# The stationary set splitting game*

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**Abstract**

The *stationary set splitting game* is a game of perfect information of length $\omega_1$ between two players, *unsplit* and *split*, in which *unsplit* chooses stationarily many countable ordinals and *split* tries to continuously divide them into two stationary pieces. We show that it is possible in ZFC to force a winning strategy for either player, or for neither. This gives a new counterexample to $\Sigma^2_2$ maximality with a predicate for the nonstationary ideal on $\omega_1$, and an example of a consistently undetermined game of length $\omega_1$ with payoff definable in the second-order monadic logic of order. We also show that the determinacy of the game is consistent with Martin's Axiom but not Martin's Maximum.

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The *stationary set splitting game* ($\mathcal{SG}$) is a game of perfect information of length $\omega_1$ between two players, *unsplit* and *split*. In each round $\alpha$, *unsplit* either accepts or rejects $\alpha$. If *unsplit* accepts $\alpha$, then *split* puts $\alpha$ into one of two sets $A$ and $B$. If *unsplit* rejects $\alpha$ then *split* does nothing. After all $\omega_1$ many rounds have been played, *split* wins if *unsplit* has not accepted stationarily often, or if both of $A$ and $B$ are stationary.

In this note we prove that it is possible to force a winning strategy for either player in $\mathcal{SG}$, or for neither, and we also show that the determinacy of $\mathcal{SG}$ is consistent with Martin's Axiom but not Martin's Maximum [4]. We also present two guessing principles, $\mathcal{C}_s$ (*club for split*) and $\mathcal{D}_u$ (*diamond for unsplit*), which imply the existence of winning strategies for *split* and *unsplit*, respectively (and are therefore incompatible; see Theorems 1.5 and 1.8). These principles may be of independent interest.

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# 1 Winning strategies

## 1.1 Strategies for *split*

A collection $\mathcal{X}$ of countable sets is *stationary* if for every function $F\colon [\bigcup \mathcal{X}]^{<\omega} \to \bigcup \mathcal{X}$ there is an element of $\mathcal{X}$ closed under $F$. A set $\mathcal{X}$ of countable sets is *projective stationary* [2] if for every stationary $S \subset \omega_1$ the set of $X \in \mathcal{X}$ with $X \cap \omega_1 \in S$ is stationary. We note that a partial order $P$ is said to be *proper* if forcing with $P$ preserves the stationarity (in the sense above) of stationary sets from the ground model (see [11]).

The following statement holds in fine structural models such as $L$. It is a strengthening of the principle (+) used in [8]. Justin Moore has pointed out to us that his Mapping Reflection Principle [9] implies the failure of (+). We note also that in the statement of (+), "projective stationary" can be replaced with "club" without strengthening the statement. We do not know if that is the case for $\mathcal{C}+$.

**1.1 Definition.** Let $\mathcal{C}+$ be the statement that there exists a projective stationary set $\mathcal{X}$ consisting of countable elementary substructures of $H(\aleph_2)$ such that for all $X$, $Y$ in $\mathcal{X}$ with $X \cap \omega_1 = Y \cap \omega_1$, either every for every club $C \subset \omega_1$ in $X$ there is a club $D \subset \omega_1$ in $Y$ with $D \cap X \subset C \cap X$, or for every for every club $D \subset \omega_1$ in $Y$ there is a club $C \subset \omega_1$ in $X$ with $C \cap X \subset D \cap X$.

Given a partial run of $\mathcal{SG}$ of length $\alpha$, we let $E_\alpha$ be the set of $\beta < \alpha$ accepted by *unsplit*, and we let $A_\alpha$, $B_\alpha$ be the partition of $E_\alpha$ chosen by *split*.

**Theorem 1.2.** *If $\mathcal{C}+$ holds then* split *has a winning strategy in $\mathcal{SG}$.*

*Proof.* Let $\mathcal{X}$ be a set of countable elementary submodels of $H(\aleph_2)$ witnessing $\mathcal{C}+$, and for each $\alpha < \omega_1$ let $\mathcal{X}_\alpha$ be the set of $X \in \mathcal{X}$ with $X \cap \omega_1 = \alpha$. Let $Z$ be the set of $\alpha < \omega_1$ such that $\mathcal{X}_\alpha$ is nonempty (since $\mathcal{X}$ is projective stationary, this set contains a club).

