alpha(\delta_n, f_\ell, j_n) = \min[\bigcap_{\beta<j_n} C_{f_\ell,\beta} \setminus (\delta_n + 1)].$$

As H is injective, it follows that

$$g_{f_1,\zeta}\restriction(\delta_n \setminus \{0\}) = g_{f_2,\zeta}\restriction(\delta_n \setminus \{0\})$$

and

$$C_{f_1,\zeta} \cap \delta_n = C_{f_2,\zeta} \cap \delta_n,$$

which means that

$$g_{f_1,\zeta}\restriction(\gamma_0^1 \setminus \{0\}) = g_{f_2,\zeta}\restriction(\gamma_0^2 \setminus \{0\})$$

and

$$C_{f_1,\zeta} \cap \gamma_0^1 = C_{f_2,\zeta} \cap \gamma_0^2.$$

This concludes the proof of the case $i = 0$.

The case where i is a limit ordinal is immediate.

Finally, suppose that (a) and (b) hold for $i < \lambda$, and let us see that they hold for $i+1$. We have that for all $n \in \omega$, $g_{f_1,j_n}\restriction\gamma_0^1 = g_{f_2,j_n}\restriction\gamma_0^2$. Note that γ_0^1 and γ_0^2 are both nonzero, as $0 \notin C_{f,0}$ for all functions $f \in {}^{\lambda}2$. By the induction hypothesis,

$$g_{f_1,j_n}\restriction(\gamma_i^1 \setminus \{0\}) = g_{f_2,j_n}\restriction(\gamma_i^2 \setminus \{0\}).$$

Putting these two facts together, we have that

$$g_{f_1,j_n}\restriction\gamma_i^1 = g_{f_2,j_n}\restriction\gamma_i^2.$$

For $\ell \in \{1,2\}$ we have $\gamma_i^\ell \in \bigcap_{n\in\omega} C_{f_\ell,j_n}$, so for all $n \in \omega$, $F(g_{f_\ell,j_n}\restriction\gamma_i^\ell) = h_{f_\ell,j_n}(\gamma_i^\ell)$, which is equal to

$$H(\langle \alpha_n^\ell, \ldots, C_{f_\ell,\beta} \cap \alpha_n^\ell, g_{f_\ell,\beta}\restriction(\alpha_n^\ell \setminus \{0\}), \ldots\rangle_{\beta<j_n}),$$

where $\alpha_n^\ell = \alpha(\gamma_i^\ell, f_\ell, j_n) = \min[\bigcap_{\beta<j_n} C_{f_\ell,\beta} \setminus (\gamma_i^\ell + 1)]$. As H is injective, it follows that $\alpha_n^1 = \alpha_n^2$ for each $n \in \omega$. Since for each $f \in {}^{\lambda}2$ and each $\beta < \beta' < \lambda$, $C_{f,\beta'}$ is contained in the limits points of $C_{f,\beta}$, it follows that $\langle \alpha_n^\ell : n < \omega\rangle$ is increasing for each $\ell \in \{1,2\}$. It follows furthermore (for the same reason) that for each $\ell \in \{1,2\}$,

$$\bigcup_{n\in\omega} \alpha_n^\ell = \min[\bigcap_{\beta<j_n} C_{f_\ell,\beta} \setminus (\gamma_i^\ell + 1)] = \gamma_{\gamma+1}^\ell,$$

so $\gamma_{i+1}^1 = \gamma_{i+1}^2$ (which is part (a)). Clause (b) follows from the fact that every $\zeta < j$ is less than some j_n, and the fact that $\gamma_{i+1}^1 = \gamma_{i+1}^2$ is the supremum of the set of ordinals $\alpha_n^1 = \alpha_n^2$ for $n \in \omega$. □

0.17 Definition. Given a set X and a cardinal λ, a collection $\mathcal{F} \subseteq [X]^\lambda$, is called a (X,λ)-cover if every subset of X of cardinality λ has a superset in $\mathcal{F}$. We let $\mathrm{cov}(X,\lambda)$ denote the least cardinality of a (X,λ)-cover consisting of sets of cardinality λ.

The following facts are immediate, for sets X and Y and cardinals λ and μ.

1. $X \subseteq Y \Rightarrow \mathrm{cov}(X,\lambda) \leq \mathrm{cov}(Y,\lambda)$.
2. $|X| \leq |Y| \Rightarrow \mathrm{cov}(X,\lambda) \leq \mathrm{cov}(Y,\lambda)$.
3. $|X| = |Y| \Rightarrow \mathrm{cov}(X,\lambda) = \mathrm{cov}(Y,\lambda)$.
4. $\lambda < \mu \Rightarrow \mathrm{cov}(\mu,\lambda) \geq \mu$.
5. $\mathrm{cov}(\lambda,\lambda) = 1$.

