Notes on Woodin's Suitable Extenders

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1 Closure properties and supercompactness

This section is based on Chapter 5 of [4].

1.1 Supercompact cardinals

Given cardinals κ and λ, we let $\mathcal{P}_\kappa(\lambda)$ denote the collection of subsets of λ of cardinality less than κ.

1.1 Definition. A cardinal κ is *supercompact* if for every cardinal $\lambda > \kappa$ there exists a normal fine measure μ on $\mathcal{P}_\kappa(\lambda)$, that is, a filter μ on $\mathcal{P}_\kappa(\lambda)$ which is

- *fine*, so that for all $x \in \mathcal{P}_\kappa(\lambda)$, $\{y \in \mathcal{P}_\kappa(\lambda) \mid x \subseteq y\} \in \mu$;
- *normal*, so that all $A \in \mu$ and for all functions $f\colon A \to \lambda$ such that $f(x) \in x$ for all $x \in A$, f is constant on a set in μ.

1.2 Exercise. A normal fine measure on $\mathcal{P}_\kappa(\lambda)$ must be κ-complete.

Given such a μ, we can form the corresponding ultrapower $\mathrm{Ult}(V, \mu)$. Elements of the ultrapower are represented by functions with domain $\mathcal{P}_\kappa(\lambda)$. Given two such functions,f and g, and a relation R in $\{\in, =\}$, the relation $[f]_\mu R [g]_\mu$ holds in the ultrapower (by definition) if and only if

$$\{x \mid f(x) R g(x)\} \in \mu.$$

By the countable completeness of μ, $\mathrm{Ult}(V, \mu)$ is wellfounded, and we identify it with its Mostowski collapse. Applying Łoś's Theorem, we get that the corresponding *supercompactness embedding*, the map sending each set a to $[f]_\mu$, where f is the function on $\mathcal{P}_\kappa(\lambda)$ with constant value a, is elementary.

1.3 Definition. For any embedding $j\colon M \to N$, where M is a model in the language of set theory, we let $\mathrm{CRT}(j)$ denote the *critical point* of j, the least ordinal α of M such that $j(\alpha) > \alpha$, if such an α exists.

1.4 Exercise. If $j\colon V \to M$ is a supercompactness embedding induced by a normal fine measure on $\mathcal{P}_\kappa(\lambda)$, then $\mathrm{CRT}(j) = \kappa$. By normality, the function f defined by $f(x) = \mathrm{ot}(x \cap \gamma)$ represents γ, for each $\gamma \le \lambda$. Furthermore, the identity function on $\mathcal{P}_\kappa(\lambda)$ represents $j[\lambda]$, from which it follows that for each $A \subseteq \mathcal{P}_\kappa(\lambda)$, $A \in \mu$ if and only if $j[\lambda] \in j(A)$.

1.5 Exercise. If $\kappa < \lambda$ are cardinals, and μ is a normal fine measure on $\mathcal{P}_\kappa(\lambda)$, then $\mathcal{P}_\kappa(\lambda) \cap L \notin \mu$. (Hint: Since L has a definable wellordering, any elementary embedding from L to L is recoverable from its action on the ordinals.)

The ultrapower is also closed under sequences of length λ. To see this, given functions $g_\gamma \colon \mathcal{P}_\kappa(\lambda) \to V$ $(\gamma < \lambda)$, consider the function f on $\mathcal{P}_\kappa(\lambda)$ defined by letting $f(x)$ be the sequence of length $\mathrm{ot}(x)$ whose η-th element is $g_\gamma(x)$ whenever $\gamma \in x$ and $\mathrm{ot}(x \cap \gamma) = \eta$. Then f represents a λ sequence, and for each $\gamma < \lambda$, the γ-th member of $[f]_\mu$ is $[g_\gamma]_\mu$.

1.6 Exercise. Suppose that $\kappa \leq \lambda$ are cardinals, and $j \colon V \to M$ is an elementary embedding such that $j(\kappa) > \lambda$, $j \restriction \mathcal{P}_\kappa(\lambda)$ is in V and $j[\lambda]$ is an element of M. Then $\{A \subseteq \mathcal{P}_\kappa(\lambda) \mid j[\lambda] \in j(A)\}$ is a normal fine measure.

1.7 Exercise. Let κ be an infinite cardinal, and let $\lambda > \kappa$ be a regular cardinal. Then $\{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$ can be partitioned into λ many disjoint stationary sets. (Hint : Fix a cofinal κ-sequence for each such α. Suppose that for each $\beta < \kappa$ there are fewer than λ many ordinals which are the βth member of the chosen sequence for stationarily many α. The union of these stationary sets must contain a club in each case. Intersecting these clubs, we get a club whose sequences are all bounded.)

The following result is Theorem 135 of [4].

Theorem 1.8 (Solovay)**.** *Suppose that $\delta < \lambda$ are regular cardinals. Then there is a set $X \subseteq \mathcal{P}_\delta(\lambda)$ such that*

- *the function $\sigma \mapsto \sup(\sigma)$ is injective on X;*
- *X is a member of every normal fine measure on $\mathcal{P}_\delta(\lambda)$.*

Proof. Let $\langle S_\alpha : \alpha < \lambda \rangle$ be a partition of

$$\{\alpha < \lambda : \mathrm{cof}(\alpha) = \omega\}$$

into pairwise disjoint stationary sets (see Exercise 1.7). For each $\eta < \lambda$ such that $\mathrm{cof}(\eta) \in (\omega, \delta)$, let Z_η be the set of $\alpha < \eta$ such that $S_\alpha \cap \eta$ is stationary in η, in the sense that S_α intersects every closed cofinal $C \subseteq \eta$. Let X be the set of Z_η's which are cofinal in η.

Let μ be a normal fine measure on $\mathcal{P}_\delta(\lambda)$, and let $j \colon V \to M$ be the corresponding embedding. Let $\eta = \sup(j[\lambda])$. Since $j[\lambda]$ is ω-closed, every cofinal ω-closed subset of η has cofinal ω-closed intersection with $j[\lambda]$. Since each S_α is stationary in λ, it follows that, in M, each $j(S_\alpha) \cap \eta$ is stationary in η. Then $j[\lambda] \in j(X)$, so $X \in \mu$. ☐

The following definition (Definition 133 of [4]; Definition 15 of [5]) reflects the situation where the supercompactness of a cardinal δ in an inner model N is witnessed by the restrictions to N of normal fine measures in V. We do not require here that N be a definable inner model. For instance, we will apply our results here in the context where some V_κ models ZFC and N is a transitive model of ZFC whose ordinal height is κ.

1.9 Definition. Suppose that N is a transitive inner model of ZFC. We say that $o^N_{\mathrm{LONG}}(\delta) = \infty$ (or N is a *weak extender model* for δ *supercompact*) if for all $\lambda > \delta$ there exists a normal fine measure μ on $\mathcal{P}_\delta(\lambda)$ such that

- μ *concentrates on* N, i.e., $N \cap \mathcal{P}_\delta(\lambda) \in \mu$;
- μ is *amenable* to N, i.e., $N \cap \mu \in N$.

1.10 Exercise. Suppose that μ is a normal fine measure on $\mathcal{P}_\kappa(\lambda)$, for some cardinals $\kappa < \lambda$, and that N is a transitive inner model of ZFC such that $N \cap \mathcal{P}_\delta(\lambda) \in \mu$ and $N \cap \mu \in N$. Then $N \cap \mu$ is a normal fine measure on $\mathcal{P}_\kappa(\lambda)$ in N.

Our first goal is to give two consequences of Definition 1.9, Theorems 1.11 and 1.32. The first is Theorem 136 of [4].

Theorem 1.11. *Suppose that N is weak extender model for δ supercompact. Then the following hold.*

- *Every set of ordinals of cardinality less than δ is contained in a set of ordinals of cardinality less than δ which is a member of N.*
- *Whenever $\lambda > \delta$ is a singular cardinal, λ is singular in N and $(\lambda^+)^N = \lambda^+$.*
- *Whenever $\lambda > \delta$ is a regular cardinal in N, $|\gamma| = \mathrm{cof}(\gamma)$.*

Proof. The first part is immediate from the definition of $o^N_{\mathrm{LONG}}(\delta) = \infty$, using fineness. Let us see that the third part implies the second. Suppose that $\lambda > \delta$ is a singular cardinal in V. If λ were regular in N, then we would have $\gamma = |\gamma| = \mathrm{cof}(\gamma)$, i.e., γ would be regular. So λ is singular in N. Now suppose that $(\lambda^+)^N < \lambda^+$. Then since $(\lambda^+)^N$ is a regular cardinal in N, $|(\lambda^+)^N| = \mathrm{cof}((\lambda^+)^N)$. This is impossible, $|(\lambda^+)^N|$ is singular.