Play for *split* as follows. In round $\alpha \in Z$, if *unsplit* accepts $\alpha$, let $\mathcal{Y}_\alpha$ be the set of all $X \in \mathcal{X}_\alpha$ such that $X$ contains a stationary subset of $\omega_1$, $E_X$, such that $E_X \cap \alpha = E_\alpha$. If $\mathcal{Y}_\alpha = \emptyset$, put $\alpha \in A_{\alpha+1}$. Otherwise, since every club subset of $\omega_1$ in every member of $\mathcal{Y}_\alpha$ intersects $E_\alpha$, there cannot be two club subsets of $\omega_1$ in $\bigcup \mathcal{Y}_\alpha$ , one disjoint from $A_\alpha$ and one disjoint from $B_\alpha$, since some club subset of $\omega_1$ in $\bigcup \mathcal{Y}_\alpha$ would be contained in both of these clubs. If any member of $\mathcal{Y}_\alpha$ contains a club subset of $\omega_1$ disjoint from $A_\alpha$, then put $\alpha$ in $A_{\alpha+1}$, and if any member of $\mathcal{Y}_\alpha$ contains a club subset of $\omega_1$ disjoint from $B_\alpha$, then put $\alpha$ in $B_{\alpha+1}$. If neither case holds, put $\alpha \in A_{\alpha+1}$.

Let $E$ be the play by *unsplit* in a run of $\mathcal{SG}$ where *split* has played by this strategy, and let $A$ and $B$ be the corresponding play by *split*. Let $C$ be a club subset of $\omega_1$ and supposing that $E$ is stationary, fix $X \in \mathcal{X}$ containing $E$, $A$, $B$ and $C$ with $X \cap \omega_1 \in E \cap C$. Then if $A \cap C \cap X \cap \omega_1 = \emptyset$, then $X \cap \omega_1 \in A \cap C$, and if $B \cap C \cap X \cap \omega_1 = \emptyset$, then $X \cap \omega_1 \in B \cap C$, which shows that $C$ does not witness that *unsplit* won this run of the game. □

The following fact, in conjunction with Theorem 1.2, shows that Martin's Axiom is consistent with the existence of a winning strategy for *split*.

**Theorem 1.3.** *The statement $\mathcal{C}+$ is preserved by forcing with c.c.c. partial orders.*

*Proof.* Let $P$ be a c.c.c. forcing and let $\mathcal{X}$ witness $\mathcal{C}+$. Let $\gamma$ be a regular cardinal greater than $\aleph_2$ and $2^{|P|}$. Let $G \subset P$ be a $V$-generic filter, and let

$$\mathcal{X}[G] = \{X[G] \cap H(\aleph_2)^{V[G]} : X \prec H(\gamma)^V, X \cap H(\aleph_2)^V \in \mathcal{X}\}.$$

Since every club subset of $\omega_1$ in $V[G]$ contains one in $V$, in order to show that $\mathcal{X}[G]$ witnesses $\mathcal{C}+$ in $V[G]$, it suffices to show that $\mathcal{X}[G]$ is projective stationary there. Fix a $P$-name $\rho$ for a function from $[H(\aleph_2)^{V[G]}]^{<\omega}$ to $H(\aleph_2)^{V[G]}$. For any countable $X \prec H(\gamma)$ with $X \cap H(\aleph_2) \in \mathcal{X}$ and $\rho \in X$, $X[G] \cap H(\aleph_2)^{V[G]}$ is in $\mathcal{X}[G]$ and closed under the realization of $\rho$. Fix a $P$-name $\tau$ for a stationary subset of $\omega_1$ and a condition $p \in P$. Let $S$ be the set of countable ordinals forced to be in $\tau$ by some condition below $p$. Then exist a countable $X \prec H(\gamma)$ with $X \cap H(\aleph_2) \in \mathcal{X}$, $X \cap \omega_1 \in S$ and $\rho \in X$ and a condition $q$ below $p$ forcing that $X[\dot{G}] \cap \omega_1$ (where $\dot{G}$ is the name for the generic filter) is in the realization of $\tau$. By genericity, then, $\mathcal{X}[G]$ is projective stationary. $\square$

We do not know how to force $\mathcal{C}+$, however, and use a different principle to force the existence of a winning strategy for *split*.

**1.4 Definition.** Let $\mathcal{C}_s$ be the statement that there exist $c_\alpha$ ($\alpha < \omega_1$ limit) such that each $c_\alpha$ is a sequence $\langle a^\alpha_\beta : \beta < \gamma_\alpha \rangle$ (for some countable $\gamma_\alpha$) of cofinal subsets of $\alpha$ of orderype $\omega$ and

- for all limit $\alpha < \omega_1$ and all $\beta < \beta' < \gamma_\alpha$, $a^\alpha_{\beta'} \setminus a^\alpha_\beta$ is finite;
- for every club $C \subset \omega_1$ and every stationary $E \subset \omega_1$ there exists an $a^\alpha_\beta$ with $\alpha \in E$ such that $a^\alpha_\beta \setminus C$ is finite and $a^\alpha_\beta \cap E$ is infinite.

