NOTES ON TODORCEVIC'S ERICE LECTURES ON FORCING WITH A COHERENT SUSLIN TREE

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1. Part I

1.1. The P-ideal dichotomy. The P-ideal dichotomy is the statement that whenever I is a P-ideal on a set X, either X is a countable union of sets orthogonal to I (i.e., intersecting no member of I infinitely), or there is an uncountable subset of X whose countable subsets are all in I. The statement is not weakened when we assume that I consists of countable sets, which we do here.

First we review a proper forcing which forces an instance of the P-ideal dichotomy. Let X and I be as above, and assume that X is not a countable union of sets orthogonal to I. Let $\kappa = (|X|^{\aleph_0})^+$. For each countable elementary submodel M of $H(\kappa)$ with X and I in M, fix an element a_M of I which contains mod finite all members of $M \cap I$. Let P be the partial order whose conditions p are pairs $(\mathcal{M}_p, Y_p)$, where $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\kappa)$ with X and I as members, and Y_p is a finite $\mathcal{M}_p$-separated (i.e., for any two members of Y_p there is an element of $\mathcal{M}_p$ that has one as an element and not the other) subset of X such that for each $y \in Y_p$ and each $M \in \mathcal{M}_p$, if $y \in M$ then $y \in a_M$, and if $y \notin M$ then y is not in any set in M orthogonal to I. The order is inclusion on both coordinates.

Now suppose that p is a condition, and N is a countable elementary submodel of $H((2^{|P|})^+)$ with P and p as elements. Let p' be the condition $(\mathcal{M}_p \cup \{N \cap H(\kappa)\}, Y_p)$. We want to see that p' is (P,N)-generic. So let D be a dense subset of P in N and let r be a condition below p'. We may assume that $r \in D$. Let M_0 be the largest model in $\mathcal{M}_r \cap N$.

Arguing in N, and identifying finite subsets of X with their increasing enumeration in terms of some wellordering of X in all models of $\mathcal{M}_r$, let $\mathcal{T}$ be the tree of finite increasing sequences t from X such that

- all members of t are greater than all members of $Y_r \cap N$,
- no member of t is in any set in M_0 orthogonal to I,
- there is an extension Z of $(Y_r \cap N)^\frown t$ of length $|Y_r|$ for which there is some condition $q \in D$ with $Y_q = Z$.

Note that $Y_r \setminus N$ is in $\mathcal{T}$. Now thin $\mathcal{T}$ (iteratively removing as few nodes as possible) to a tree $\mathcal{T}'$ such that for each node t of $\mathcal{T}'$ of length less than $|Y_r \setminus N|$ (including the emptyset), the set of $x \in X$ such that $t^\frown\langle x\rangle \in \mathcal{T}'$ is not orthogonal to I. This thinning takes $|Y_r \setminus N|$ many rounds starting, one for each non-terminal level of the tree, proceeding from the top down. Note that $Y_r \setminus N$ is still in $\mathcal{T}'$, since for

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each proper initial segment t of $Y_r \setminus N$, t is in some elementary submodel M of $H(\kappa)$, and the next element of Y above the maximum of t is not in any set in M orthogonal to I.

Now we can choose a cofinal branch through $\mathcal{T}'$ consisting of elements of N, with the property that the elements of the branch are all in a_M for all $M \in \mathcal{M}_r \setminus N$. To see this, note that at each point of our construction the set of possible extensions in $\mathcal{T}'$ must contain an infinite element of I, and all but finitely many of the members of I will be in $\bigcap_{M \in \mathcal{M} \setminus N} a_M$.

This completes the proof.

1.2. **PFA(S) and the P-ideal dichotomy.** Now suppose that S is a coherent Suslin tree, λ is a cardinal and $\dot{I}$ is an S-name for a P-ideal on λ such that λ is not a countable union of sets orthogonal to $\dot{I}$. Again, let κ be $(\lambda^{\aleph_0})^+$. For each countable elementary submodel M of $H(\kappa)$ with S and $\dot{I}$ as members, we choose a name $\dot{a}_M$ for a countable subset of λ such that the members of $S_{\omega_1 \cap M}$ (where S_α denotes the αth level of S) decide $\dot{a}_M$, and the realization of $\dot{a}_M$ is forced to

- contain mod finite all members of the realization of $\dot{I}_M$.
- be contained mod finite in some member of I containing mod finite all members of the realization of $\dot{I}_M$,

where $\dot{I}_M$ is the name for the realization of all the names in M for members of I. We can find such a name by filling an appropriate (ω, ω)-gap corresponding to each member of $S_{\omega_1 \cap M}$. Since we assume that $\dot{I}$ is a name for an ideal containing all finite subsets of λ, $\dot{a}_M$ is in fact a name for a member of the realization of $\dot{I}$.