Finally, let us check the third part. Fix γ, and suppose that μ is a normal fine measure on $\mathcal{P}_\delta(\gamma)$ such that $N \cap \mathcal{P}_\delta(\gamma) \in \mu$ and $N \cap \mu \in N$. Let $X \in N$ be a set as given by Solovay's theorem, with respect to δ and γ. Let $\nu = \{A \subseteq \gamma \mid \{\sigma \in X \mid \sup(\sigma) \in A\}$. Since $X \in \mu$ and the function $\sigma \mapsto \sup(\sigma)$ is injective on X, ν is a δ-complete nonprincipal ultrafilter on γ. Furthermore, ν contains every club $C \subseteq \gamma$, since if we define f on $\{\sigma \in X \mid \sup(\sigma) \notin C\}$ by letting $f(\sigma)$ be the least element of σ greater than $\sup(\sigma \cap C)$, then f is regressive but cannot be constant on a set in μ.

By the first part of theorem, since γ is regular in N, $\mathrm{cof}(\gamma) \geq \delta$. Fix a club $C \subseteq \gamma$ of cardinality $\mathrm{cof}(\gamma)$. Then $\{\sigma \in X \mid \sup(\sigma) \in C\}$ is a set in μ, and its union has cardinality at most $\mathrm{cof}(\gamma)$. Since μ is a fine measure, it follows that $\mathrm{cof}(\gamma) = |\gamma|$. $\square$

Recall that the cardinals $\beth_\alpha$ are defined by $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\beta = \sup_{\alpha<\beta} \beth_\beta$ when β is a limit ordinal. For any ordinal α, $|V_{\omega+\alpha}| = \beth_\alpha$. A $\beth$-*fixed point* is a cardinal κ for which $\beth_\kappa = \kappa$, i.e., for which $|V_\kappa| = \kappa$.

The following is based on Lemma 134 of [4].

Lemma 1.12. *Suppose that N is a transitive inner model of* ZFC*, $\delta < \kappa$ are cardinals, κ is a $\beth$-fixed point, and μ is a normal fine measure on $\mathcal{P}_\delta(\kappa)$ such that*

$$N \cap \mathcal{P}_\delta(\kappa) \in \mu$$

and $\mu \cap N \in N$. Let $j\colon V \to M$ be the elementary embedding given by μ. Then

$$j(N \cap V_\delta) \cap V_\kappa = N \cap V_\kappa$$

and

$$j{\restriction}(V_\kappa \cap N) \in j(V_{\kappa+1} \cap N).$$

Proof. Since $|V_\kappa| = \kappa$, $|V_\alpha| < \kappa$ for all $\alpha < \kappa$. Therefore, for each $\alpha < \kappa$, $V_\alpha \cap N$ has cardinality less than κ in N, which means that $V_\kappa \cap N$ has cardinality κ in N. Fix a bijection $\pi\colon \kappa \to N \cap V_\kappa$ in N.

Letting $j_N\colon N \to P$ be the embedding computed in N from $\mu \cap N$, we get that $j_N(\pi)[j[\kappa]] = j_N[V_\kappa \cap N]$, so the transitive collapse of $j_N(\pi)[j[\kappa]]$ is $V_\kappa \cap N$, which is $V_\kappa \cap P$. Thus $j_N[\kappa]$ is in the j_N-image of the set of $Y \in \mathcal{P}_\delta(\kappa)$ for which the transitive collapse of $\pi[Y]$ is $N \cap V_{\mathrm{ot}(Y)}$. It follows that the set of $Y \in \mathcal{P}_\delta(\kappa)$ for which the transitive collapse of $\pi[Y]$ is $N \cap V_{\mathrm{ot}(Y)}$ (which is equal to $(N \cap V_\delta) \cap V_{\mathrm{ot}(Y)}$) is in $\mu \cap N$.

Now, applying the same fact with μ, we get that the transitive collapse of $j(\pi)[j[\kappa]]$ is $j(N \cap V_\delta) \cap V_\kappa$. However, $j(\pi)[j[\kappa]] = j[N \cap V_\kappa]$, whose transitive collapse is $N \cap V_\kappa$.

Finally, since $N \cap \mathcal{P}_\delta(\kappa) \in \mu$, $j[\kappa] \in j(N \cap \mathcal{P}_\delta(\kappa))$. Since $\pi \in N$, it follows that $j[V_\kappa \cap N]$ is in $j(V_{\kappa+1} \cap N)$. Since $j{\restriction}(V_\kappa \cap N)$ is the inverse of the transitive collapse map on $j[V_\kappa \cap N]$, the last part of the lemma holds. $\square$

The following is Lemma 138 of [4] and half of Theorem 21 of [5]. The converse is also true (see Remark 137 of [4] and Theorem 21 of [5]).

Lemma 1.13. *Suppose that N is a weak extender model for δ supercompact. Then for every $\beth$-fixed point $\gamma > \delta$ and every $a \in V_\gamma$ there there exist a cardinal $\bar{\gamma}$ and an elementary embedding*

$$j\colon V_{\bar{\gamma}+\omega} \to V_{\gamma+\omega}$$

such that the following hold, where $\bar{\delta}$ is the critical point of j:

- *$j(\bar{\delta}) = \delta$;*
- *for some $\bar{a} \in V_\delta$, $j(\bar{a}) = a$;*
- *$j(N \cap V_{\bar{\gamma}}) = N \cap V_\gamma$;*
- *$j{\restriction}(V_{\bar{\gamma}+\omega} \cap N) \in N$.*

Proof. Fix cardinals $\kappa > \gamma > \delta$ such that $|V_\kappa| = \kappa$. Let μ be a normal fine measure on $\mathcal{P}_\delta(\kappa)$ such that $\mathcal{P}_\delta(\kappa) \cap N \in \mu$ and $\mu \cap N \in N$. Let $j \colon V \to M$ be the corresponding embedding. By Lemma 1.12,

$$j(N \cap V_\delta) \cap V_\kappa = N \cap V_\kappa.$$

Since M is closed under sequences of length κ, $j \restriction V_{\gamma+\omega}$ is in M, and it suffices to check that $j \restriction V_{\gamma+\omega}$ witnesses in M that the second part of the lemma holds for $\langle j(\gamma), j(a)\rangle$ relative to $j(N)$.

The first two clauses are immediate. To verify the third, note first that by the consequence of Lemma 1.12 given above,

$$(j \restriction V_{\gamma+\omega})(j(N) \cap V_\gamma) = j(N \cap V_\gamma) = j(N) \cap V_{j(\gamma)}.$$

Finally, we wish to see that $(j \restriction V_{\gamma+\omega}) \restriction (V_{\gamma+\omega} \cap j(N)) \in j(N)$, which amounts to showing that $j \restriction (V_{\gamma+\omega} \cap j(N)) \in j(N)$, which is the same as $j \restriction (V_{\gamma+\omega} \cap N) \in j(N)$, by the consequence of Lemma 1.12 given above. That $j \restriction (V_{\gamma+\omega} \cap N) \in j(N)$ follows from the last part of Lemma 1.12. $\square$

1.2 Long Extenders

This subsection is based loosely on Section 3.1 of [4].

1.14 Definition. Given finite sets of ordinal $s \subseteq t$, define the projection map

$$\pi_{t,s} \colon [\mathrm{Ord}]^{|t|} \to [\mathrm{Ord}]^{|s|}$$

as follows. Suppose that $t = \{\gamma_0, \ldots, \gamma_{n-1}\}$ (listed in increasing order), and that $a \subseteq n$ is such that $s = \{\gamma_i : i \in a\}$. Then for each $\{\alpha_0, \ldots, \alpha_{n-1}\} \in [\kappa]^n$ (listed in increasing order), we let $\pi_{t,s}(\{\alpha_0, \ldots, \alpha_{n-1}\}) = \{\alpha_i : i \in a\}$.