Lemma 0.18. *Suppose that λ and μ are cardinals, and α is an ordinal.*

1. *if $\lambda \leq \mu$, then $cov(\mu^+,\lambda) = cov(\mu,\lambda) + \mu^+$.*
2. *if μ is a limit ordinal, $\lambda < \mu$ and $\langle \mu_i : i < cf(\mu)\rangle$ is an increasing sequence with limit μ and $\mu_0 > \lambda$, then $cov(\mu,\lambda) \leq \Pi_{i<cf(\mu)} cov(\mu_i,\lambda)$.*

3. $cov(\lambda^{+\alpha}, \lambda) \leq (\lambda^{+\alpha})^{|\alpha|}$.

Proof. For the first conclusion, $\geq$ follows from facts (1) and (4) listed above. The other direction follows from the fact that each subset of μ^+ of cardinality λ is bounded. For the second conclusion, take a product of the covers for each μ_i. For the third conclusion, argue by induction on α. The case $\alpha = 0$ is fact (5) above. To go from β to $\beta + 1$, note that $\mathrm{cov}(\lambda^{\beta+1}, \lambda) = \mathrm{cov}(\lambda^{\beta}, \lambda) + \lambda^{\beta+1}$ (by the first conclusion of this lemma) which is less than or equal to $(\lambda^{\beta})^{|\beta|} + \lambda^{\beta+1}$ (by the induction hypothesis) which is less than or equal to $(\lambda^{\beta+1})^{\beta+1}$. If α is a limit of the sequence $\langle \alpha_i : i < cf(\alpha)\rangle$, then $\mathrm{cov}(\lambda^{+\alpha}, \lambda) \leq \Pi_{i<cf(\alpha)}\mathrm{cov}(\lambda^{+\alpha_i}, \lambda)$ (be the second conclusion of this lemma) which is less than or equal to $((\lambda^{+\alpha})^{|\alpha|})^{cf(\alpha)} = (\lambda^{+\alpha})^{|\alpha|}$. □

The next five lemmas are the first five parts of Shelah's Lemma 1.14, revised to accommodate the sixth part. In all cases, I believe I have used a hypothesis no stronger than the one used by Shelah.

Lemma 0.19. *Suppose that λ and μ are cardinals, with λ regular and uncountable, and let $\bar{\chi} = \langle \chi_i : i < \lambda\rangle$ be a sequence of nonzero cardinals. Let S be a subset of λ, and suppose that Unif$(\lambda, S, \mu, \bar{\chi})$ holds. Then Unif$(\lambda, S, cov(\mu, \lambda), \lambda, \bar{\chi})$ holds.*

Proof. Let F witness Unif$(\lambda, S, \mu, \bar{\chi})$, let $\langle A_i : i < \mathrm{cov}(\mu, \lambda)\rangle$ be a (μ, λ)-cover, and enumerate each A_i by $\langle \alpha_{i,j} : j < \lambda\rangle$. For each $\eta \in D(\mathrm{cov}(\mu, \lambda), \lambda)$, let

$$F^*(\eta) = F(\langle \alpha_{\eta(0),\eta(1+i)} : i < length(\eta)\rangle).$$

Then given $h \in D_\lambda(\bar{\chi})$, let $\eta \in D_\lambda(\mu)$ be such that $\{i \in S : F(\eta{\restriction}i) = h(i)\}$ contains a club relative to S. Let $\eta^* \in D_\lambda(\mathrm{cov}(\mu, \lambda), \lambda)$ be such that $range(\eta) \subseteq A_{\eta^*(0)}$ and such that $\eta^*(1+i) = \eta(i)$. Then $F^*(\eta^*{\restriction}i) = F(\eta{\restriction}i)$ for all $i \geq \omega$. □

Lemma 0.20. *Suppose that λ, χ, μ_0 and μ_1 are cardinals, with λ regular and uncountable, $\chi \geq \lambda$ and $cov(\chi, \lambda) \leq \mu_0$. Let S be a subset of λ. Then Unif$(\lambda, S, \mu_0, \mu_1, \chi)$ holds if and only if Unif$(\lambda, S, \mu_0, \mu_1, \lambda)$ holds.*

Proof. The forward direction follows from Lemma 0.6 (4). For the reverse direction, let F witness Unif$(\lambda, S, \mu_0, \mu_1, \lambda)$. Let e_0 and e_1 be functions from μ_0 to μ_0 such that $\alpha \mapsto (e_0(\alpha), e_1(\alpha))$ is a bijection from μ_0 to $\mu_0 \times \mu_0$. Let $\langle A_i : i < \mu_0\rangle$ be a (χ, λ)-cover, and for each $i < \mu_0$ let $q_i \colon \lambda \to A_i$ be a bijection. For each $\eta \in D(\mu_0, \mu_1)$, let $k(\eta)$ be the sequence produced by replacing the first member of η with $e_1(\eta(0))$, and let $F^*(\eta) = q_{e_0(\eta(0))}(F(k(\eta)))$. Then given $h \in D(\chi)$, let $i < \lambda$ be such that the range of h is contained in A_i. Then there is an $\eta \in D_\lambda(\mu_0, \mu_1)$ such that $F(\eta{\restriction}i) = q_i^{-1}(h(i))$ for a club of i relative to S. Let $\eta^* \in D_\lambda(\mu_0, \mu_1)$ be such that $e_0(\eta^*(0)) = i$, $e_1(\eta^*(0)) = \eta(0)$ and $\eta^*(j) = \eta(j)$ for all nonzero j in the domain of η. Then for all $j < \lambda$, $F^*(\eta^*{\restriction}j) = q_i(F(k(\eta^*{\restriction}j)))$, which is equal to $h(j)$ for a club of j relative to S. □

