Two $\mathbb{P}_{\max}$ arguments

Paul B. Larson

October 26, 2023

Abstract

We sketch proofs of two of Woodin's results on $\mathbb{P}_{\max}$

1 Recovering the generic filter from any new set

In this section we give a proof of the following fact.

Theorem 1.1. *If $V \models \mathsf{AD}^+$, $G \subseteq \mathbb{P}_{\max}$ is a V-generic filter and, in $V[G]$, $B \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$, then $G \in L(\mathbb{R})[B]$.*

Proof. Fix the objects introduced in the statement of the theorem. Recall that (by definition) $A_G = \bigcup\{a : \langle (M,I), a\rangle \in G\}$ and, in $V[G]$, for each $p = \langle (M,I), a\rangle \in G$ there is a unique iteration $j_{p,G} \colon (M,I) \to (M^*, I^*)$ such that $j(a) = A_G$ and $I^* = M^* \cap \mathrm{NS}_{\omega_1}$. We use the following facts:

- in $V[G]$, G is the set of $\langle (M,I), a\rangle \in \mathbb{P}_{\max}$ for which there exists an iteration $j \colon (M,I) \to (M^*, I^*)$ such that $j(a) = A_G$ and $I^* = M^* \cap \mathrm{NS}_{\omega_1}$;
- for any $p = \langle (M,I), a\rangle \in G$, if $b \in \mathcal{P}(\omega_1)^M$ is such that

  $$\langle (M,I), b\rangle \in \mathbb{P}_{\max},$$

  then, by the argument for the weak homogeneity of $\mathbb{P}_{\max}$,

  $$A_G \in L(\langle (M,I), a\rangle, j_{p,G}(b));$$

- in $V[G]$, every club subset of ω_1 contains a club in $L(\mathbb{R})$ (the countable indiscernibles of some real), so the model $L(\mathbb{R})[B]$ correctly computes stationarity for subsets of ω_1.

It suffices then to see that, in $V[G]$, for all $B \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$, there exist

- $p = \langle (M,I), a\rangle \in G$,
- $x \in \mathcal{P}(\omega)^M$ and
- $b \in \mathcal{P}(\omega_1)^M$

such that $j_{p,G}(b) = B$ and $\omega_1^M = \omega_1^{L[x,b]}$.

Since $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega)$ in $V[G]$, it suffices (by the genericity of G) to show that, for each $\mathbb{P}_{\max}$ condition $p = \langle (M, I), a\rangle$ and each set $b \in \mathcal{P}(\omega_1)^M$, there exist a condition $q = \langle (N, J), a'\rangle < \langle (M, I), a\rangle$ and an $x \in \mathcal{P}(\omega)$ such that either $j(b) \in L[x]$ or $\omega_1^N = \omega_1^{L[j(b),x]}$, where j is the iteration witnessing that $q < p$.

Since each $\mathbb{P}_{\max}$ condition can be iterated into a limit structure (which satisfies the conditions in the claim below), it suffices to show the following.

Claim 1.2. *Suppose that $(\bar{M}, \bar{I}) = \langle (M_i, I_i) : i \in \omega\rangle$, J, a and b are such that*

- *for each $i \in \omega$,*
  - *M_i is a countable transitive model of* ZFC*,*
  - *$\omega_1^{M_i} = M_0$,*
  - *$\langle (M_i, I_i), a\rangle \in \mathbb{P}_{\max}$,*
  - *$M_i \in H(\aleph_2)^{M_{i+1}}$,*
  - *$I_i = I_{i+1} \cap M_i$ and $I_i \subseteq NS_{\omega_1}^{M_{i+1}}$,*
  - *there exists a $y_i \in \mathcal{P}(\omega)^{M_{i+1}}$ such that the least y_i-indiscernible above $\omega_1^{M_0}$ is greater than the ordinal height of M_i, and, every club subset of $\omega_1^{M_0}$ in M_i contains a tail of the y_i-indiscernibles below $\omega_1^{M_0}$,*
- *$b \in \mathcal{P}(\omega_1)^{M_0}$,*
- *J is a normal precipitous ideal on ω_1.*

Then there exists an iteration $j\colon \langle (M_i, I_i) : i \in \omega\rangle \to \langle (\hat{M}_i, \hat{I}_i) : i \in \omega\rangle$ such that

- *for each $i \in \omega$, $\hat{I}_i = \hat{M}_i \cap J$,*
- *there exists an $x \subseteq \omega$ such that either $j(b) \in L[x]$ or $\omega_1 = \omega_1^{L[x,j(b)]}$.*