1.15 Definition. A *pre-extender* is a collection $\langle E_s : s \in [\eta]^{<\omega}\rangle$, for some ordinal η such that, for some ordinal $\hat{\eta} \leq \eta$, the following conditions are satisfied.

- Each E_s is an ultrafilter with $[\hat{\eta}]^{|s|} \in E_s$.
- (Coherence) For all finite $s \subseteq t \subset \eta$, for each $A \subseteq [\hat{\eta}]^{|s|}$,

$$A \in E_s \Leftrightarrow \pi^{-1}_{t,s}[A] \in E_t.$$

- (Normality) if $s \in [\eta]^{<\omega}$, $A \in E_s$ and $f \colon A \to \mathrm{Ord}$ is such that $f(x) < \max(x)$ for all $a \in A$, then there exists $t \supseteq s$ in $[\eta]^{<\omega}$ such that the set of $x \in [\hat{\eta}]^{|t|}$ for which $f(\pi_{t,s}(x)) \in x$ is in E_t.

The ordinal η is called the *length* of E, and denoted $\mathrm{LTH}(E)$.

Given a model (N, E), an N-*pre-extender* is a collection $\langle E_s : s \in [\eta]^{<\omega}\rangle$ as above where each E_s is an N-ultrafilter, and the sets A and f in the definition are restricted to sets in N.

1.16 Exercise. Show directly from the definitions (i.e., without using the embedding defined below), that if $\langle E_s : s \in [\eta]^{<\omega} \rangle$ is an N-pre-extender, for some $\eta > 0$ and a model (N, E) of a suitable fragment of ZF, then $E_{\{0\}}$ is a principal ultrafilter generated by $\{0\}$.

1.17 Exercise. Show directly from the definitions (i.e., without using the embedding defined below), that if $\langle E_s : s \in [\eta]^{<\omega} \rangle$ is an N-pre-extender, γ is an ordinal with $\gamma + 1 < \eta$, and A is a set of ordinal singletons, then $A \in E_{\{\gamma\}}$ if and only if $\{\{\alpha + 1\} \mid \alpha \in A\} \in E_{\{\gamma+1\}}$.

1.18 Remark. Sometime we prefer to write a pre-extender E as a function with domain $[\eta]^{<\omega}$, and we write $E(s)$ for E_s.

An N-pre-extender E gives rise to an elementary embedding $j_E \colon N \to M_E$ where the elements of M_E are represented by pairs of the form $[f, s]$ for $f \colon [\hat{\eta}]^{|s|} \to N$ in N and $s \in [\eta]^{<\omega}$. Given a relation R in $\{\in, =\}$, we define $[f, s]_E R [g, t]_E$ to hold in in M_E if and only if

$$\{x \in [\kappa]^{|s \cup t|} \mid f(\pi_{s \cup t, s}(x)) R g(\pi_{s \cup t, t}(x))\} \in E_{s \cup t}.$$

For each $x \in V$, $j_E(x)$ is represented by each suitable pair $[f, s]$ for which f takes the constant value x. As above, this gives an elementary embedding. By normality, for each $n \in \omega$ and $s \in [\eta]^n$, if i_n is the identity function on n-tuples from $\hat{\eta}$, then $[i_n, s]$ represents s in the ultrapower. This implies that for all $s \in [\eta]^{<\omega}$ and $A \subseteq [\hat{\eta}]^{|s|}$, $A \in E_s$ if and only if $s \in j_E(A)$. This in turn implies that $[g, s]_E = j_E(g)(s)$ for all suitable g and s.

1.19 Definition. Given an N-pre-extender E, the *strength* of E is the ordinal $\sup\{\nu \mid V_\nu^N \subseteq M_E\}$, which is denoted by $\rho(E)$.

1.20 Definition. We say that an N-pre-extender E is *trivial* if each E_s is a principal ultrafilter; otherwise, it is *non-trivial.*

1.21 Exercise. Suppose that E is an N-pre-extender. Show that E is non-trivial if and only if $E_{\{\gamma\}}$ is nonprincipal, for some $\gamma < \mathrm{LTH}(E)$.

1.22 Exercise. Show that if γ is minimal such that $E_{\{\gamma\}}$ is nonprincipal, then γ is the critical point of j_E.

We sometimes say that the critical point γ of j_E is the *critical point* of E and write $\mathrm{CRT}(E) = \gamma$.

Given $\gamma < \eta$ and $\alpha < \hat{\eta}$, $j_E(\alpha) = \gamma$ if and only if $E_{\{\gamma\}}$ is a principal ultrafilter generated by $\{\alpha\}$ (i.e., $\{\alpha\} \in E_{\{\gamma\}}$), as these are both equivalent to the statement that the set of β for which the constant function α equals the identity function is in $E_{\{\gamma\}}$.

1.23 Definition. Given an N-pre-extender E, the least ordinal α for which there is no $\gamma < \mathrm{LTH}(E)$ with $\{\alpha\} \in E_{\{\gamma\}}$ is called the *support* of E and denoted $\mathrm{SPT}(E)$.

Alternately, $\mathrm{SPT}(E)$ is also the least ordinal γ with $j_E(\gamma) \geq \mathrm{LTH}(E)$. Since, as remarked above, each $A \subseteq [\hat{\xi}]^{|s|}$ is in E_s if and only $s \in j_E(A)$, it follows that $[\mathrm{SPT}(E)]^{|s|} \in E_s$ for all $s \in [\xi]^{<\omega}$.

1.24 Definition. An N-pre-extender E is said to be *short* if $\mathrm{SPT}(E) = \mathrm{CRT}(E)$; when $\mathrm{SPT}(E) > \mathrm{CRT}(E)$, E is *long*.

1.25 Definition. Given a transitive model N, an *N-extender* is an N-pre-extender whose image model M_E is wellfounded.

1.26 Remark. If E is an extender with $|\mathrm{SPT}(E)| = \kappa$, and M_E is illfounded, then M_E is illfounded below the image of κ^+. To see this, suppose that the pairs $[f_i, s_i]$ $(i \in \omega)$ induce a descending ω-sequence in the ultrapower. Then each f_i can be taken to take value 0 outside of $[\mathrm{SPT}(E)]^{<\omega}$, and the union X of the ranges of the f_i's then has cardinality less than κ^+. Letting $\pi \colon X \to \mathrm{ot}(X)$ be the transitive collapse map on X, the sequence $[\pi \circ f_i, s_i]$ $(i \in \omega)$ now represents a descending ω-sequence in the ultrapower.

Suppose that we have an elementary embedding $j \colon N \to M$ with critical point κ, and an ordinal η. In this situation, we get a an N-pre-extender (the *N-pre-extender of length η derived from j*) by letting $\hat{\eta}$ be the least ordinal α such that $j(\alpha) \geq \eta$, and setting $E_s = \{A \in [\hat{\eta}]^{|s|} \mid s \in j(A)\}$ for each $s \in [\eta]^{<\omega}$. Then $k([f,s]_E) = j(f)(s)$ defines an elementary embedding $k \colon M_E \to M$ (so if M is wellfounded, then so is M_E, and then E is an N-extender). The critical point of k at least η. It follows that if γ is an ordinal such that $(2^\gamma)^M < \eta$, then $\mathcal{P}(\gamma)^M = \mathcal{P}(\gamma)^{M_E}$.

Note, however, that we cannot conclude that these are equal to $\mathcal{P}(\gamma)^V$.

1.27 Example. Consider an embedding $j \colon V \to M$ by a single measure on κ. Then $j = j_E$ for E the derived extender of length η, for any ordinal $\eta > \kappa$. Furthermore, $j(\kappa)$ is strongly inaccessible in M, and, letting $\gamma = (2^\kappa)^M$, $\mathcal{P}(\gamma)^V$ is not contained in M.