Lemma 0.21. *Suppose that λ, χ, μ_0 and μ_1 are cardinals, with λ regular and uncountable, $\chi \geq \lambda$ and $cov(\chi, \lambda) \leq \mu_0$. Suppose that λ is not a strong limit cardinal, and that $\chi \geq 2$. Let S be a subset of λ. Then Unif$(\lambda, S, \mu_0, \mu_1, \chi)$ holds if and only if Unif$(\lambda, S, \mu_0, \mu_1, 2)$ holds.*

Proof. The forward direction follows from Lemma 0.6 (4). For the reverse direction, it suffices to consider the case $\chi = \lambda$, by Lemma 0.20. Since λ is not a strong limit, this case follows from Lemma 0.13 and parts (2) and (4) of Lemma 0.6. $\square$

Lemma 0.22. *Suppose that λ, μ_0 and μ_1 are cardinals, with λ regular and uncountable, and $\lambda \leq \mu_1$. Let $\bar{\chi} = \langle \chi_i : i < \lambda \rangle$ be a sequence of nonzero cardinals. Let S be a subset of λ, and suppose that Unif$(\lambda, S, \mu_0, \mu_1, \bar{\chi})$ holds. Then Unif$(\lambda, S, \mu_0 + cov(\mu_1, \lambda), \lambda, \bar{\chi})$ holds.*

Proof. If $\mu_0 \geq \mathrm{cov}(\mu_1, \lambda)$ then this follows from Lemma 0.6 (4). Supposing otherwse, $\mu_0 + \mathrm{cov}(\mu_1, \lambda) = \mu_0 \cdot \mathrm{cov}(\mu_1, \lambda) = \mathrm{cov}(\mu_1, \lambda)$, so we may fix functions $e_0 \colon \mathrm{cov}(\mu_1, \lambda) \to \mu_0$ and $e_1 \colon \mathrm{cov}(\mu_1, \lambda) \to \mathrm{cov}(\mu_1, \lambda)$ such that $\alpha \mapsto (e_0(\alpha), e_1(\alpha))$ is a bijection from $\mathrm{cov}(\mu_1, \lambda)$ to $\mu_1 \times \mathrm{cov}(\mu_1, \lambda)$. Let F witness Unif$(\lambda, S, \mu_0, \mu_1, \bar{\chi})$, let $\langle A_i : i < \mathrm{cov}(\mu_1, \lambda) \rangle$ be a (μ_1, λ)-cover, and enumerate each A_i by $\langle \alpha_{i,j} : j < \lambda \rangle$. For each $\eta \in D(\mathrm{cov}(\mu, \lambda), \lambda)$, $k(\eta)$ be the sequence obtained by replacing $\eta(0)$ with $e_0(\eta(0))$, and $\eta(i)$ for each nonzero i with $\alpha_{e_1(\eta(0)), \eta(i)}$, and let

$$F^*(\eta) = F(k(\eta)).$$

Then given $h \in D_\lambda(\bar{\chi})$, let $\eta \in D_\lambda(\mu_0, \mu_1)$ be such that $\{i \in S : F(\eta \restriction i) = h(i)\}$ contains a club relative to S.

Let $i < \lambda$ be such that the range of η is contained in A_i. Let $\eta^* \in D_\lambda(\mu_0 + \mathrm{cov}(\mu_1, \lambda), \mu_1)$ be such that $e_1(\eta^*(0)) = i$, $e_0(\eta^*(0)) = \eta(0)$ and $\alpha_{i, \eta^*(j)} = \eta(j)$ for all nonzero j in the domain of η. Then for all $j < \lambda$, $F^*(\eta^* \restriction j) = F(k(\eta^* \restriction j))$, which is equal to $h(j)$ for a club of j relative to S. $\square$

Lemma 0.23. *Suppose that λ, μ_0 and μ_1 are cardinals, with λ regular, uncountable and not a strong limit. Let $\bar{\chi} = \langle \chi_i : i < \lambda \rangle$ be a sequence of nonzero cardinals. Suppose that $\mu_0 \geq cov(\mu_1, \lambda)$ and $\lambda < \mu_1$. Let S be a subset of λ, and suppose that Unif$(\lambda, S, \mu_0, \mu_1, \bar{\chi})$ holds. Then Unif$(\lambda, S, \mu_0, 2, \bar{\chi})$ holds.*