Proof of Claim. There are two cases, depending on whether or not there exist an ordinal $\gamma < \omega_1$ and iterations j_0 and j_1 of $(\bar{M}, \bar{I})$ such that $\gamma \in j_1(b)$ and $\gamma \notin j_0(b)$. If there are no such γ, j_0 and j_1, then $j(b) \in L[x]$ for any suitable iteration j of $(\bar{M}, \bar{I})$, where x is any subset of ω for which

$$\langle (M_i, I_i) : i \in \omega\rangle \in H(\aleph_1)^{L[x]}.$$

Supposing then that there exist such γ, j_0 and j_1, we can fix such a triple with γ as small as possible, and j_0 and j_1 as short as possible so that γ is less than both $j(\omega_1^{M_0})$ and $j'(\omega_1^{M_0})$. It follows that j_0 and j_1 both have successor length. Let j_0' and j_1' be the corresponding iterations with their last steps removed. Let $\langle (M_i^0, I_i^0) : i \in \omega\rangle$ and $\langle (M_i^1, I_i^1) : i \in \omega\rangle$ be the corresponding final sequences for j_0' and j_1'.

Recall that for each iteration j of $(\bar{M}, \bar{I})$ of any length α, $j(\omega_1^{M_0})$ is the αth ordinal ($\geq \omega_1^{M_0}$) which is an indiscernible of each y_i, and the next such indiscernible is the supremum of the ordinals of the final models of the iteration. It follows in particular that $\omega_1^{M_0^0} = \omega_1^{M_0^1}$ (so, by the minimality of γ, $j_0'(b) = j_1'(b)$), and, for some $i' \in \omega$, γ is below the least

$y_{i'}$-indiscernible above $\omega_1^{M_0^0}$. It follows that there exist a $y_{i'}$-term $t_\phi^{y_{i'}}$ (for some formula ϕ), a finite set c of y_i-indiscernibles below $\omega_1^{M_0^0}$ and a finite set d of $y_{i'}$-indiscernibles above $\omega_1^{M_0^0}$ such that $\gamma = t_\phi^{y_{i'}}(c, \omega_1^{M_0^0}, d)$.

Consider now the function $f\colon \omega_1^{M_0^0} \to \omega_1^{M_0^0}$ which, whenever α is a $y_{i'}$-indiscernible above the members of c, returns $t_\phi^{y_{i'}}(c, \alpha, d')$ (for any set of $y_{i'}$-indiscernibles above α of the same size as d), and returns 0 otherwise. Then f is in $L[y_{i'}^{\#}]$ and therefore in $M^0_{i'+1}$ and $M^1_{i'+1}$, and $j(f)(\omega_1^{M_0^0}) = \gamma$ for any elementary embedding induced by a normal filter for either $\langle (M_i^0, I_i^0) : i \in \omega\rangle$ or $\langle (M_i^1, I_i^1) : i \in \omega\rangle$.

Let $X = \{\alpha : f(\alpha) \in j_0'(b)\}$. Since $\gamma \not\in j_0(b)$, $\omega_1^{M_0^0} \setminus X$ is $I^0_{i'+1}$-positive in M^0_{i+1}. Since $\gamma \in j_0(b)$, X is $I^1_{i'+1}$-positive in M^0_{i+1}.

We would like to see that either X is $I^0_{i'+1}$-positive in $M^0_{i'+1}$ or $\omega_1^{M_0^0} \setminus X$ is $I^1_{i'+1}$-positive in $M^1_{i'+1}$. If neither of these is the case, then a tail of the $y_{i'+1}$-indiscernibles are elements of a corresponding club witnessing the corresponding fact in each of these two models. That is, for a tail of $y_{i'+1}$-indiscernibles α below $\omega_1^{M_0^0}$, $f(\alpha)$ is both in and not in $j_0'(b)$. This is of course impossible.

Now let $\langle (M_i', I_i') : i \in \omega\rangle$ be one of $\langle (M_i^0, I_i^0) : i \in \omega\rangle$ and $\langle (M_i^1, I_i^1) : i \in \omega\rangle$ such that X is $I_{i'+1}$-positive and co-$I_{i'+1}$-positive in $M'_{i'+1}$. Let x (in V) be a subset of ω for which

$$\langle (M_i', I_i') : i \in \omega\rangle \in H(\aleph_1)^{L[x]}.$$

Let E be any subset of ω_1 for which $\omega_1 = \omega_1^{L[E]}$. Let C be the club of countable ordinals greater than or equal to $\omega_1^{M_0'}$ which are indiscernibles for each y_i. Let j be an iteration of $\langle (M_i', I_i') : i \in \omega\rangle$ such that

- each I_i'-positive set is mapped to a J-positive set (note that this requires attention only at limit stages of the iteration) and
- the corresponding image of X is put in the normal filter at stage $\alpha + 1$ if and only if $\alpha \in E$.