Given an N-pre-extender $E = \{E_s : s \in [\eta]^{<\omega}\}$ and $\delta < \eta$, we let

$$E{\restriction}\delta = \{E_s : s \in [\delta]^{<\omega}\}.$$

As above, there is then an elementary embedding from $M_{E\restriction\delta}$ to M_E with critical point at least δ. An ordinal $\delta < \eta$ is called a *generator* of E if the induced embedding from $M_{E\restriction\delta}$ to $M_{E\restriction(\delta+1)}$ is not the identity map. If we let η^* be the supremum of the ordinals of the form $\gamma + 1$, for each generator γ, then $j_E \colon V \to M_E$ is the same embedding as $j_{E\restriction\eta^*} \colon V \to M_{E\restriction\eta^*}$.

1.28 Exercise. Show that for any non-trivial pre-extender E, $\rho(E) \leq \mathrm{LTH}(E)$. (Hint : first reduce to the case where $\mathrm{SPT}(E)$ and $\mathrm{LTH}(E)$ are both limit ordinals.)

One construction that we will use many times is forming the restriction of a pre-extender to an inner model.

1.29 Definition. If E is a pre-extender of length ξ and N is a transitive model of a sufficient fragment of ZF, the N-pre-extender $E \restriction N$ is the set

$$\{E_s \cap N : s \in [\xi]^{<\omega}\}.$$

1.30 Exercise. Suppose that E is a pre-extender and N is a transitive inner model of ZFC with $E \restriction N \in N$. Show that for every bounded subset A of $\mathrm{SPT}(E)$ in N, $j_E(A) = j_{E \restriction N}(A)$.

1.31 Exercise. Suppose that E is a pre-extender and N is a transitive inner model of ZFC with $E \restriction N \in N$. Let α be such that $j_E(\alpha) \geq \rho(E \restriction N)$. Show that

$$j_E(N \cap V_\alpha) \cap V_{\rho(E \restriction N)} = N \cap V_{\rho(E \restriction N)}.$$

1.3 Closure properties

The following is Theorem 140 of [4], the main theorem of Section 5.1.

Theorem 1.32. *Suppose that N is a weak extender model for δ supercompact, and let $\gamma > \delta$ be a cardinal of N. Suppose that*

- *M is a transitive set,*
- *$j \colon H(\gamma^+)^N \to M$ is an elementary embedding with $\mathrm{CRT}(j) \geq \delta$,*
- *$\lambda \in [\gamma, j(\gamma)]$,*
- *$\mathcal{P}(\lambda) \cap M \subseteq N$.*

Let F be the N-pre-extender of length λ given by j. Then $\mathrm{Ult}(N, F)$ is well-founded and $F \in N$.

Proof. Let F be the N-pre-extender of length λ given by j. We represent F as a function with domain $[\lambda]^{<\omega}\}$, where each $F(s)$ is the set

$$\{A \in \mathcal{P}([\gamma]^{<\omega})^N \mid s \in j(A)\}.$$

We show first that $F \in N$. Fix a $\beth$-fixed point $\kappa > \gamma$ such that $j, M \in V_\kappa$. By Lemma 1.13, there exist

$$\bar{\delta} < \bar{\gamma} \leq \bar{\lambda} < \bar{\kappa} < \delta,$$

a transitive set $\bar{M} \in N$, an elementary embedding

$$\bar{j} \colon H(\bar{\gamma}^+)^N \to \bar{M}$$

with $\bar{j} in V_{\bar{\kappa}}$ and an elementary embedding

$$\pi \colon V_{\bar{\kappa}+1} \to V_{\kappa+1}$$

such that the following hold.

1. $\mathrm{CRT}(\pi) = \bar{\delta}$
2. $\pi(\langle\bar{\delta}, \bar{\gamma}, \bar{\lambda}, \bar{\kappa}\rangle) = \langle\delta, \gamma, \lambda, \kappa\rangle$
3. $\pi(\bar{M}) = M$
4. $\pi(\bar{j}) = j$
5. $\pi(N \cap V_{\bar{\kappa}}) = N \cap V_\kappa$
6. $\pi\restriction(N \cap V_{\bar{\kappa}+1}) \in N$

Let $\bar{F}$ be the N-pre-extender of length $\bar{\lambda}$ derived from $\bar{j}$. Since $\bar{\lambda} < \bar{\kappa}$, item (5) above implies that $\bar{F} \in V_{\bar{\kappa}}$ and $\pi(\bar{F}) = F$. Item (6) then implies that if $\bar{F} \in N$ then $F \in N$. We will show that $\bar{F} \in N$.

Suppose that $s \in [\bar{\lambda}]^{<\omega}$, and that $A \in \mathcal{P}([\bar{\gamma}]^{|s|}) \cap N$. Then $A \in \bar{F}(s)$ if and only if $s \in \bar{j}(A)$, which holds if and only if $\pi(s) \in \pi(\bar{j}(A))$. Since $\pi(\bar{j}) = j$, we have that $A \in \bar{F}(s)$ if and only if $\pi(s) \in j(\pi(A))$.

Let E be the N-extender of length κ given by π. We represent E as a function with domain $[\kappa]^{<\omega}$, where each $E(s)$ is the set $\{A \in \mathcal{P}([\bar{\kappa}]^{<\omega})^N \mid s \in \pi(A)\}$. Then $E \in N$, by item (6), and $E\restriction\gamma$ (i.e., $E\restriction[\gamma]^{<\omega}$) is in $H(\gamma^+)^N$. Therefore, $j(E\restriction\gamma)$ is in M. Let $H = j(E\restriction\gamma)\restriction\lambda$. Then H is a function with domain $[\lambda]^{<\omega}$, and each $H(s)$ is an M-ultrafilter on $[\bar{\kappa}]^{<\omega}$, as $\bar{\kappa} < \mathrm{CRT}(j) = \delta$. So H is coded in M by a subset of λ, and, since $\mathcal{P}(\lambda) \cap M \subseteq N$, H is in N.

We want to see that H is an extender in N, i.e., that $\mathrm{Ult}(N, H)$ is wellfounded. Supposing otherwise, there exists in N a sequence $\langle [f_i, a_i] : i < \omega\rangle$ representing a descending ω-sequence in $\mathrm{Ult}(N, H)$, where each a_i is a finite subset of λ and each f_i is a function from $[\bar{\kappa}]^{|a_i|}$ to $(\bar{\kappa}^+)^N$. We would like to see that there is a sequence $\langle b_i : i \in \omega\rangle$ of finite subsets of γ such that $\langle [f_i, b_i] : i < \omega\rangle$ represents a descending ω-sequence in $\mathrm{Ult}(N, E\restriction\gamma)$. This would give a contradiction, as $\mathrm{Ult}(N, E\restriction\gamma)$ embeds into $V_{\kappa+1}$. Suppose that there is no sequence of b_i's as desired. Then by the elementarity of j (which fixes each f_i), in M there is no sequence $\langle b_i : i \in \omega\rangle$ of finite subsets of $j(\gamma)$ such that $\langle [f_i, b_i] : i < \omega\rangle$ represents a descending ω-sequence in $\mathrm{Ult}(M, j(E\restriction\gamma))$. Again by the elementarity of j, in M every tree of height ω of cardinality at most $j(\gamma)$ either has an infinite path or a ranking function. The existence (in V) of the sequence $\langle a_i : i \in \omega\rangle$ shows then that in M there is a sequence $\langle b_i : i \in \omega\rangle$ of finite subsets of λ such that $\langle [f_i, b_i] : i < \omega\rangle$ represents a descending ω-sequence in $\mathrm{Ult}(M, j(E\restriction\gamma))$ Therefore, $\mathrm{Ult}(N, H)$ is wellfounded.

Let M_H be (the Mostowski collapse of) $\mathrm{Ult}(N, H)$, and let $j_H \colon N \to M_H$ be the associated embedding. We claim that for each $A \in \mathcal{P}([\bar{\gamma}]^{<\omega}) \cap N$,

$$j(\pi(A)) \cap [\lambda]^{<\omega} = j_H(j(A)) \cap [\lambda]^{<\omega}.$$

To see this, let $M_{E\restriction\gamma}$ be the Mostowski collapse of $\mathrm{Ult}(N, E\restriction\gamma)$, and let

$$j_E \colon N \to M_{E\restriction\gamma}$$

be the induced embedding. By item (5) above, and the definition of E, $j_E(A) = \pi(A)$ for every $A \in \mathcal{P}([\bar{\gamma}]^{<\omega})^N$. By item (6), and since δ is a strong limit in N, $\pi\restriction\mathcal{P}([\bar{\gamma}]^{<\omega})^N \in H(\gamma^+)^N$. It follows that for every $A \in \mathcal{P}([\bar{\gamma}]^{<\omega})^N$,

$$j(\pi(A)) = j(j_E(A)) = j(j_E\restriction\mathcal{P}([\bar{\gamma}]^{<\omega})^N)(j(A)).$$

It suffices then to see that for every $B \subset [\gamma]^{<\omega}$ in M,

$$j(j_E\restriction\mathcal{P}([\bar{\gamma}]^{<\omega})^N)(B) \cap [\lambda]^{<\omega} = j_H(B) \cap [\lambda]^{<\omega}.$$

This follows from the definition of H.

