Nonmeager universally null sets

Jörg Brendle and Paul Larson

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This note gives an example of a universally measurable set without the Baire property from the weakest hypotheses that we know to be sufficient.[1] A nonmeager set with the Baire property contains a perfect set, and therefore is not universally null. A nonmeager universally null set therefore cannot have the Baire property. Although it is open whether consistently every universally measurable set has the Baire property, every universally null set is meager in models formed by adding more than $\aleph_1$ many random reals to a model of the Continuum Hypothesis.

Since $\mathrm{cov}(\mathrm{Meager}) \leq \min\{\mathfrak{d}, \mathrm{non}(\mathrm{Null})\}$ in ZFC (see [1]), the result below is a small improvement on the result from [4] in which a nonmeager universally null set is produced from the hypothesis $\mathrm{cov}(\mathrm{Meager}) = \mathrm{cof}(\mathrm{Meager})$. Our result also implies for example, that there is a counterexample in the Blass-Shelah model (one of the standard models for the Filter Dichotomy), of size $\mathfrak{c}$. The Filter Dichotomy implies the nonexistence of medial limits, which also give rise to universally measurable set without the Baire property (see [3]).

Theorem 0.1. *If* $\min\{\mathfrak{d}, \mathrm{non}(\mathrm{Null})\} = \mathrm{cof}(\mathrm{Meager})$ *then there exists a nonmeager universally null set of cardinality* $\mathrm{cof}(\mathrm{Meager})$.

Proof. Let κ be $\mathrm{cof}(\mathrm{Meager})$. Work in ω^ω. List $\{A_\alpha : \alpha < \kappa\}$ a base for the meager ideal and let $\{f_\alpha : \alpha < \kappa\}$ be a dominating family. For each $\alpha < \kappa$, let $T_\alpha \subseteq \omega^{<\omega}$ be a superperfect tree such that $A_\alpha \cap [T_\alpha] = \emptyset$. For each such α, since T_α is superperfect, $[T_\alpha]$ is homeomorphic to ω^ω, and thus has the same dominating number. Since $\mathfrak{d} = \kappa$, we can pick for each $\alpha < \kappa$ an $x_\alpha \in [T_\alpha]$ such that x_α is not dominated mod-finite by any member of $\{f_\beta : \beta < \alpha\}$.

We claim that $X = \{x_\alpha : \alpha < \kappa\}$ is the required set. Clearly X is non-meager. To see that X is universally null, let μ be a Borel probability measure on ω^ω. There is g in ω^ω such that $\mu(B_g) = 1$, where $B_g = \{x : x \leq^* g\}$. Let α be such that $g \leq^* f_\alpha$. Then $\{x_\beta : \beta > \alpha\} \cap B_g = \emptyset$. Thus $\mu(\{x_\beta : \beta > \alpha\}) = 0$. Since $\mathrm{non}(N) \geq \kappa$, $\mu(\{x_\beta : \beta \leq \alpha\}) = 0$. It follows that $\mu(X) = 0$. $\square$

The hypotheses of the theorem above are equivalent to the conjunction of $\mathfrak{d} = \mathrm{cof}(\mathrm{Meager})$ and $\mathrm{non}(\mathrm{Null}) \geq \mathrm{cof}(\mathrm{Meager})$. For all we know, either $\mathfrak{d} = \mathrm{cof}(\mathrm{Meager})$ or $\mathrm{non}(\mathrm{Null}) \geq \mathrm{cof}(\mathrm{Meager})$ could already imply the existence of a nonmeager universally null set of cardinality $\mathrm{cof}(\mathrm{Meager})$. We do not know for instance whether there is such a set of size $\aleph_2$ in the Miller model. Note however that $\mathrm{non}(\mathrm{Meager}) \leq \mathrm{cof}(\mathrm{Meager})$ (in ZFC) and if $\mathrm{non}(\mathrm{Meager}) < \mathrm{non}(\mathrm{Null})$, then every nonmeager set of cardinality $\mathrm{non}(\mathrm{Meager})$ is universally null.

[1]The material in this note is from the Fall of 2009.

References

[1] T. Bartoszyński, H. Judah, **Set theory**, A K Peters, Ltd., Wellesley, MA, 1995

[2] A. Blass, S. Shelah, *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic 33 (1987), no. 3, 213243

[3] P. Larson, *The Filter Dichotomy and medial limits*, Journal of Mathematical Logic 9 (2009) 2, 159-165

[4] P. Larson, I. Neeman, S. Shelah, *Universally measurable sets in generic extensions*, Fund. Math. 208 (2010) 2, 173-192