A theorem of Todorcevic on universal Baireness

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Abstract

Given a cardinal κ, a set $A \subset \omega^\omega$ is κ-universally Baire if there exist trees S, T which project to A and its complement respectively, and which continue to project to complements after the forcing $Coll(\omega, \kappa)$. Todorcevic [3] has shown that under Martin's Maximum every set of reals of cardinality $\aleph_1$ is $\aleph_1$-universally Baire but not $\aleph_2$-universally Baire. Here we present alternate proofs of these two facts.

Lemma 0.1. *Suppose that S is a tree on $\omega \times \gamma$ for some ordinal γ, and suppose that some forcing P adds a new real to the projection of S. Then the projection of S contains a perfect set in the ground model.*

Proof. Let X be a countable elementary submodel of $H((2^{|P|})^+)$ containing P and S and a name τ for a new element of the projection of S. Then there exists a perfect set of filters contained in $X \cap P$ deciding a perfect set of values for τ along with element of γ^ω witnessing that these realizations of τ are in the projection of S. Then all of these realizations of τ are in the projection of S. $\square$

Recall that Martin's Maximum implies that the nonstationary ideal on ω_1 (NS_{ω_1}) is precipitous, and that $2^{\aleph_1} = \aleph_2$ [1].

Theorem 0.2. *Suppose that NS_{ω_1} is precipitous. Let A be an uncountable set of reals not containing a perfect set. Then A is not $2^{\aleph_1}$-universally Baire.*

Proof. Let A be an uncountable set of reals with no perfect subset, and towards a contradiction let S, T be trees witnessing that A is $2^{\aleph_1}$-universally Baire. Since A has no perfect subset, the projection of S in the collapse extension has to be exactly A. Pick an ω_1-sequence $\bar{a}$ of reals from A. Let $G \subset \mathcal{P}(\omega_1)/NS_{\omega_1}$ be a V-generic filter, and let $j \colon V \to M$ be the corresponding embedding. Since A is $2^{\aleph_1}$-universally Baire, the ω_1^V-st element x of $j(\bar{a})$ is in the projection of T, and so it is in the projection of $j(T)$. Furthermore, since M is wellfounded, $M \models x \in p[j(T)]$. Then in V stationarily many members of $\bar{a}$ are in the projection of T, giving a contradiction. $\square$

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Almost-disjoint coding for sets of reals of cardinality κ holds if for every set $\{a_\alpha : \alpha < \kappa\}$ of infinite subsets of ω with pairwise finite intersection and every $A \subset \kappa$ there is a $b \subset \omega$ such that for all $\alpha < \kappa$, $b \cap a_\alpha$ is infinite if and only if α is in A. Martin's Axiom implies that almost-disjoint coding holds for sets of reals of cardinality less than the continuum [2].

Theorem 0.3. *If almost-disjoint coding holds for sets of reals of cardinality κ, then every set of reals of cardinality κ is κ-universally Baire.*

Proof. Let $A = \{a_\alpha : \alpha < \kappa\}$ be a set of reals of cardinality κ. It suffices to show that if $V[G]$ is an extension by $Coll(\omega, \kappa)$, then every new real in $V[G]$ is in the projection of a tree in V whose projection is disjoint from A. Each new real in $V[G]$ is the realization of a name which is forced by the empty condition to be a new real. Fix such a name τ.

For each real x, let x^* is the set of finite initial segments of x.

For each $p \in Coll(\omega, \kappa)$, there are 2^ω many reals which are realizations of τ by filters containing p. Pick one such real, $y_p \notin A$, for each p. Applying almost-disjoint coding, let z be a set of finite sequences from ω intersecting each y_p ($p \in Coll(\omega, \kappa)$) infinitely often and each a^*_α ($\alpha < \kappa$) finitely often. Then for each condition p and each integer n, there is a condition $p' \leq p$ forcing that $\tau^*_g \cap z$ has size at least n. By genericity, then, the realization of τ will have infinitely many initial segments in z.

The set of reals having infinitely many initial segments in z is an analytic set disjoint from A. $\square$

References

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