Then $E \in j[x, j(b)]$, since E is the set of α such that, letting η be the $(\alpha+1)$st element of C, $t_\phi^{y_{i'}}(c, \eta, d') \in j(b)$ whenever d' is a finite set of $y_{i'}$ indiscernibles above η with the same size as d.

□

2 Preserving $\mathrm{cof}(\alpha) \geq \omega_2$

In this section we adapt the proof of $\mathsf{MM}^{++}(\mathfrak{c})$ in $\mathbb{P}_{\max}$ extensions of models of $\mathsf{AD}_{\mathbb{R}}$ to show that cofinality greater than ω_1 is preserved in such models. Recall that Woodin has proved a stronger conclusion (every bounded subset of Θ of cardinality $\aleph_1$ in the $\mathbb{P}_{\max}$ extension is contained in a ground model set having cardinality $\aleph_1$ there) assuming only AD^+ in the ground model.

We use the following theorem (Theorem 9.38 in the original version of the $\mathbb{P}_{\max}$ book).

Theorem 2.1. *Suppose that $V = L(\mathcal{P}(\mathbb{R}))$ and AD^+ holds. Let X be a set of ordinals. Then there exists a set Y of ordinals such that $X \in L[Y]$ and, for any bounded $t \subseteq \omega_1$ there exists a transitive model N of* ZFC *such that*

- *$L[Y, t] \subseteq N$;*
- *$V_\gamma^N = L[Y, t] \cap V_\gamma$, where γ is the least strongly inaccessible cardinal of $L[Y, t]$,*
- *there is a countable ordinal which is a Woodin cardinal in N.*

Theorem 2.2. *If $V \models \mathsf{AD}_\mathbb{R} + \mathsf{AD}^+$, $\alpha < \Theta$ has cofinality at least ω_2 in V and $G \subseteq \mathbb{P}_{\max}$ is a V-generic filter then, in $V[G]$, $\mathrm{cof}(\alpha) \geq \omega_2$.*

Proof. Let $\preceq$ be a prewellordering of ω^ω of length α, let $p = \langle (M, I), a\rangle$ be a $\mathbb{P}_{\max}$ condition and let $f : \omega_1^M \to \omega^\omega$ in M. Since $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$ holds in $V[G]$, it suffices to find a condition $q = \langle (M', I'), a'\rangle < p$ such that

- (M', I') is $\preceq$-iterable and
- for some $x \in \omega^\omega \cap M'$, $y \preceq x$ holds for all y in the range of $j(f)$, where j is the iteration of (M, I) sending a to a'.

Since our asumptions imply that all sets of reals in V are Suslin, we may fix trees S and T on $\omega \times \mathrm{Ord}$ projecting to $\preceq$ and its complement. Let Y be as in Theorem 2.1 above, with respect to some set of ordinals coding S, T and p.

By the Solovay measure argument from the $\mathsf{MM}^{++}(\mathfrak{c})$ proof, there exists a countable $\sigma \subseteq \omega^\omega$ such that $L(Y\sigma)$ satisfies AD^+ along with the statement that the projection of S is a prewellordering of ω^ω whose length has cofinality greater than ω_1.

Let g be a $L(Y, \sigma)$-generic filter for $\mathrm{Col}(\omega, {<}\omega_1)^{L(Y,\sigma)}$-generic over $L(Y, \sigma)$. The partial order $\mathrm{Col}(\omega, {<}\omega_1)$ adds a partition of ω_1 into $\aleph_1$ many stationary sets. In $L[Y, g]$ there exists an iteration j of (M, I) such that the image of each I-positive set is stationary in $L(Y, \sigma)[g]$. Since $\mathrm{Col}(\omega, {<}\omega_1)^{L(Y,\sigma)}$ has cardinality $\aleph_1$ in $L(Y, \sigma)$, there exists an $x \in \omega^\omega$ which is $\preceq$-above all members of the range of $j(f)$. Then the j-image of each I-positive set in M is stationary in $L[Y, g, x]$. Let t be a bounded subset of ω_1 coding g and x, and let N be as given by Theorem 2.1.

As in the $\mathsf{MM}^{++}(\mathfrak{c})$ proof, we can convert N into a $\mathbb{P}_{\max}$ condition as desired. First force over N with $\mathrm{Col}(\omega_1, {<}\delta)^N$, where δ is the least Woodin cardinal of N, and then force over this extension with a c.c.c. forcing making $\mathsf{MA}_{\aleph_1}$ hold. Letting N^* be this forcing extension, let N' be $V_\kappa \cap N^*$, where κ is the least strongly inaccessible cardinal of N^*, let I' be $\mathrm{NS}_{\omega_1}^{N'}$ and let $a' = j(a)$. □