Proof. The reverse direction follows from Lemma 0.6 (4). For the forward direction, first note that Lemma 0.22 gives us Unif$(\lambda, \mu_0, \lambda, \bar{\chi})$ from Unif$(\lambda, \mu_0, \mu_1, \bar{\chi})$. Then Unif$(\lambda, \mu_0, 2, \bar{\chi})$ follows from Lemma 0.7, since the two "μ_1"'s are the same in this case (in checking this fact, it may help to break into cases, depending on whether λ is a strong limit or not). $\square$

Let us return now to the second part of Shelah's Remark 1.9A. We want to see that Unif$(\lambda, S, \mu_0, \mu_1, \bar{\chi}) \Leftrightarrow$ Unif$(\lambda, S, \mu_0, \mu_0, \bar{\chi})$, when μ_0 and μ_1 are as in Lemma 0.7 and $\mathrm{cov}(\mu_0, \lambda) = \mu_0$. Since $\mu_0 \geq \mu_1$, the forward direction follows from Lemma 0.6 (4). For the other direction, Unif$(\lambda, S, \mu_0, \mu_0, \bar{\chi})$ implies Unif$(\lambda, \mu_0, 2, \bar{\chi})$ by Lemma 0.23, and Unif$(\lambda, \mu_0, 2, \bar{\chi})$ implies Unif$(\lambda, \mu_0, \mu_1, \bar{\chi})$ by Lemma 0.6 (4).

The following is Shelah's Conclusion 1.15.

Conclusion 0.24. *Suppose that μ is a cardinal less than $\aleph_{\omega_1}$ and that $\mu^{\aleph_0} < 2^{\aleph_1}$. Then Unif$(\omega_1, \mu, \mu, 2)$ fails.*

Proof. Suppose that $\mathrm{Unif}(\omega_1, \mu, \mu, 2)$ holds. Then by Lemmas 0.13 and 0.6(2), $\mathrm{Unif}(\omega_1, \mu, \mu, 2^{\aleph_0})$ holds. Then by Lemma 0.19, $\mathrm{Unif}(\omega_1, \mathrm{cov}(\mu, \aleph_1), \aleph_1, 2^{\aleph_0})$ holds. Let $\alpha < \omega_1$ be such that $\mu = \omega_\alpha$. Then by Lemma 0.18 (3),

$$\mathrm{cov}(\mu, \omega_1) \leq \aleph_\alpha^{|\alpha|} \leq \mu^{\aleph_0} < 2^{\aleph_1},$$

so we can apply Theorem 0.15 with "μ" as $\mathrm{cov}(\mu, \omega_1)$, getting the failure of $\mathrm{Unif}(\omega_1, \mathrm{cov}(\mu, \omega_1), 2^{\aleph_0}, 2^{\aleph_0})$ (note that $2^{<\aleph_1} = 2^{\aleph_0}$) and thereby a contradiction. $\square$

Theorem 0.26 below is Shelah's Theorem 1.16, with the hypothesis $\mu > \lambda$ dropped. As we see in the second line of the proof, this hypothesis follows from the other assumptions of the theorem. With this hypothesis removed, Theorem 0.26 is a strengthening our Theorem 0.15 The proof is a modification of the proof of Theorem 0.15.

0.25 Definition. Let $(*)_{2^\lambda, \mu, \lambda^+}$ denote the statement that there exists a family $\{S_i :< 2^\lambda\}$ of elements of $[\mu]^{\lambda^+}$ such that $|S_i \cap S_j| < \aleph_0$ for all distinct i, $j < 2^\lambda$.

Theorem 0.26. *Suppose that μ and λ are cardinals, with λ a regular uncountable cardinal. Suppose that $2^{<\lambda} < 2^\lambda$ and that Unif$(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$ holds. Then $(*)_{2^\lambda, \mu, \lambda^+}$ holds.*

Proof. As before, the conclusion is immediate if $\mu \geq 2^\lambda$, so assume otherwise. Similarly, by Theorem 0.15, we have that $2^\lambda \leq \mu^{\aleph_0}$. Since $(2^{<\lambda})^{\aleph_0} = 2^{<\lambda} < 2^\lambda$, it follows that $\mu > 2^{<\lambda}$.[1]

For each $\alpha < \lambda^+$, let $\langle B_i^\alpha : i < \lambda \rangle$ be a continuous, $\subseteq$-increasing sequence of sets of cardinality less than λ, with union α.

Let Mod be the set of sequences

$$\langle \alpha, \ldots, \ldots C_\beta, g_\beta, \ldots \rangle_{\beta \in B_i^\alpha}$$

for some $i < \lambda$, where each g_β is a function from $B_i^\alpha \setminus \{0\}$ to ${}^{<\lambda}2$, and each C_β is a closed subset of i. Note that one can recover the corresponding α and i from any given member of Mod. Furthermore, Mod has cardinality $2^{<\lambda}$, so we can fix a bijection H from Mod to ${}^{<\lambda}2$.