The principle $\mathcal{C}_s$ also holds in fine structural models such as $L$. The winning strategy for *split* given by $\mathcal{C}_s$ is very similar to the one given by $\mathcal{C}+$.

**Theorem 1.5.** *If $\mathcal{C}_s$ holds then* split *has a winning strategy in $\mathcal{SG}$.*

*Proof.* Let $a^\alpha_\beta$ ($\alpha < \omega_1$ limit, $\beta < \gamma_\alpha$) witness $\mathcal{C}_s$. Play for *split* as follows. In round $\alpha$, $\alpha$ a limit, if *unsplit* has accepted $\alpha$ and if some $a^\alpha_\beta$ intersects $A_\alpha$ infinitely and $B_\alpha$ finitely, then put $\alpha$ in $B_{\alpha+1}$. If some $a^\alpha_\beta$ intersects $B_\alpha$ infinitely and $A_\alpha$ finitely, then put $\alpha$ in $A_{\alpha+1}$. Since the $a^\alpha_\beta$'s ($\beta < \gamma_\alpha$) are $\subset$-decreasing mod finite, both cases cannot occur. If neither case occurs, put $\alpha$ in $A_{\alpha+1}$.

Let $E$ be the play by *unsplit* in a run of $\mathcal{SG}$ where *split* has played by this strategy, and let $A$ and $B$ be the corresponding play by *split*. Let $C$ be a club subset of $\omega_1$ and supposing that $E$ is stationary, fix $a^\alpha_\beta$ with $\alpha \in E$ such that $a^\alpha_\beta \setminus C$ is finite and $a^\alpha_\beta \cap E$ is infinite. Then if $A \cap a^\alpha_\beta$ is finite, then $\alpha \in A \cap C$, and if $B \cap a^\alpha_\beta$ is finite, then $\alpha \in B \cap C$, which shows that $C$ does not witness that *unsplit* won this run of the game. $\square$

A partial order $P$ is said to be *strategically* $\omega$-closed if there exists a function $f\colon P^{<\omega} \to \mathcal{P}(P)$ such that whenever $\langle p_i : i \leq n\rangle$ is a finite descending sequence in $P$, $f(\langle p_i : i \leq n\rangle)$ is a dense subset below $p_n$ and, whenever $\langle p_i : i < \omega\rangle$ is a descending sequence in $P$ such that for each $n$ there exists a $j$ with

$$p_j \in f(\langle p_i : i \leq n\rangle),$$

the sequence has a lower bound in $P$. It is easy to see that strategic $\omega$-closure is equal to the property that for every countable $X \prec H((2^{|P|})^+)$ and every $(X, P)$-generic filter $g$ contained in $X$ there is a condition in $P$ extending $g$.

Let us say that a set $a$ *captures* a pair $E, C$ if $a \setminus C$ is finite and $a \cap E$ is infinite. Given $A \subset \omega_1$, let $\mathbb{C}(A)$ be the partial order which adds a club subset of $A$ by initial segments. We force $\mathcal{C}_s$ by first adding a potential $\mathcal{C}_s$-sequence by initial segments, and then iterating to kill off every counterexample.

We refer the reader to [11] for background on countable support iterations of proper forcing.

**Theorem 1.6.** *Suppose that CH and $2^{\aleph_1} = \aleph_2$ hold. Let $\bar{P} = \langle P_\eta, \underset{\sim}{Q}_\eta : \eta < \omega_2\rangle$ be a countable support iteration such that $P_0$ is the partial order consisting of sequences $\langle c_\alpha : \alpha < \delta \text{ limit}\rangle$, for some countable ordinal $\delta$, such that each $c_\alpha$ is a sequence $\langle a^\alpha_\beta : \beta < \gamma_\alpha\rangle$ (for some countable ordinal $\gamma_\alpha$) of cofinal subsets of $\alpha$ of ordertype $\omega$, deceasing by mod-finite inclusion (and $P_0$ is ordered by extension). Suppose that the remainder of $\bar{P}$ satisfies the following conditions.*