Since $\bar{\gamma} < \delta \leq \mathrm{CRT}(j)$, $j_H(j(A)) \cap \lambda^{|s|} = j_H(A) \cap \lambda^{|s|}$. Putting everything together, we get that for all $s \in [\bar{\lambda}]^{<\omega}$ and all $A \in \mathcal{P}([\bar{\gamma}]^{|s|}) \cap N$, $A \in \bar{F}(s)$ if and only if $\pi(s) \in j_H(A)$. Since H and $\pi\restriction[\bar{\lambda}]^{<\omega}$ are in N, this implies that $\bar{F} \in N$, and thus that $F \in N$.

By Remark 1.26, since $F \in N$, $\mathrm{Ult}(N, F)$ is wellfounded.

□

The following corollary is Theorem 143 of [4]. Recall that by Kunen's theorem, there can be no nontrivial elementary embedding $j\colon V_{\gamma+2} \to V_{\gamma+2}$. Combined with the corollary, this says that if N is a weak extender model for δ supercompact, then there is no elementary embedding from N to N with critical point at least δ.

Corollary 1.33. *Suppose the N is a weak extender model for δ supercompact. Let γ be an ordinal, and suppose that $j\colon N \cap V_{\gamma+1} \to N \cap V_{j(\gamma)+1}$ is an elementary embedding with critical point at least δ. Then $j \in N$.*

Theorem 1.34 below (Theorem 148 of [4]), however, shows that if there exists a supercompact cardinal δ, then there exists a weak extender model for δ supercompact and a nontrivial elementary embedding from N to N with critical point below δ.

Theorem 1.34. *Suppose that δ is a supercompact cardinal. Then there is a weak extender model N for δ supercompact such that*

1. *N is closed under ω-sequences;*
2. *there is a non-trivial elementary embedding from N to N;*
3. *there is a subset of N which is not set-generic over N.*

A simplified version of Theorem 1.34, obtaining only conclusion (2), appears as Example 27 of [5]. The model N is the ω-th iterate of V by a normal measure μ on a measurable cardinal $\kappa < \delta$. Letting $j_0\colon V \to M_0$ be the embedding giving by μ, $j_0\restriction N$ is a nontrivial elementary embedding from N to N. Letting $j_\omega\colon V \to N$ be the embedding given by the iteration, the hard part of the example (which we will skip) shows that for each $\gamma > \delta$ for which $j_\omega(\gamma) = \gamma$, if ν is a normal fine measure on $\mathcal{P}_\delta(\gamma)$, then $j_\omega(\nu) = \nu \cap N$.

1.4 The Extender Algebra

1.35 Definition. Given cardinals $\kappa < \delta$ and a set $A \subseteq V_\delta$, κ is *$<\delta$-A-strong* if for each $\gamma < \delta$ there exists an extender E such that

- $\mathrm{CRT}(E) = \kappa$
- $j_E(\kappa) > \gamma$
- $\rho(E) \geq \gamma$
- $j_E(A) \cap V_\gamma = A \cap V_\gamma$

The following definition of Woodin cardinals is not the most commonly given one, though it is equivalent.

1.36 Definition. A cardinal δ is *Woodin* if for each $A \subseteq V_\delta$ there is a cardinal $\kappa < \delta$ which is $<\delta$-A-strong.

1.37 Exercise. Show that Woodin cardinals, as defined above, are strongly inaccessible.

We say that a set $\mathcal{E}$ of extenders *witnesses* that δ is Woodin if for each $A \subseteq V_\delta$ there is a $\kappa < \delta$ such that for all $\gamma < \delta$ there is an extender $E \in \mathcal{E}$ as in Definition 1.35.

1.38 Remark. If δ is a Woodin cardinal, then there is a set $\mathcal{E}$ of extenders witnessing that δ is Woodin such that $\mathcal{E} \subseteq V_\delta$ and each extender in E is short. Furthermore, for any $\alpha < \delta$, the set of extenders formed by removing from $\mathcal{E}$ all extenders with critical point less than α still witnesses that δ is Woodin. Finally, if κ is the critical point of a short extender E, then $j_E(\kappa) \leq (2^\kappa \times |\mathrm{LTH}(E)|)^+$, from which it follows that adding the condition $\mathrm{LTH}(E) \geq \gamma$ to the definition of $<\delta$-A-strong would not change this definition (at least if δ is a strong limit cardinal, which Woodin cardinals are) and would not change the class of sets of extenders witnessing that δ is Woodin.

We define the *Extender Algebra* $\mathbb{B}(\mathcal{E}, \delta)$ relative to a set of extenders $\mathcal{E}$ and a cardinal δ. Let $\mathbb{W}_\delta$ be the set of all expressions in the propositional language with sentence symbols c_α for each $\alpha < \delta$, with the operations of negation and conjunctions and disjunctions of cardinality less than δ. For each $\sigma \in \mathbb{W}_\delta$, let $D(\sigma)$ be the least ordinal β such that the sentence symbols appearing in σ are all among $\{c_\alpha : \alpha < \beta\}$; note that $D(\sigma)$ is always less than δ.

For any complete Boolean algebra $\mathbb{C}$, any function $\pi \colon \delta \to \mathbb{C}$ induces a function π^* from $\mathbb{W}$ to $\mathbb{C}$. Let $\mathbb{I}(\mathcal{E}, \delta)$ be the set of pairs (σ_1, σ_2) from $\mathbb{W}_\delta$ such that for some $E \in \mathcal{E}$ with $\mathrm{LTH}(E) \geq D(\sigma_2)$, σ_1 is the disjunction of a sequence of formulas $\langle \tau_\alpha : \alpha < \mathrm{CRT}(E) \rangle$, and σ_2 is the disjunction of

$$j_E(\langle \tau_\alpha : \alpha < \mathrm{CRT}(E) \rangle) \restriction \gamma,$$

for some $\gamma < j_E(\mathrm{CRT}(E))$. Finally, define an equivalence relation $\sim_{\mathcal{E},\delta}$ on on $\mathbb{W}_\delta$ by setting $\tau_1 \sim_{\mathcal{E},\delta} \tau_2$ if for every complete Boolean algebra $\mathbb{C}$ and every function

$f\colon \delta \to \mathbb{C}$, if $\pi^*(\sigma_1) = \pi^*(\sigma_2)$ for every pair $(\sigma_1, \sigma_2) \in \mathbb{I}$, then $\pi^*(\tau_1) = \pi^*(\tau_2)$. Let $\mathbb{B}(\mathcal{E}, \delta) = \mathbb{W}_\delta / \sim_{\mathcal{E},\delta}$, and for each $\tau \in \mathbb{W}_\delta$, let $[\tau]_{\mathcal{E},\delta}$ denote the $\sim_{\mathcal{E},\delta}$-class of τ.

Lemma 1.39. *If $\mathcal{E} \subseteq V_\delta$ is a set of extenders witnessing that δ is Woodin, then $\mathbb{B}(\mathcal{E}, \delta)$ is δ-c.c.*

Proof. Let $S = \langle \tau_\alpha : \alpha < \delta \rangle$ be a sequence of elements of $\mathbb{W}_\delta$. We will show that $\{[\tau_\alpha]_{\mathcal{E},\delta} : \alpha < \delta\}$ does not form an antichain in $\mathbb{B}(\mathcal{E}, \delta)$. Let $\kappa < \delta$ be $<\delta$-S-strong, and let $\gamma > D(\tau_\kappa)$. Since $\mathcal{E}$ witnesses that δ is Woodin, there is an $E \in \mathcal{E}$ such that

- $\mathrm{CRT}(E) = \kappa$;
- $\mathrm{LTH}(E) \geq \gamma$;
- $j_E(\langle \tau_\alpha : \alpha < \kappa \rangle)(\kappa) = \tau_\kappa$.