Let F witness $\mathrm{Unif}(\lambda, \mu, 2^{<\lambda}, 2^{<\lambda})$. For each function $f \in {}^\lambda 2$ we can define by recursion on $\beta < \lambda^+$ functions $h_{f,\beta} \colon \lambda \to {}^{<\lambda}2$, $g_{f,\beta} \in D_\lambda(\mu, 2^{<\lambda})$ and club sets $C_{f,\beta} \subseteq \lambda$ as follows.

[1] The proof contains the following observation, which is interesting but doesn't appear to be necessary: By Hausdorff's formula, if κ is a cardinal and n is an integer, then $(\kappa^{+n})^{\aleph_0} = \kappa^{+n} + \kappa^{\aleph_0}$. If $\mu = (2^{<\lambda})^n$ for some integer n, then,

$$\mu^{\aleph_0} = \mu + (2^{<\lambda})^{\aleph_0} = \mu + 2^{<\lambda} = \mu < 2^\lambda \leq \mu^{\aleph_0},$$

giving a contradiction. So $\mu \geq (2^{<\lambda})^{+\omega}$.

- $h_{f,0} = g_{f,0} = f$ and $C_{f,0} = \lambda \setminus \{0\}$;
- for $\lambda > 0$:
  - $h_{f,\gamma}(i) = H(\langle \alpha, \ldots, C_{f,\beta} \cap \alpha, g_{f,\beta} \restriction (\alpha \setminus \{0\}), \ldots \rangle_{\beta \in B_i^\gamma}$, where $\alpha = \alpha(i, f, \gamma) = \min \bigcap_{\beta \in B_i^\gamma} C_{f,\beta} \setminus (i+1)$;
  - $g_{f,\gamma}$ is such that some club C is subset of

$$\{\delta < \lambda \mid F(g_{f,\gamma} \restriction \delta) = h_{f,\gamma}(\delta)\},$$

  and $C_{f,\gamma}$ is the set δ which are of limits points of C and of every $C_{f,\beta}$ with $\beta \in B_\delta^\gamma$.

The key claim is the following: if $f_0 \in {}^\lambda 2$ and $\langle j_n : n < \omega \rangle$ is an increasing sequence of elements of λ, then

$$|\{f \in {}^\lambda 2 : \forall n \in \omega \, g_{f,j_n}(0) = g_{f_0,j_n}(0)\}| \leq 2^{<\lambda}.$$

Given the claim, the remainder of the proof is as follows. For each $f_0 \colon \lambda \to 2$, let

$$Y'_{f_0} = \{f \in {}^\lambda 2 : |\{j < \lambda^+ : g_{f,j}(0) = g_{f_0,j}(0)\}| \geq \aleph_0\}.$$

The set of increasing ω-sequences from λ^+ has cardinality $(\lambda^+)^{\aleph_0}$, which (since λ is regular and uncountable) is at most $2^{<\lambda} + \lambda^+$, which (since $\lambda < \mu < 2^\lambda$) is less than 2^λ. Furthermore, for each $f_0 \in {}^\lambda 2$, $|Y'_{f_0}|$ has cardinality at most $(\lambda^+)^{\aleph_0} \times 2^{<\lambda}$, which is also less than 2^λ.[2] The set of pairs $(f_0, f_1) \in {}^\lambda 2$ such that $f_0 \in Y'_{f_1}$ forms an equivalence relation on ${}^\lambda 2$. Let F^* be a subset of ${}^\lambda 2$ which intersects each equivalence class in exactly one point. Since each equivalence class has cardinality less than 2^λ, $|F^*| = 2^\lambda$. Then

$$\{\{(g_{f,j}(0), j) : j < \lambda^+\} : f \in F^*\}$$

is a family of 2^λ many subsets of $\mu \times \lambda^+$ of cardinality λ^+, having pairwise finite intersection. This finishes the proof of the theorem, assuming the claim.

Finally, we prove the claim. To do this, suppose towards a contradiction that $f_0 \in {}^\lambda 2$ and $\langle j_n : n \in \omega \rangle$ witness the failure of the claim. Let

$$Y_{f_0} = \{f \in {}^\lambda 2 : \forall n \in \omega \, g_{f,j_n}(0) = g_{f_0,j_n}(0)\},$$

which by our assumption has cardinality greater than $2^{<\lambda}$. Choose $i < \lambda$ large enough so that $j_n \in B(j_m, i)$ whenever $n < m$. For each $f \in Y_{f_0}$, let $\alpha(f) = \min \bigcap_{n<\omega} C_{f,j_n} \setminus (i+1)$. Define an equivalence relation E on Y_{f_0} by setting $f_1 E f_2$ if and only if and for all $n < \omega$,

- $\alpha(f_1) = \alpha(f_2)$;
- $f_1 \restriction \alpha(f_1) = f_2 \restriction \alpha(f_2)$;

[2]Here our calculation is different from Shelah's.

