Turing Determinacy, Countable Choice and Ultrafilters

Paul Larson *
Miami University

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We let Turing Determinacy (TD) be the statement that every Turing-invariant set of reals contains a Turing cone, and Countable Choice for reals ($\mathsf{CC}_{\mathbb{R}}$) be the statement that every countable family of nonempty sets of reals has a Choice function. It is apparently an open question whether TD implies $\mathsf{CC}_{\mathbb{R}}$. We note here that the conjunction $TD \wedge \neg\mathsf{CC}_{\mathbb{R}}$ implies the existence of a nonprincipal ultrafilter on ω.

To see this, let $\{A_i : i \in \omega\}$ witness the failure of $\mathsf{CC}_{\mathbb{R}}$. For each $x \in \omega^\omega$, there is a minimal i such that x is not Turing above any member of A_i. Let $f(x)$ be this i, and note that f is a Turing invariant function. For each $B \subseteq \omega$, $f^{-1}[B]$ is Turing invariant. Let U be the set of $B \subseteq \omega$ for which $f^{-1}[B]$ contains a Turing cone. Then U is a nonprincipal ultrafilter on ω.

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