Wadge Determinacy implies Countable Choice for reals

Paul Larson *
Miami University

December 25, 2020

Given sets $A, B \subseteq \omega^\omega$, the Wadge game $\mathcal{W}(A, B)$ is the length-ω integer game in which Player I builds $x \in \omega^\omega$, Player II builds $y \in \omega^\omega$, and II wins if $x \in A \leftrightarrow y \in B$. Wadge Determinacy is the statement that $\mathcal{W}(A, B)$ is determined, for all $A, B \subseteq \omega^\omega$. We write $A \leq_{\mathrm{L}} B$ to indicate that Player II has a winning strategy in $\mathcal{W}(A, B)$ and $A <_{\mathrm{L}} B$ for $A \leq_{\mathrm{L}} B$ and $\neg(B \leq_{\mathrm{L}} A)$.

We let Countable Choice for reals ($\mathsf{CC}_{\mathbb{R}}$) be the statement that every countable family of nonempty subsets of ω^ω has a choice function. The purpose of this note is to prove the following.

Theorem 0.1. *Wadge Determinacy implies* $\mathsf{CC}_{\mathbb{R}}$.

Our proof of the theorem uses the following lemma.

Lemma 0.2. *Suppose that Wadge Determinacy holds, and that $\langle A_i : i \in \omega\rangle$ is a sequence of subsets of ω^ω such that $A_i <_{\mathrm{L}} A_{i+1}$ for all $i \in \omega$. Then there exist a function $f : \omega \to \omega$ and a sequence $\langle \sigma_i : i \in \omega\rangle$ such that, for each $i \in \omega$, $f(i) > i$ and σ_i is a winning strategy for Player II in $\mathcal{W}(A_i, A_{f(i)})$.*

Proof. For each $i \in \omega$, let A'_i denote $\{\langle i\rangle \frown x : x \in A\}$. Consider the games $\mathcal{W}(\bigcup_{i\in\omega} A'_{2i}, \bigcup_{i\in\omega} A'_{2i+1})$ and $\mathcal{W}(\bigcup_{i\in\omega} A'_{2i+1}, \bigcup_{i\in\omega} A'_{2i})$. By Wadge Determinacy, there exist winning strategies Σ_1 and Σ_2 for Player II in these two games (respectively). Composing these strategies gives the lemma, with $f(2i) = \Sigma_2(\Sigma_1(2i))$ and $f(2i + 1) = \Sigma_2(\Sigma_1(2i + 1))$ for each $i \in \omega$. □

Now, let $\langle A_i : i \in \omega\rangle$ be a sequence of nonempty subsets of ω^ω. Given $x, y \in \omega^\omega$, we write $x \otimes y$ for the element of ω^ω taking value $x(n)$ at $2n$ and $y(n)$ at $2n+1$, for all $n \in \omega$. Given $A, B \subseteq \omega^\omega$, we write $A \otimes B$ for $\{x \otimes y : x \in A,\ y \in B\}$.

Using the diagonalization argument from the proof of Lemma 1.2 of [1], we can find sets $B_i \subseteq \omega^\omega$ $(i \in \omega)$ such that, for each $i \in \omega$, $A_i \otimes B_i <_{\mathrm{L}} B_{i+1}$. It follows that

$$A_i \otimes B_i <_{\mathrm{L}} A_{i+1} \otimes B_{i+1}$$

holds for all $i \in \omega$.

*Partially supported by NSF grant DMS-1764320.

Let $\pi_i \colon \omega \to \omega$ $(i \in \omega)$ be injections with disjoint ranges. For each $i \in \omega$, let $C_i = \{x \in \omega^\omega : \forall j \leq i\, x \circ \pi_j \in A_i \otimes B_i\}$. Then $C_i <_{\mathrm{L}} C_{i+1}$ holds for all $i \in \omega$.

Applying the lemma to $\langle C_i : i \in \omega \rangle$, we get a function $f \colon \omega \to \omega$ and a sequence $\langle \sigma_i : i \in \omega \rangle$ such that, for each $i \in \omega$, $f(i) > i$ and σ_i is a winning strategy for Player II in $\mathcal{W}(C_i, C_{f(i)})$. Starting with any $x_0 \in C_0$, this gives a sequence $\langle x_i : i < \omega \rangle$ such that, for each positive $i \in \omega$, $x_i \in C_{f^i(0)}$. From this one can find a choice function for $\langle A_i : i \in \omega \rangle$.

References

[1] R.M. Solovay. The independence of DC from AD. In : **Large Cardinals, Determinacy and Other Topics**, Lecture Notes in Logic volume 49, Cambridge University Press, 2020