- for all $n < \omega$,
  - $g_{f_1,j_n} \restriction \alpha(f_1) = g_{f_2,j_n} \restriction \alpha(f_2)$;
  - $C_{f_1,j_n} \restriction \alpha(f_1) = C_{f_2,j_n} \restriction \alpha(f_2)$.

Then E is an equivalence relation with at most $\lambda \times (2^{<\lambda})^{\aleph_0} \times (2^{<\lambda})^{\aleph_0} = 2^{<\lambda}$ classes, so there are distinct functions f_1 and f_2 from λ to 2 such that $f_1 E f_2$.

Let j be the supremum of $\{j_n : n < \omega\}$. For each $\ell \in \{1,2\}$, let $C^\ell = \bigcap_{n<\omega} C_{f_\ell,j_n}$ and let $\langle \gamma^\ell_i : i < \lambda \rangle$ be an increasing enumeration of C^ℓ. For notational convenience, let $\gamma^\ell_\lambda = \lambda$ for $\ell \in \{1,2\}$.

We prove by induction on $i \leq \lambda$ that

a) $\gamma^1_i = \gamma^2_i$;

b) for all $\zeta < j$,
$$g_{f_1,\zeta} \restriction (\gamma^1_i \setminus \{0\}) = g_{f_2,\zeta} \restriction (\gamma^2_i \setminus \{0\})$$
and
$$C_{f_1,\zeta} \cap \gamma^1_i = C_{f_2,\zeta} \cap \gamma^2_i.$$

Letting $i = \lambda$ and $\zeta = 0$, this shows that $f_1 = f_2$ (since $g_{f,0} = f$ for all $f \in {}^\lambda 2$), giving a contradiction and thereby proving the claim.

The induction has three cases: $i = 0$, i limit and i successor. For each $n \in \omega$ we let $\delta_n = \min C^0_{j_n} = \min C^1_{j_n}$.

Case $i = 0$. For each $\ell \in \{1,2\}$, $\langle C_{f_\ell,j_n} : n < \omega \rangle$ is a $\subseteq$-decreasing sequence of clubs, so $\gamma^\ell_0 = \sup\{\delta_n : n < \omega\}$. This shows (a). To see (b), fix $\zeta < j$ and $n \in \omega$ such that $\zeta < \jmath_n$ and if
$$g_{f_1,\zeta} \restriction (\gamma^1_0 \setminus \{0\}) \neq g_{f_2,\zeta} \restriction (\gamma^2_0 \setminus \{0\})$$
then
$$g_{f_1,\zeta} \restriction (\delta_n \setminus \{0\}) \neq g_{f_2,\zeta} \restriction (\delta_n \setminus \{0\})$$
and if
$$C_{f_1,\zeta} \cap \gamma^1_0 \neq C_{f_2,\zeta} \cap \gamma^2_0$$
then
$$C_{f_1,\zeta} \cap \delta_n \neq C_{f_2,\zeta} \cap \delta_n.$$

We have assumed that $g_{f_1,j_n} \restriction \delta_n = g_{f_2,j_n} \restriction \delta_n$. Also, for each $\ell \in \{1,2\}$, $\delta_n \in C_{f_\ell,j_n}$, so
$$F(g_{f_\ell,j_n} \restriction \delta_n) = h_{f_\ell,j_n}(\delta_n) = H(\langle \alpha, \ldots, C_{f_\ell,\beta} \cap \alpha, g_{f_\ell,\beta} \restriction (\alpha \setminus \{0\}), \ldots \rangle_{\beta \in B^{j_n}_{\delta_n}}),$$
where
$$\alpha = \alpha(\delta_n, f_\ell, j_n) = \min[\bigcap_{\beta \in B^{j_n}_{\delta_n}} C_{f_\ell,\beta} \setminus (\delta_n + 1)].$$
As H is injective, it follows that
$$g_{f_1,\zeta} \restriction (\delta_n \setminus \{0\}) = g_{f_2,\zeta} \restriction (\delta_n \setminus \{0\})$$

and

$$C_{f_1,\zeta} \cap \delta_n = C_{f_2,\zeta} \cap \delta_n,$$

which means that

$$g_{f_1,\zeta}\restriction(\gamma_0^1 \setminus \{0\}) = g_{f_2,\zeta}\restriction(\gamma_0^2 \setminus \{0\})$$

and

$$C_{f_1,\zeta} \cap \gamma_0^1 = C_{f_2,\zeta} \cap \gamma_0^2.$$

This concludes the proof of the case $i = 0$.

The case where i is a limit ordinal is immediate.