- *For each nonzero $\eta < \omega_2$ there is a $P_\eta$-name $\tau_\eta$ for a subset of $\omega_1$ such that if $(\tau_\eta)_{G_\eta}$ (where $G_\eta$ is the restriction of the generic filter to $P_\eta$) is stationary in the $P_\eta$ extension and there exists a club $C \subset \omega_1$ in this extension such that no $a^\alpha_\beta$ with $\alpha \in \tau_{G_\eta}$ captures the pair $\tau_{G_\eta}, C$, then $\underset{\sim}{Q}_\eta$ is $\mathbb{C}(\omega_1 \setminus (\tau_\eta)_{G_\eta})$ (and otherwise, $\underset{\sim}{Q}_\eta$ is $\mathbb{C}(\omega_1)$).*
- *For every pair $E, C$ of subsets of $\omega_1$ in any $P_\eta$-extension ($\eta < \omega_2$), if $E$ is stationary in this extension and $C$ is club and no $a^\alpha_\beta$ with $\alpha \in E$ captures $E, C$, then there is a $\rho \in [\eta, \omega_2)$ such that if $E$ is stationary in the $P_\rho$ extension, then $\underset{\sim}{Q}_\rho$ is $\mathbb{C}(\omega_1 \setminus E)$.*

*Then $\bar{P}$ is strategically $\omega$-closed, and $\mathcal{C}_s$ holds in the $\bar{P}$-extension. Furthermore, in the $\bar{P}$ extension, $\diamondsuit(S)$ holds for every stationary $S \subset \omega_1$.*

*Proof.* Let $X$ be a countable elementary submodel of $H((2^{|\bar{P}|})^+)$ with $\bar{P} \in X$, let $g$ be an $X$-generic filter contained in $\bar{P} \cap X$. Let $\gamma_{X\cap\omega_1}$ be the ordertype of $X \cap \omega_2$, and for each $\beta < \gamma_{X\cap\omega_1}$, let $\eta_\beta$ be the $\beta$th member of $X \cap \omega_2$. For each $\beta < \gamma_{X\cap\omega_1}$, let $a^{X\cap\omega_1}_\beta$ be a cofinal subset of $X \cap \omega_1$ of ordertype $\omega$ such that, letting $g_\eta$ denote the restriction of $g$ to $P_\eta$,

- for all $\beta' < \beta < \gamma_{X\cap\omega_1}$, $a^{X\cap\omega_1}_\beta \setminus a^{X\cap\omega_1}_{\beta'}$ is finite;
- $a^\alpha_\beta$ is eventually contained in every club subset of $\omega_1$ in $X[g_{\eta_\beta}]$ and intersects infinitely every stationary subset of $\omega_1$ in every $X[g_{\eta_{\beta'}}]$, $\beta' \in [\beta, \gamma_{X\cap\omega_1})$.

It remains to see that we can extend $g$ to a condition whose first coordinate is given by adding $c_{X\cap\omega_1} = \langle a^\alpha_\beta : \beta < \gamma_{X\cap\omega_1}\rangle$ to the union of the first coordinates of the elements of $g$, and whose $\eta$th coordinate, for each nonzero $\eta \in X \cap \omega_2$ is the condition given by the union of $\{X \cap \omega_1\}$ and the set of realizations of the $\eta$th coordinates of the members of $g$. We do this by induction on $\eta$, letting $g'_\eta$ be our extended condition in $P_\eta$.

For each $\eta \in \omega_2 \cap X$, there is a $P_\eta$-name $\sigma \in X$ for a club subset of $\omega_1$ such that if, in the $P_\eta$-extension $(\tau_\eta)_{G_\eta}$ is stationary and there exists a club $C$ such that $\tau_{G_\eta}, C$ is not captured by any $a^\alpha_\beta$ with $\alpha \in (\tau_\eta)_{G_\eta}$, then $\sigma_{G_\eta}$ is such a $C$. However, the realizations of $\tau_\eta$ and $\sigma$ by $g$ are captured by $a^{X\cap\omega_1}_{o.t.(\eta\cap\omega_2)}$, so $g'_\eta$ forces that $\tau_{G_\eta}, \sigma_{G_\eta}$ is captured by $a^{X\cap\omega_1}_{o.t.(\eta\cap\omega_2)}$. It follows that $g'_\eta$ forces that either $\underset{\sim}{Q}_\eta$ is $\mathbb{C}(\omega_1)$, or $X \cap \omega_1$ is not in $\tau_{G_\eta}$. In either case, the union of the members of $g \cap \underset{\sim}{Q}_\eta$ be can extended to a condition in $\underset{\sim}{Q}_\eta$ by adding $\{X \cap \omega_1\}$.