It follows then that the pair

$$(\vee\{\tau_\alpha : \alpha < \kappa\}, \vee\{\tau_\alpha : \alpha \leq \kappa\})$$

is in $\mathbb{I}(\mathcal{E}, \delta)$, which means that

$$\vee\{\tau_\alpha : \alpha < \kappa\} \sim_{\mathcal{E},\delta} \vee\{\tau_\alpha : \alpha \leq \kappa\}.$$

Therefore, $[\tau_{\mathrm{CRT}(E)}]_{\mathcal{E},\delta}$ is not incompatible with each $[\tau_\alpha]_{\mathcal{E},\delta}$ with $\alpha < \kappa$. $\square$

The following is Theorem 171 of [4]. Unlike typical applications of the Extender Algebra, the proof of Theorem 171 does not involve iteration trees. All A which are bounded subsets of δ satisfy the conditions of the theorem, as, given a set of extenders witnessing that δ is Woodin, the set of extenders E in the set with critical point above $\sup(A)$ also witnesses that δ is Woodin.

Theorem 1.40. *Suppose that δ is a strong limit cardinal, $\mathcal{E} \subseteq V_\delta$ is a set of short extenders, N is a transitive inner model satisfying ZFC, and the set $\mathcal{E}^* = \{E{\restriction}N \mid E \in \mathcal{E}\}$ is an element of N which witnesses in N that δ is a Woodin cardinal. Then, for every $A \subseteq \delta$ such that*

$$j_E(A \cap \mathrm{CRT}(E)) \cap \mathrm{LTH}(E) = A \cap \mathrm{LTH}(E)$$

for every $E \in \mathcal{E}$, A is N-generic for $\mathbb{B}(\mathcal{E}^*, \delta)^N$.

Proof. The first point is that $\sim^N_{\mathcal{E}^*,\delta} = \sim_{\mathcal{E},\delta} \cap N$. To see this, note first of all $\mathbb{I}(\mathcal{E}^*, \delta)^N = \mathbb{I}(\mathcal{E}, \delta) \cap N$, so $\sim_{\mathcal{E},\delta} \cap N \subseteq \sim^N_{\mathcal{E}^*,\delta}$. For the other direction, suppose that $\tau_1 \sim^N_{\mathcal{E}^*,\delta} \tau_2$. It suffices to see that that, in V, if $\mathbb{B}_0$ is the two-element Boolean algebra $\{0, 1\}$, then for all $\pi\colon \delta \to \mathbb{B}_0$, if $\pi^*(\sigma_1) = \pi^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in \mathbb{I}^N$, then $\pi^*(\tau_1) = \pi^*(\tau_2)$. Applying $\undertilde{\Sigma}^1_2$-absoluteness in a forcing extension of N in which δ is countable, we see that it suffices to verify this claim for all $\pi\colon \delta \to \mathbb{B}_0$ existing in set-generic extensions of N.

Now, suppose that $\mathbb{C}$ is a complete Boolean algebra in N, and t is a $\mathbb{C}$-name in N for a function $\nu\colon \delta \to \mathbb{B}_0$ with the property that $\nu^*(\sigma_1) = \nu^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in \mathbb{I}(\mathcal{E}^*, \delta)^N$. Define the function $\pi_t\colon \delta \to \mathbb{C}$ by setting

$$\pi_t(\alpha) = [\![t(\alpha) = 1]\!].$$

Then $\pi_t^*(\sigma_1) = \pi_t^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in \mathbb{I}(\mathcal{E}^*, \delta)^N$, so $\pi_t^*(\tau_1) = \pi_t^*(\tau_2)$. Then it is forced that $\nu^*(\tau_1) = \nu^*(\tau_2)$.

Now fix $A \subseteq \delta$ as in the statement of the theorem. Let G be the filter on $\mathbb{B}(\mathcal{E}^*, \delta)^N$ generated by the terms c_α ($\alpha \in A$) and $\neg c_\alpha$ ($\alpha \notin A$). We want to see that G is N-generic. Let $\chi_A\colon \delta \to \mathbb{B}_0$ be the characteristic function of A. Then $(\chi_A^*)^{-1}[\{1\}]$ is the set of $\sigma \in \mathbb{W}_\delta$ satisfied by A (note that this is computed correctly in $L[\sigma, A]$.) Let us see first that $\chi_A^*(\sigma_1) = \chi_A^*(\sigma_2)$ for all $(\sigma_1, \sigma_2) \in \mathbb{I}(\mathcal{E}^*, \delta)^N$. Fixing such σ_1, σ_2, let $E \in \mathcal{E}$ be such that $E \restriction N$ witnesses that $(\sigma_1, \sigma_2) \in \mathbb{I}(\mathcal{E}^*, \delta)^N$. Then since $j_E(A \cap \mathrm{CRT}(E)) \cap \mathrm{LTH}(E) = A \cap \mathrm{LTH}(E)$ and $\mathrm{LTH}(E) \geq D(\sigma_2)$, A satisfies σ_2 if and only if $j_E(A)$ does. Since σ_1 is a subdisjunction of σ_2, which is a subdisjunction of $j_E(\sigma_1)$, and since A satisfies σ_1 if and only if $j_E(A)$ satisfies $j_E(A)$, A satisfies σ_1 if and only if it satisfies σ_2. Thus $\chi_A^*(\sigma_1) = \chi_A^*(\sigma_2)$.

Therefore, χ_A induces a Boolean homomorphism $\chi_A^{**}\colon \mathbb{B}(\mathcal{E}^*, \delta)^N \to \mathbb{B}_0$, and the corresponding filter $(\chi_A^{**})^{-1}[\{1\}]$ is N-generic. □

1.5 Where comparison must fail

The following theorem (Theorem 127 of [4]), gives a reformulation of supercompactness in terms of extenders.

Theorem 1.41 (Magidor). *A cardinal δ is supercompact if and only if for all $\gamma > \delta$ there is an extender E such that $\mathrm{SPT}(E) < \delta$, $\rho(E) \geq \gamma$ and $j_E(\mathrm{CRT}(E)) = \delta$.*

Proof. First suppose that δ is supercompact. Fix $\gamma < \delta$. By increasing γ if necessary we may assume that $\gamma = |V_\gamma|$. Since δ is supercompact there exists an elementary embedding $j\colon V \to M$ with critical point δ such that $j(\delta) > \gamma$ and $M^{V_{\gamma+1}} \subseteq M$.

Let E be the extender of length $j(\gamma)$ derived from j. That is,

$$E = \{E_s : s \in [j(\gamma)]^{<\omega}\},$$

where each $E_s = \{A \subseteq [\gamma]^{|s|} \mid s \in j(A)\}$. Since $j \restriction V_{\gamma+1} \in M$, $E \in M$. Furthermore, in M, $\mathrm{SPT}(E) = \gamma < j(\delta)$, $\rho(E) = j(\gamma)$ (since $|V_\gamma| = \gamma$) and $j_E(\mathrm{CRT}(E)) = j(\delta)$. It follows by the elementarity of j that in V there exists an extender F such that $\mathrm{SPT}(F) < \delta$, $\rho(F) = \gamma$ and $j_F(\mathrm{CRT}(F)) = \delta$.

Towards showing the reverse direction, fix $\gamma_0 > \delta$. We want to find a normal fine measure on $\mathcal{P}_\delta(\gamma_0)$. Assume towards a contradiction that there is no such measure, and that γ_0 is the least cardinal for which this is true. Fix a $\beth$-fixed point $\gamma > \gamma_0$. Then there is an extender E such that $\mathrm{SPT}(E) < \delta$, $\rho(E) \geq \gamma$

and $j_E(\mathrm{CRT}(E)) = \delta$. Replacing E with $E\restriction\gamma$ if necessary we may assume that $\rho(E) = \mathrm{LTH}(E) = \gamma$. Let $j_E : V \to M$ be the corresponding embedding.