Finally, suppose that (a) and (b) hold for $i < \lambda$, and let us see that they hold for $i+1$. We have that for all $n \in \omega$, $g_{f_1,j_n}\restriction\gamma_0^1 = g_{f_2,j_n}\restriction\gamma_0^2$. Note that γ_0^1 and γ_0^2 are both nonzero, as $0 \notin C_{f,0}$ for all functions $f \in {}^{\lambda}2$. By the induction hypothesis,

$$g_{f_1,j_n}\restriction(\gamma_i^1 \setminus \{0\}) = g_{f_2,j_n}\restriction(\gamma_i^2 \setminus \{0\}).$$

Putting these two facts together, we have that

$$g_{f_1,j_n}\restriction\gamma_i^1 = g_{f_2,j_n}\restriction\gamma_i^2.$$

For $\ell \in \{1,2\}$ we have $\gamma_i^\ell \in \bigcap_{n\in\omega} C_{f_\ell,j_n}$, so for all $n \in \omega$, $F(g_{f_\ell,j_n}\restriction\gamma_i^\ell) = h_{f_\ell,j_n}(\gamma_i^\ell)$, which is equal to

$$H(\langle\alpha_n^\ell, \ldots, C_{f_\ell,\beta} \cap \alpha_n^\ell, g_{f_\ell,\beta}\restriction(\alpha_n^\ell \setminus \{0\}), \ldots\rangle_{\beta\in B^{j_n}_{\delta_n}}),$$

where $\alpha_n^\ell = \alpha(\gamma_i^\ell, f_\ell, j_n) = \min[\bigcap_{\beta\in B^{j_n}_{\delta_n}} C_{f_\ell,\beta} \setminus (\gamma_i^\ell + 1)]$. As H is injective, it follows that $\alpha_n^1 = \alpha_n^2$ for each $n \in \omega$. Since for each $f \in {}^{\lambda}2$ and each $\beta < \beta' < \lambda$, $C_{f,\beta'}$ is contained in the limits points of $C_{f,\beta}$, it follows that $\langle\alpha_n^\ell : n < \omega\rangle$ is increasing for each $\ell \in \{1,2\}$. It follows furthermore (for the same reason) that for each $\ell \in \{1,2\}$,

$$\bigcup_{n\in\omega} \alpha_n^\ell = \min[\bigcap_{\beta<j} C_{f_\ell,\beta} \setminus (\gamma_i^\ell + 1)] = \gamma_{\gamma+1}^\ell,$$

so $\gamma_{i+1}^1 = \gamma_{i+1}^2$ (which is part (a)). Clause (b) follows from the fact that every $\zeta < j$ is less than some j_n, and the fact that $\gamma_{i+1}^1 = \gamma_{i+1}^2$ is the supremum of the set of ordinals $\alpha_n^1 = \alpha_n^2$ for $n \in \omega$. $\square$

Conclusion 0.27. *If λ is regular and uncountable, μ is a cardinal such that $2^{<\lambda} \leq \mu < 2^\lambda$ and $cov(\mu, \lambda) < 2^\lambda$, then $Unif(\lambda, \mu, \mu, 2)$ fails.*

The following is Definition 5.1 of Chapter 1 of [1].

0.28 Definition. Given cardinals α, β, δ and γ, $\mathrm{cov}(\alpha, \beta, \delta, \gamma)$ is the least cardinal μ such that there is a family P consisting of μ many subsets of α, each of cardinality less than β, such that for every $t \subseteq \alpha$ of cardinality less than δ, t is contained in the union of a subfamily of P of cardinality less than γ.

Note that $\mathrm{cov}(\mu, \lambda^+, \lambda^+, 2)$ is the same as our $\mathrm{cov}(\mu, \lambda)$. Also, $\mathrm{cov}(\lambda, \kappa, \kappa, 2) \leq \mu$ follows trivially from $|\mathcal{S}_{<\kappa^+}(\lambda)| = \lambda^\kappa \leq 2^{<\lambda} \leq \mu$, which holds in the context of Conclusion 0.27 ($\mathcal{S}_{<\kappa^+}(\lambda)$ is the set of subsets of λ of cardinality less than κ^+).

The proof of Conclusion 0.27 uses Theorem 2.1(2) of [2].

Theorem 0.29. *If $\mu > \lambda \geq \kappa$, $\theta = cov(\mu; \lambda^+, \lambda^+, \kappa)$, $cov(\lambda, \kappa, \kappa, 2) \leq \mu$ (or $\leq \theta$) and $\lambda \geq 2^{<\kappa}$ (or just $\theta \geq 2^{<\kappa}$) then $cov(\mu, \lambda^+, \lambda^+, 2)^{<\kappa} = cov(\mu, \lambda^+, \lambda^+, 2)$.*

Proof of Conclusion 0.27. If $\mathrm{Unif}(\lambda, \mu, \mu, 2)$ holds, then by Lemmas 0.6(2) and 0.13, $\mathrm{Unif}(\lambda, \mu, \mu, 2^{<\lambda})$ holds. Then by Lemma 0.19, $\mathrm{Unif}(\lambda, \mathrm{cov}(\mu, \lambda), \lambda, 2^{<\lambda})$ holds, and by Lemma 0.6(4), $\mathrm{Unif}(\lambda, \mathrm{cov}(\mu, \lambda), 2^{<\lambda}, 2^{<\lambda})$ holds. By Lemma 0.26, $(*)_{2^\lambda, \mathrm{cov}(\mu,\lambda), \lambda^+}$ holds.