To see that $\diamondsuit(S)$ holds for every stationary $S \subset \omega_1$ in the $\bar{P}$ extension, fix such an $S$ in the $P_\alpha$ extension for some $\alpha < \omega_2$. Since $\bar{P}$ is $(\omega, \infty)$ distributive, there exists in this extension a set $\langle e^\delta_\beta : \delta, \beta < \omega_1\rangle$ such that for every $\delta < \omega_1$ and every $x \subset \delta$ there are uncountably many $\beta$ such that $e^\delta_\beta = x$. Then, letting $T \in \mathcal{P}(\omega_1)^{V[G_\alpha]}$ be the set such that the realization of $\underset{\sim}{Q}_\alpha$ is $\mathbb{C}(T)$, $\underset{\sim}{Q}_\alpha$ adds a $\diamondsuit$ sequence $\langle b_\delta : \delta \in S\rangle$ defined by letting $b_\delta$ be $e^\delta_\beta$, where the $\beta$th element of $T$ above $\beta$ is the first element of the generic club for $\underset{\sim}{Q}_\alpha$ above $\delta$. To see that this is a $\diamondsuit$ sequence, note that since $S$ is stationary in the $\bar{P}$ extension, there are stationarily many elementary submodels $X$ of any sufficiently large $H(\theta)^{V[G]}$ in this extension with $X \cap \omega_1 \in S$. Then $X \cap (G/G_\alpha)$ is a $(X \cap V[G_\alpha], \bar{P}/P_\alpha)$-generic filter which can be extended to a condition in $\bar{P}/P_\alpha$ by adding $X \cap \omega_1$ to each coordinate, and extended again to make any element of $T \setminus ((X \cap \omega_1) + 1)$ the least element of the generic club for $\underset{\sim}{Q}_\alpha$ above $X \cap \omega_1$. That $\langle b_\beta : \beta \in S\rangle$ is a $\diamondsuit$ sequence then follows by genericity. $\square$

Section 2 shows that proper forcing does not always preserve the existence of a winning strategy for *split*.

## 1.2 A strategy for *unsplit*

In this section we show that it is consistent for *unsplit* to have a winning strategy in $\mathcal{SG}$. We do this via the following guessing principle.

**1.7 Definition.** Let $\mathcal{D}_u$ be the statement that there exists a diamond sequence $\langle \sigma_\alpha : \alpha < \omega_1\rangle$ such that for every $E \subset \omega_1$ there is a club $C \subset \omega_1$ such that either

$$\forall \alpha \in C((E \cap \alpha = \sigma_\alpha) \Rightarrow \alpha \in E)$$

or

$$\forall \alpha \in C((E \cap \alpha = \sigma_\alpha) \Rightarrow \alpha \notin E).$$

**Theorem 1.8.** *If $\mathcal{D}_u$ holds then* unsplit *has a winning strategy in $\mathcal{SG}$.*

*Proof.* Let $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$ witness $\mathcal{D}_u$. Play for *unsplit* by accepting $\alpha$ if and only if $\sigma_\alpha = A_\alpha$. At the end of the game, the set of $\alpha$ such that $\sigma_\alpha = A_\alpha$ is stationary, and there is a club $C$ such that either for all $\alpha$ in $C$, if $\sigma_\alpha = A_\alpha$, then $\alpha$ is in $A$, or for all $\alpha$ in $C$, if $\sigma_\alpha = A_\alpha$, then $\alpha$ is in $B$. In either case, *split* has lost. $\square$

Our iteration to force $\mathcal{D}_u$ employs the same strategy as the iteration for $\mathcal{C}_s$. We first force to add a $\diamondsuit$-sequence $\langle \sigma_\alpha : \alpha < \omega_1 \rangle$ by initial segments, and we then iterate to make this sequence witness $\mathcal{D}_u$, iteratively forcing a club through the set of $\alpha < \omega_1$ such that $\sigma_\alpha \neq E \cap \alpha$ or $\alpha \in E$ for each $E \subset \omega_1$ such that the sets $\{\alpha \in E \mid \sigma_\alpha = E \cap \alpha\}$ and $\{\alpha \in \omega_1 \setminus E \mid \sigma_\alpha = E \cap \alpha\}$ are both stationary.

More specifically, we have the following. Given a sequence $\Sigma = \langle \sigma_\alpha : \alpha < \omega_1 \rangle$ such that each $\sigma_\alpha$ is a subset of $\alpha$, and given $E \subset \omega_1$, let $A(\Sigma, E)$ be the set of $\alpha \in E$ such that $\sigma_\alpha = E \cap \alpha$, and let $B(\Sigma, E)$ be the set of $\alpha \in \omega_1 \setminus E$ such that $\sigma_\alpha = E \cap \alpha$.