Since $\gamma > \gamma_0$ and $V_\gamma \subseteq M$, by the minimality of γ_0 as above, γ_0 is definable in M from δ. Therefore, there exists a $\bar{\gamma}_0$ such that $j_E(\bar{\gamma}_0) = \gamma_0$. Let $\bar{\delta}$ be the critical point of E. Since $j_E(\bar{\gamma}_0) = \gamma_0 < \gamma$ and $V_\gamma \subseteq M$, $j_E[\bar{\gamma}_0] \in M$, which means that j_E induces a normal fine measure on $\mathcal{P}_{\bar{\delta}}(\bar{\gamma}_0)$. Then $j_E(\mu)$ is a normal fine measure on $\mathcal{P}_\delta(\gamma_0)$, giving a contradiction. □

We say that a class $\mathcal{E}$ of extenders *witnesses* that δ is supercompact if for each $\gamma > \delta$ there is an extender $E \in \mathcal{E}$ such that $\mathrm{SPT}(E) < \delta$, $\rho(E) \geq \gamma$ and $j_E(\mathrm{CRT}(E)) = \delta$.

The following lemma is a relativized version of one half of Theorem 1.41.

Lemma 1.42. *Suppose that N is a transitive set model of ZFC of ordinal height κ, and that $V_\kappa \models \mathrm{ZFC}$. Suppose that $\mathcal{E}$ is a set of extenders such that*

- *every element of $\mathcal{E}$ is an initial segment of an extender E with $\rho(E) = \mathrm{LTH}(E)$ and $\rho(E)$ a strongly inaccessible cardinal;*
- *$\{E\restriction N : E \in \mathcal{E}\}$ witnesses in N that δ is supercompact, for some cardinal δ of N.*

Then N is a weak extender model for δ supercompact, with respect to V_κ.

Proof. Fix a cardinal γ_0 of N above δ. We want to find a normal fine measure μ on $\mathcal{P}_\delta(\gamma_0)$ such that $\mathcal{P}_\delta(\gamma_0) \cap N \in \mu$ and $\mu \cap N \in N$. Assume towards a contradiction that there is no such measure, and that γ_0 is the least cardinal for which this is true. Fix a $\beth$-fixed point $\gamma > \gamma_0$. Then there is an extender $E \in \mathcal{E}$ with $\rho(E) = \mathrm{LTH}(E)$ and $\rho(E)$ a strongly inaccessible cardinal, and an initial segment E' of E such that $E'\restriction N \in N$ and, in N, $\mathrm{SPT}(E'\restriction N) < \delta$, $\rho(E'\restriction N) \geq \gamma$ and $j_{E'\restriction N}(\mathrm{CRT}(E'\restriction N)) = \delta$. Replacing E' with $E'\restriction\gamma$ if necessary we may assume that $\rho(E'\restriction N) = \mathrm{LTH}(E'\restriction N) = \gamma$ in N, and that $\rho(E') = \gamma$. Let $j_{E'} : V \to M$ be the corresponding embedding. By Exercise 1.31,

$$j_E(N \cap V_{\mathrm{SPT}(E')}) \cap V_\gamma = N \cap V_\gamma.$$

Since $\gamma > \gamma_0$ and $V_\gamma \subseteq M$, by the minimality of γ_0 as above, γ_0 is definable in M from δ and $j_{E'}(N \cap V_{\mathrm{SPT}(E')})$. Therefore, there exists a $\bar{\gamma}_0$ such that $j_{E'}(\bar{\gamma}_0) = \gamma_0$. Let $\bar{\delta}$ be the critical point of E'. Since $j_{E'}(\bar{\gamma}_0) = \gamma_0 < \gamma$ and $V_\gamma \subseteq M$, $j_{E'}[\bar{\gamma}_0] \in M$, which means that $j_{E'}$ induces a normal fine measure on $\mathcal{P}_{\bar{\delta}}(\bar{\gamma}_0)$. By Exercise 1.30, $j_{E'}[\bar{\gamma}_0] = j_{E'\restriction N}[\bar{\gamma}_0]$. Since $\rho(E'\restriction N) = \gamma$, $j_{E'\restriction N}[\bar{\gamma}_0]$ is in the image model of $j_{E'\restriction N}$. Again by Exercise 1.30, we have that $\mathcal{P}_{\bar{\delta}}(\bar{\gamma}_0) \cap N \in \mu$ and $\mu \cap N \in N$. Then $j_{E'}(\mu)$ is a normal fine measure on $\mathcal{P}_\delta(\gamma_0)$, $\mathcal{P}_\delta(\gamma_0) \cap N \in \mu$ and $\mu \cap N \in N$, giving a contradiction. □

The following is Definition 153 of [4].

1.43 Definition. We let $\mathcal{M}_S$ denote the set of all transitive sets M for which

1. $M \models ZFC$;

2. $M \cap \mathrm{Ord}$ is a strong cardinal;

3. there exists $\delta \in M$ such that $\mathcal{E}^*$ witnesses that δ is supercompact in M;

4. there exist $\delta_0 > \kappa_0$ and $\mathcal{E}_0 \subseteq \mathcal{E}^*$ in M such that $\mathcal{E}_0$ witnesses in M that δ_0 is a Woodin cardinal,

where

- κ_0 is the least strong cardinal,
- $\mathcal{E}$ is the set of all initial segments of extenders E with $\rho(E) = \mathrm{LTH}(E)$ and $\rho(E)$ strongly inaccessible, and
- $\mathcal{E}^* = \{E \restriction M \mid E \in \mathcal{E} \wedge E \restriction M \in M\}$.

1.44 Remark. By Lemma 1.42, if $M \in \mathcal{M}_S$ and δ is as in part (3) of Definition 1.43, then M is a weak extender model for δ supercompact with respect to $V_{M \cap \mathrm{Ord}}$.

By Theorem 1.40, we have the following.

Lemma 1.45. *If $M \in \mathcal{M}_S$, then every subset of the least strong cardinal is set-generic over M.*

1.46 Definition. A cardinal κ is *extendible* if for each ordinal η there exists an elementary embedding of $V_{\kappa+\eta}$ into some V_λ, with critical point κ.

1.47 Definition. A cardinal κ is *huge* if there exists an elementary embedding $j\colon V \to M$ with $\mathrm{CRT}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$.

1.48 Exercise. Prove that if κ is a huge cardinal then

$$V_\kappa \models \text{“there is an extendible cardinal”}.$$

(Hint: First show that there is a $\lambda < \kappa$ such that for all $\alpha < \kappa$ there is an elementary embedding $j\colon V_{\lambda+\alpha} \to V_\beta$, for some β (possibly larger than κ).

1.49 Definition. Suppose that κ is a strongly inaccessible cardinal, and that $N \subseteq V_\kappa$. We say that $(V_\kappa, N) \models$ “*there is an N-extendible cardinal*” if there exists $\delta < \kappa$ such that for all $\alpha < \kappa$ there is an elementary embedding

$$j\colon V_{\delta+\alpha} \to V_{j(\delta)+j(\alpha)}$$

with $\mathrm{CRT}(j) = \delta$, $\alpha < j(\delta) < \kappa$ and, for all $\beta < \alpha$,

$$j(N \cap V_{\delta+\beta}) = N \cap V_{j(\delta)+j(\beta)}.$$

The following is Lemma 157 of [4].

Lemma 1.50. *Suppose that κ is a huge cardinal. Then for each set $N \subseteq V_\kappa$, $(V_\kappa, N) \models$ "there are cofinally many N-extendible cardinals".*

Proof. As κ is huge, there is an elementary embedding $j\colon V \to M$ with $\mathrm{CRT}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$. We have then that $V_{j(\kappa)} \subseteq M$ and $j \restriction V_{j(\kappa)} \subseteq M$. By the elementarity of j (and the fact that $\mathrm{CRT}(j) = \kappa$), it suffices to show that $(V_{j(\kappa)}, j(N)) \models$ "there is a $j(N)$-extendible cardinal". Supposing towards a contradiction that this fails, there exists $\alpha_0 < j(\kappa)$ such that there is no elementary embedding $k\colon V_{\kappa+\alpha_0} \to V_{k(\kappa+\alpha_0)}$ for which

- $\mathrm{CRT}(k) = \kappa$;
- $\alpha_0 < k(\kappa) < j(\kappa)$;
- for all $\beta < \alpha_0$, $k(j(N) \cap V_{\kappa+\beta}) = j(N) \cap V_{k(\kappa)+k(\beta)}$.