For each such M, let $\dot{\xi}_M$ be the canonical (nice) name for the least element of λ not in any subset of λ orthogonal to $\dot{I}$ realized by a name in M.

We now define the forcing P. A condition in p is a function whose domain is a finite $\in$-chain $\mathcal{M}_p$ of countable elementary submodels of $H(\kappa)$ with S and $\dot{I}$ as members, and range contained in S, such that for each $M \in \mathcal{M}_p$, $p(M)$ is not in M but is in all members of $\mathcal{M}_p \setminus M$, and that $p(M)$ decides the value of $\dot{\xi}_M$ (note that $p(M)$ will also decide the value of $\dot{a}_M$, though this is less important). The function p must have the further property that if M, N are in $\mathcal{M}_p$ and $p(N) < p(M)$, then $p(M)$ forces that $\dot{\xi}_N \in \dot{a}_M$.

Now suppose that p_0 is a condition in P, and N is a countable elementary submodel of $H((2^{|P|})^+)$ with P and p_0 as members. Let p_1 be the condition $p_0 \cup \{(N \cap H(\kappa), t')\}$, where t' is any element of $S \setminus N$. Now let s_0 be any element of $S_{N \cap \omega_1^M}$. We need to see that (p_1, s_0) is $(P \times S, N)$-generic.

Let (r, s_1) be an element of $P \times S$ below (p_1, s_0). We may assume that $(r, s_1) \in D$, and that the height of s_1 is greater than the height of any member of the range of r. Fix $\gamma_0 \in \omega_1 \cap N$ such that no member of the range of r disagrees with s_0 at any point in the interval $[\gamma_0, \omega_1 \cap N)$. Enumerate the models of the domain of r (as ordered by the $\in$-relation) as $\langle Q_i : i < |r| \rangle$.

For any condition $p \in P$, let Ξ_p be the function with the same domain as p where $\Xi_p(Q)$ is the value of $\dot{\xi}_Q$ as decided by $p(Q)$.

For each $t \in T$, let $\mathcal{T}_t$ be the tree of consisting of all initial segments of increasing sequences e from X which are the ranges of $\Xi_{p \setminus (r \cap N)}$, for some $p \in P$ end-extending $(r \cap N)$ such that $(p, t) \in D$, $|p| = |r|$ and

- for each $i < |r|$, if M is the the ith element of the domain of p, then $p(M)$ agrees with $r(Q_i)$ up to γ_0, and $p(M)$ agrees with t after γ_0 if and only if $r(Q_i)$ agrees with s_1 after γ_0.

Let a be the set of $i < |r|$ such that $r(Q_i)$ does not disagree with s_1 on ordinals greater than or equal to γ_0.

Since D is closed under strengthening the right coordinate, $\mathcal{T}_t \subseteq \mathcal{T}_{t'}$ whenever $t \geq_S t'$.

For each $t \in S$, thin $\mathcal{T}_t$ to a tree $\mathcal{T}'_t$ (iteratively removing as few nodes as possible, level by level) such that for each $\sigma \in \mathcal{T}_t$ (including the empty sequence),

- if
  - $|r \cap N| + |\sigma| + 1 \in a$,
  - B_σ is the set of immediate successors of σ in $\mathcal{T}'_t$,

  then B is forced by the union of t beyond γ_0 with $r(Q_{|r\cap N|+|\sigma|+1})\restriction\gamma_0$ to have infinite intersection with some countable set C forced by this condition to be in $\dot{I}$ (which since this union is M-generic is the same as saying that the union does not force B to be orthogonal to $\dot{I}$).