**Theorem 1.9.** *Suppose that CH + $2^{\aleph_1} = \aleph_2$ holds, and let $\bar{P}$ be a countable support iteration $\langle P_\alpha, \underset{\sim}{Q}_\alpha : \alpha < \omega_2 \rangle$ such that $P_0$ is the partial order consisting of sequences $\langle \sigma_\beta : \beta < \gamma \rangle$, for some countable ordinal $\gamma$, such that each $\sigma_\beta$ is a subset of $\beta$, ordered by extension. Let $\Sigma$ be the sequence added by $P_0$ and suppose that the remainder of $\bar{P}$ satisfies the following conditions.*

- *Each $\underset{\sim}{Q}_\alpha$ is either $\mathbb{C}(\omega_1)$ or $\mathbb{C}(\omega_1 \setminus B(\Sigma, E))$ for some $E \subset \omega_1$ such that $A(\Sigma, E)$ and $B(\Sigma, E)$ are both stationary.*
- *For every $E \subset \omega_1$ in any $P_\alpha$-extension ($\alpha < \omega_2$) there is a $\gamma \in [\alpha, \omega_2)$ such that if $A(\Sigma, E)$ and $B(\Sigma, E)$ are both stationary in the $P_\gamma$ extension, then $\underset{\sim}{Q}_\gamma$ is $\mathbb{C}(\omega_1 \setminus B(\Sigma, E))$.*

*Then $\bar{P}$ is strategically $\omega$-closed, and in the $\bar{P}$-extension, $\mathcal{D}_u$ holds. Furthermore, in the $\bar{P}$ extension, $\diamondsuit(S)$ holds for every stationary $S \subset \omega_1$.*

*Proof.* The iteration $\bar{P}$ is clearly strategically $\omega$-closed, since for any countable $X \prec H((2^{|\bar{P}|})^+)$ and any $(X, \bar{P})$-generic filter $g$ contained in $X$, one can extend $g$ to a condition by making $\sigma_{X \cap \omega_1}$ unequal to the realization by $g$ of any name in $X$ for a subset of $\omega_1$, and adding $X \cap \omega_1$ to all the clubs being added by the $\underset{\sim}{Q}_\alpha$'s, $\alpha \in X \cap \omega_2$. It is clear also that in the $\bar{P}$-extension there is no $E \subset \omega_1$ such that $A(\Sigma, E)$ and $B(\Sigma, E)$ are both stationary.

To see that at least one of $A(\Sigma, E)$ and $B(\Sigma, E)$ is stationary for each $E \subset \omega_1$, we first note the following.

**Claim** 1. *Suppose that $E \subset \omega_1$ is a member of the $P_\alpha$ extension, for some $\alpha < \omega_2$, and $A(\Sigma, E)$ is stationary in this extension. Then $A(\Sigma, E)$ remains stationary in the $\bar{P}$ extension.*

Note that $A(\Sigma, E)$ has countable intersection with $B(\Sigma, F)$, for every $F \subset \omega_1$. Fix $X \prec H(((2^{|\bar{P}|})^+)^V)^{V[G_\alpha]}$ (where $G_\alpha$ is the restriction of the generic filter $G$ to $P_\alpha$) with $X \cap \omega_1 \in A(\Sigma, E)$ and $A(\Sigma, E) \in X$. Then any $(X, \bar{P}/P_\alpha)$-generic filter contained in $X$ can be extended to a condition by adding $X \cap \omega_1$ to the clubs being added at every stage of $\bar{P}$ after the first.

Similar reasoning shows the following two facts, which complete the proof that $\Sigma$ witnesses $\mathcal{D}_u$ in the $\bar{P}$ extension.

**Claim** 2. *Suppose that $E \subset \omega_1$ is a member of the $P_\alpha$ extension, for some $\alpha < \omega_2$, and not a member of the $P_\gamma$ extension, for any $\gamma < \alpha$. Then $A(\Sigma, E) \cup B(\Sigma, E)$ is stationary in the $P_\alpha$ extension.*