We may assume that $\alpha_0 > 0$. Let k denote $j \restriction V_{\kappa+\alpha_0}$. Then k witnesses that

$$(V_\kappa, N) \prec (V_{j(\kappa)}, j(N)).$$

Applying j, we get that $j(k)$ has critical point $j(\kappa)$ and witnesses that

$$j((V_\kappa, N)) \prec j((V_{j(\kappa)}, j(N))),$$

i.e., that

$$(V_{j(\kappa)}, j(N)) \prec (M \cap V_{j(j(\kappa))}, j(j(N))).$$

Now, $k \in M \cap V_{j(j(\kappa))}$, $\mathrm{CRT}(k) = \kappa$, and

$$\alpha_0 < j(\kappa) = k(\kappa) < j(k)(j(\kappa)).$$

Now fix $\beta < \alpha_0$. Since $\mathrm{CRT}(j(k)) = j(\kappa) > \kappa + \beta$,

$$j(k)(j(N) \cap V_{\kappa+\beta}) = j(N) \cap V_{\kappa+\beta}.$$

Then

$$k(j(k)(j(N) \cap V_{\kappa+\beta})) = k(j(N) \cap V_{\kappa+\beta}).$$

Now,

$$\begin{aligned}
k(j(N) \cap V_{\kappa+\beta}) &= j(j(N) \cap V_{\kappa+\beta}) \\
&= j(j(N \cap V_{\kappa+\beta}) \cap V_{\kappa+\beta}) \\
&= j(k(N \cap V_{\kappa+\beta}) \cap V_{\kappa+\beta}) \\
&= j(k(N \cap V_{\kappa+\beta})) \cap j(V_{\kappa+\beta}) \\
&= j(k)(j(N \cap V_{\kappa+\beta})) \cap k(V_{\kappa+\beta}) \\
&= j(k)(j(N) \cap V_{j(\kappa+\beta)}) \cap V_{k(\kappa+\beta)}
\end{aligned}$$

It follows then that in $M \cap V_{j(j(\kappa))}$, there is an elementary embedding k with domain $V_{\kappa+\alpha_0}$ such that

- $\mathrm{CRT}(k) = \kappa$;
- $\alpha_0 < k(\kappa) < j(k)(j(\kappa))$;
- $j(k)(j(\kappa)) > k(\kappa + \beta)$;
- for each $\beta < \alpha_0$, $k(j(k)(j(N) \cap V_{\kappa+\beta})) = j(k)(j(N) \cap V_{j(\kappa)}) \cap V_{k(\kappa+\beta)}$.

Then by the elementarity of $j(k)$, and the fact that

$$\mathrm{CRT}(j(k)) = j(\kappa) > \kappa + \alpha_0,$$

there is an elementary embedding k with domain $V_{\kappa+\alpha_0}$ such that

- $\mathrm{CRT}(k) = \kappa$;
- $\alpha_0 < k(\kappa) < j(\kappa)$;
- $j(\kappa) > k(\kappa + \beta)$;
- for each $\beta < \alpha_0$,

$$k(j(N) \cap V_{\kappa+\beta}) = (j(N) \cap V_{j(\kappa)}) \cap V_{k(\kappa+\beta)} = j(N) \cap V_{k(\kappa+\beta)},$$

giving a contradiction to the choice of α_0. □

Theorem 1.51. *Suppose that there exist proper class many huge cardinals. Let ψ be the statement that there exist cofinally many ordinals κ_0 such that for some ordinal $\kappa_1 > \kappa_0$, κ_0 is an extendible cardinal in V_{κ_1}. Then every member of $\mathcal{M}_S$ satisfies ψ. Furthermore, for each ordinal γ which is Σ_2-definable there exists a transitive set N such that*

- *$N \models \mathrm{ZFC}$ and $V_\gamma \in N$;*
- *$N \models$ "There is an extendible cardinal.";*
- *no member of $\mathcal{M}_S^N$ satisfies ψ.*

Proof. First, suppose that M_0 is an element of $\mathcal{M}_S$. Then $M_0 \cap \mathrm{Ord}$ is a strong cardinal of V. Since there are proper class many huge cardinals, $M_0 \cap \mathrm{Ord}$ is then a limit of huge cardinals. By Lemma 1.50, there are cofinally many $\kappa_0 \in M \cap \mathrm{Ord}$ for which there exist $\kappa_1 \in M_0 \cap \mathrm{Ord}$ such that $\kappa_0 < \kappa_1$ and

$$(V_{\kappa_1}, M_0 \cap V_{\kappa_1}) \models \mathrm{ZFC} + \text{“}\kappa_0 \text{ is an } (M_0 \cap V_{\kappa_1})\text{-extendible cardinal”}.$$

By Remark 1.44, $M_0 \cap V_{\kappa_1}$ is a weak extender model for δ supercompact with respect to V_{κ_1}. By Corollary 1.33, then, for each such pair κ_0, κ_1,

$$M_0 \cap V_{\kappa_1} \models \mathrm{ZFC} + \text{“}\kappa_0 \text{ is an extendible cardinal”}.$$

So $M_0 \models \psi$.

Now fix a Σ_2-definable ordinal γ. If κ is a $\beth$-fixed point, and some ordinal satisfies a Σ_2 formula in V_κ, then the same ordinal satisfies this formula in V. Applying this fact we have that by increasing γ if necessary we can suppose that γ is a $\beth$-fixed point, and that, for some sentence ϕ, γ is the least ordinal η such that $V_\eta \models \phi$. Recalling Exercise 1.48, let ξ be the least ordinal η for which there exists a transitive set N with

- $V_\gamma \in N$;
- $N \models \mathrm{ZFC} +$ "there is an extendible cardinal";
- $N \cap \mathrm{Ord} = \eta$,

and let N be such a set with respect ξ.

Let a be a subset of γ in N which codes V_γ. Since V_γ is the least rank satisfying ϕ, γ is below the least strong cardinal. Fix $M_1 \in \mathcal{M}_S^N$. By Lemma 1.45, $M_1[a]$ is a set-generic extension of M_1. Supposing towards a contradiction that $M_1 \models \psi$, we have that there exist $\kappa \in M_1 \cap \mathrm{Ord}$ such that $a \in V_\kappa$ and

$$M_1[a] \cap V_\kappa \models \mathrm{ZFC} + \text{"there is an extendible cardinal"}.$$

This contradicts the minimality of $N \cap \mathrm{Ord}$, since $V_\gamma \in M_1[a] \cap V_\kappa$. $\square$

We briefly sketch the reason that Theorem 1.51 is a failure of comparison. Suppose that there exist proper class many huge cardinals. Then there exists a partial extender model $L_\alpha[E]$, witnessing the existence of large cardinal roughly at the level of supercompact cardinals and built using extenders from V, such that some $L_\kappa[E]$ is a member of $\mathcal{M}_S$ and is Σ_1 definable in $L_\alpha[E]$ using a predicate for E. Similarly, there is a another such partial extender model $L_\beta[F]$ constructed from the point of view of a transitive set N as in Theorem 158, and some $L_\lambda[F]$ is a member of $\mathcal{M}_S^N$ and Σ_1 definable in $L_\beta[F]$ using a predicate for F. Furthermore, there is a Σ_1 formula θ (in a predicate for a partial extender sequence) such that

- $L_\kappa[E]$ satisfies θ in $L_\alpha[E]$ with respect to E
- $L_\lambda[F]$ satisfies θ in $L_\beta[F]$ with respect to F
- any structure in $L_\beta[F]$ satisfying θ with respect to F is a member of $\mathcal{M}_S^N$, and
- any structure in $L_\alpha[E]$ satisfying θ with respect to E is a member of $\mathcal{M}_S$.

Then, by Theorem 1.51, the Σ_1 theory of $L_\alpha[E]$ in E is not contained in the Σ_1 theory of $L_\beta[F]$ in F, and the Σ_1 theory of $L_\beta[F]$ in F is not contained in the Σ_1 theory of $L_\alpha[E]$ in E.

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