Claim. *The range of $\Xi_{r\setminus N}$ is in $\mathcal{T}_{s_1}$*

Proof : For each $t \in S$, let $\mathcal{T}^0_t = \mathcal{T}_t$. For each ordinal $j < |r \setminus N|$ and each $t \in S$, $\mathcal{T}^{j+1}_t$ is formed from $\mathcal{T}^j_t$ by thinning removing those sequences from $\mathcal{T}^j_t$ of length $|r \setminus N| - j - 1$ whose set of immediate successors is not sufficiently large. It suffices then to fix $j < |r \setminus N|$, to suppose that the range of $\Xi_{r\setminus N}$ is in $\mathcal{T}^j_{s_1}$ and show that it is in $\mathcal{T}^{j+1}_{s_1}$. To do this, let $i = |r \setminus N| - j - 1$, and let σ_i be the first i many member of the range of $\Xi_{r\setminus N}$.

Let U be the set of $t \in S$ such that $\sigma_i \in \mathcal{T}^j_t$. Then $U \in Q_{i+1}$.

For each $t \in U$, let B^t_σ be the set of immediate successors of σ_i in $\mathcal{T}^j_t$.

If there exist $t \geq_S t'$ in $S \cap Q_{i+1}$ below s_1 such that t' forces B_t not to be orthogonal to I, then we are done. Otherwise, there is a name in Q_{i+1} for the union of the sets B^t_σ along the generic branch, and this set must be forced by some initial segment of s_1 in Q_{i+1} to be orthogonal as it is an increasing union of uncountably many orthogonal sets. But s_1 forces that $\Xi_r(Q_{i+1})$ is not in this set, and $\Xi_r(Q_{i+1}) \in B^{s_1}_\sigma$, giving a contradiction. This concludes the proof of the claim.

Then $\mathcal{T}_{s_1}$ has height $|r|$, so the set of $s \in S$ extending $s_1\restriction\gamma_0$ for which $\mathcal{T}_s$ has height $|r|$ contains s_1, so we can find such an s_2 in N which is an initial segment of s_1. Then we can find a condition of size $|r|$ in $N \cap \mathcal{T}_{s_2}$ such that the corresponding ξ's are in the required realizations of the names $\dot{a}_M$, minus their finite errors. We do this by finding in N a branch p_2 (i.e., p_2 is the set of left-coordinates of the branch) through $\mathcal{T}_{s_2}$ with the property that for each $M \in \mathcal{M}_{p_2}$ and each $Q \in \mathcal{M}_r \setminus N$, if $p_2(M) < r(Q)$, then $\dot{\xi}_M$ as decided by $p_2(M)$ is in the set $\dot{a}_Q$ as decided by $r(Q)$. Note that as we do this, if $i < |r|$ and $r(Q_i)$ disagrees with s_1 above γ_0, then the same will be true for the ith level of p_2, so the hypotheses of the above implication will not be satisfied. In the other case, the set of values $\dot{\xi}_M$ for potential models M at the ith level (according to $\mathcal{T}_{s_2}$) is forced by the union of s_2 beyond γ_0 with $r(Q_i)\restriction\gamma_0$ to have infinite intersection with some countable set C forced by this condition to be in $\dot{I}$. Then for each $Q \in r \setminus N$ such that $r(Q)$ agrees with $r(Q_i)$ up to γ_0, $\dot{a}_N$ (as decided by $s_1\restriction[\gamma_0, N \cap \omega_1] \cup r(Q_i)\restriction\gamma_0$ contains all but finitely much of C, so there is some member of $C \cap B$ which is in all of these sets. Choose the ith model M of P_2 so that the realization of $\dot{\xi}_M$ is such a member. This completes the proof that (p_1, s_0) is $(P \times S, N)$-generic.

Finally, let us suppose that $\langle M_\alpha : \alpha < \omega_1 \rangle$ is a generic sequence for P, with a corresponding function p whose domain is this sequence and whose range is contained in S. The set of conditions $p(M_\alpha)$ is somewhere dense in S, and any branch through S below this condition will force that the realizations of the names $\dot{\xi}_{M_\alpha}$ for which $p(M_\alpha)$ is in the generic branch will be an uncountable set whose countable subsets are all in the realization of $\dot{I}$. Since we could carry out this entire argument below any node of S, a dense set of nodes in S force the existence of such an uncountable set and this completes the proof that under PFA(S) the P-ideal dichotomy holds after forcing with S.

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