To see Claim 2, let $\tau$ be a $P_\alpha$-name for a subset of $\omega_1$ which is forced to be unequal to any such subset in any $P_\gamma$ extension, for any $\gamma < \alpha$. Fix $X \prec H(((2^{|\bar{P}|})^+))^V$ with $\tau \in X$. Let $g$ be an $(X, P_\alpha)$-generic filter, and note that the realization of $\tau \restriction (X \cap \omega_1)$ by $g$ is different from the realizations of $\rho \restriction (X \cap \omega_1)$ by $g$ for any $P_\gamma$-name $\rho \in X$ for a subset of $\omega_1$, for any $\gamma \in X \cap \alpha$. It follows that adding the realization of $\tau \restriction (X \cap \omega_1)$ by $g$ to the union of the first coordinate projection of $g$ gives a condition in $P_0$ forcing that $X \cap \omega_1$ is not in any $\Sigma(B, \rho_{G_\gamma})$, for any for any $P_\gamma$-name $\rho \in X$ for a subset of $\omega_1$, for any $\gamma \in X \cap \alpha$. Therefore, we can add $X \cap \omega_1$ to the clubs being added in every other stage of $\bar{P}$ in $X \cap \alpha$, and get a condition extending every condition in $g$.

**Claim** 3. *Suppose that $E \subset \omega_1$ is a member of the $P_\alpha$ extension, for some $\alpha < \omega_2$, and $A(\Sigma, E)$ is nonstationary in this extension. Then $B(\Sigma, E)$ remains stationary in the $\bar{P}$ extension.*

This is similar to the previous claims, noting that every subsequent stage of $\bar{P}$ forces a club though the complement of a set with countable intersection with $B(\Sigma, E)$.

The proof that $\diamondsuit(S)$ holds for every stationary $S \subset \omega_1$ in the $\bar{P}$ extension is (literally) the same as in the proof of Theorem 1.6. $\square$

Note that that the iterations $\bar{P}$ in Theorems 1.6 and 1.9 are strategically $\omega$-closed.

### 1.3 $\Sigma^2_2$ maximality

The statements that *split* and *unsplit* have winning strategies in $\mathcal{SG}$ are each $\Sigma^2_2$ in a predicate for $NS_{\omega_1}$, and they are obviously not consistent with each other. Woodin (see [6]) has shown that if there is a proper class of measurable Woodin cardinals, then there exists in a forcing extension a transitive class model of ZFC satisfying all $\Sigma^2_2$ sentences $\phi$ such that $\phi$ + CH can be forced over the ground model. The results here show that this result cannot be extended to include a predicate for $NS_{\omega_1}$. This was known already, in that $\diamondsuit^*$ (in the sense of [7]) and "the restriction of $NS_{\omega_1}$ to some stationary set is $\aleph_1$ dense" were both known to be consistent with $\diamondsuit$ (the second of these is due to Woodin, uses large cardinals and is unpublished, though a related proof, also due to Woodin, appears in [3]). Our example is simpler and doesn't use large cardinals; it also gives (we believe, for the first time) a counterexample consisting of two sentences each consistent with "$\diamondsuit(S)$ holds for every stationary set $S \subset \omega_1$."

## 1.4 A determined variation

There are many natural variations of $\mathcal{SG}$. We show that one such variation is determined.

**Theorem 1.10.** *Let $\mathcal{G}$ be the following game of length $\omega_1$. In round $\alpha$, player $I$ puts $\alpha$ into one of two sets $E_0$ and $E_1$, and player $II$ puts $\alpha$ into one of two sets $A_0$ and $A_1$. After all $\omega_1$ rounds have been played, $II$ wins if one of the following pairs of set are both stationary.*

- $E_0 \cap A_0$ *and* $E_0 \cap A_1$
- $E_1 \cap A_0$ *and* $E_1 \cap A_1$

*Then $II$ has a winning strategy in $\mathcal{G}$.*

*Proof.* Let $B_{00}$, $B_{01}$, $B_{10}$ and $B_{11}$ be pairwise disjoint stationary subsets of $\omega_1$. In round $\alpha$, if $\alpha$ is in $B_{ij}$, let $II$ put $\alpha$ in $A_i$ if $I$ put $\alpha$ in $E_0$ and in $A_j$ otherwise. Then after all $\omega_1$ many rounds have been played, suppose that $A_i \cap E_0$ is nonstationary. Then $B_{i0}$ and $B_{i1}$ are both contained in $E_1$ modulo $NS_{\omega_1}$, which means that $E_1 \cap A_0$ and $E_1 \cap A_1$ are both stationary. Similarly, if $A_i \cap E_1$ is nonstationary then $B_{0i}$ and $B_{1i}$ are both contained in $E_0$ modulo $NS_{\omega_1}$, which means that $E_0 \cap A_0$ and $E_0 \cap A_1$ are both stationary. $\square$