By Lemma 0.15, $\mathrm{cov}(\mu, \lambda)^{\aleph_0} \geq 2^\lambda$. By Theorem 2.1(2) of [2] (and the remarks before this proof), $\mathrm{cov}(\mu, \lambda)^{\aleph_0} = \mathrm{cov}(\mu, \lambda)$, which contradicts $\mathrm{cov}(\mu, \lambda)^{\aleph_0} \geq 2^\lambda > \mathrm{cov}(\mu, \lambda)$. □

The assumptions for the second part of the following conclusion seem to assume that $2^\theta \leq \mu$. I don't see why this should necessarily be true.

Conclusion 0.30. *If $\theta < \lambda$ are regular cardinals, $2^\theta = 2^{<\lambda} < 2^\lambda$ and μ is a cardinal such that $\lambda \leq \mu < 2^\lambda$ and $(*)_{2^\lambda, \mu, \lambda^+}$ fails, then*

1. *Unif$(\lambda, \mu, 2^\theta, 2^\theta)$ fails;*
2. *if $cov(\mu, \lambda) \leq \mu$ (or just $cov(2^\theta, \lambda) \leq \mu$ (??)) then Unif$(\lambda, \mu, \mu, \lambda)$ fails.*

Proof. The first part follows immediately from Theorem 0.26, once we know that $\mu > \lambda$. In any case, the failure of part one would give $\mu^{\aleph_0} \geq 2^\lambda$. If $\mu = \lambda$, this would contradict the assumptions that λ has uncountable cofinality and that $2^\lambda > 2^{<\lambda}$.

For the second part, assuming $\mathrm{cov}(\mu, \lambda) \leq \mu$, $\mathrm{Unif}(\lambda, \mu, \mu, \lambda)$ is the same as $\mathrm{Unif}(\lambda, \mu, \mu, 2^\theta)$, by Lemma 0.20. (??) □

I assume that the λ in the (original) first part of the following conclusion is supposed to be 2^λ, otherwise the first part is trivial, by the first line.

Conclusion 0.31. *If $\theta < \lambda$ are regular cardinals, $2^\theta = 2^{<\lambda} < 2^\lambda$ and $\theta \geq \beth_\omega$ then*

1. *for every $\mu < \lambda$, Unif$(\lambda, \mu, 2^\theta, 2^\theta)$ fails;*
2. *if $cov(\mu, \lambda) < 2^\lambda$ then Unif$(\lambda, \mu, 2^\theta, \lambda)$ fails.*

Proof. Theorem 1.10 gives us the first part, unless $\mu^{\aleph_0} \geq 2^\lambda$, so assume this to be the case. This also implies that $\mu > 2^\theta$. By Conclusion 0.30 then it suffices to show that $(*)_{2^\lambda, \mu, \lambda^+}$ fails. This follows from the main result of [3], which says (using $\mu \geq \theta \geq 1$) that $\mu^{[\kappa]} = \mu$ for all sufficiently large regular cardinals $\kappa \leq \beth_\omega$, where $\mu^{[\kappa]}$ is the smallest cardinality of a subset $\mathcal{P}$ of $\mathcal{S}_{\leq\kappa}(\mu)$ such that every member of $\mathcal{S}_{\leq\kappa}(\mu)$ is included in a union of less than κ many members of $\mathcal{P}$. To see that this suffices, suppose that $\mathcal{P}$ is such a set for some such κ, and let $\mathcal{P}_1$ be the set of subsets of members of $\mathcal{P}$ of cardinality κ. Then $\mathcal{P}_1$ has cardinality at most $\mu \times 2^\kappa \leq \mu + \beth_\omega = \mu$. If $\{S_i : i < 2^\lambda\}$ witnesses $(*)_{2^\lambda,\mu,\lambda^+}$ we can pick for each $i < 2^\lambda$ an $a_i \in [S_i]^\kappa$, a $\zeta_i^* < \kappa$ and $b_{i,\zeta} \in \mathcal{P}$ for $\zeta < \zeta_i^*$ such that $a_i \subseteq \bigcup_{\zeta < \zeta_i^*} b_{i,\zeta}$. For each $i < 2^\lambda$, there is a c_i which is a subset of $a_i \cap b_{i,\zeta}$ of cardinality κ, for some $\zeta < \zeta_i^*$. Each $c_i \in \mathcal{P}_i$, and as $|\mathcal{P}_i| \leq \mu < 2^\lambda$, two distinct c_i's must be the same, contradicting the assumption on $\{S_i : i < 2^\lambda\}$.

The second part follows from the first part and Conclusion 0.30 (2). (??) $\square$

Shelah notes that (even for smaller λ ?) it is not clear if $(*)_{2^\lambda,\mu,\lambda^+}$ is consistent with ZFC.

References

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