# 2 Indeterminacy from forcing axioms

The axiom PFA$^{+2}$ says that whenever $P$ is a proper partial order, $D_\alpha$ ($\alpha < \omega_1$) are dense subsets of $P$ and $\sigma_1$, $\sigma_2$ are $P$-names for stationary subsets of $\omega_1$, there is a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, and such that $\{\alpha < \omega_1 \mid \exists p \in G\ p \Vdash \check{\alpha} \in \sigma_i\}$ is stationary for each $i \in \{1, 2\}$. Theorems 1.6 and 1.9 together show that PFA$^{+2}$ implies the indeterminacy of $\mathcal{SG}$. Furthermore, a straightforward argument shows that the following statement implies the nonexistence of a winning strategy for *unsplit* in $\mathcal{SG}$, where $Add(1, \omega_1)$ is the partial order that adds a subset of $\omega_1$ by initial segments : for any pair $\sigma_1$,$\sigma_2$ of $Add(1, \omega_1)$-names for stationary subsets of $\omega_1$, there is a filter $G \subset Add(1, \omega_1)$ realizing both $\sigma_1$ and $\sigma_2$ as stationary sets. This statement is trivially subsumed by PFA$^{+2}$, but also holds in the collapse of a sufficiently large cardinal to be $\omega_2$, and thus is consistent with CH.

The axiom Martin's Maximum [4] says that whenever $P$ is a partial order such that forcing with $P$ preserves stationary subsets of $\omega_1$ and $D_\alpha$ ($\alpha < \omega_1$) are dense subsets of $P$, there is a filter $G \subset P$ such that $G \cap D_\alpha \neq \emptyset$ for each $\alpha < \omega_1$.

**Theorem 2.1.** *Martin's Maximum implies that $\mathcal{SG}$ is undetermined.*

*Proof.* Fix a strategy $\Sigma$ for *unsplit* in $\mathcal{SG}$, and let $E$, $A$, and $B$ be the result of a generic run of $\mathcal{SG}$ where *unsplit* plays by $\Sigma$ (the partial order consists of countable partial plays where *unsplit* plays by $\Sigma$, ordered by extension). If the complement of $E$ has stationary intersection with every stationary subset of $\omega_1$ in the ground model, one can force to kill the stationarity of $E$ in such a way that the induced two step forcing preserves stationary subsets of $\omega_1$ and produces a run of $\mathcal{SG}$ where *unsplit* plays by $\Sigma$ and loses. If the complement of $E$ does not have stationary intersection with some stationary $F \subset \omega_1$ in the ground model, then there is a partial run of the game $p$ and a name $\tau$ for a club such that $p$ forces that $E$ will contain $F \cap \tau_G$. Then there exists in the ground model a run of $\mathcal{SG}$ extending $p$ in which *unsplit* plays by $\Sigma$ and loses: *split* picks a pair of disjoint stationary subsets $F_0$, $F_1$ of $F$, and plays so that

- for every $\alpha < \omega_1$, some initial segment of the play forces some ordinal greater than $\alpha$ to be in $\tau$,
- whenever *unsplit* accepts $\alpha \in F$, *split* puts $\alpha$ in $A$ if $\alpha \in F_0$ and puts $\alpha \in B$ if $\alpha \in F_1$.

Now fix a strategy $\Sigma$ for *split* in $\mathcal{SG}$, and generically add a regressive function $f$ on $\omega_1$ by initial segments. Let $E^\alpha = f^{-1}(\alpha)$ and let $A^\alpha, B^\alpha$ be the responses given by $\Sigma$ to a play of $E^\alpha$ by *unsplit*. Note that each $E^\alpha$ will be stationary.

Suppose that there exist an $\alpha < \omega_1$ and stationary sets $S$, $T$ in the ground model such that $(S \cap E^\alpha) \setminus A^\alpha$ and $(T \cap E^\alpha) \setminus B^\alpha$ are both nonstationary. Then there is a condition $p$ in our forcing (i.e., a regressive function on some countable ordinal) such that $p$ forces that $(S \cap E^\alpha) \subset A^\alpha$ and $(T \cap E^\alpha) \subset B^\alpha$, modulo nonstationarity (and so in particular $S$ and $T$ have nonstationary intersection). Let $\tau$ be a name for a club disjoint from $(S \cap E^\alpha) \setminus A^\alpha$ and $(T \cap E^\alpha) \setminus B^\alpha$. Extend $p$ to a filter $f$ (identified with the corresponding function) realizing $\tau$ as a club subset of $\omega_1$, at successor stages extending to add a new element to the realization of $\tau$, and at limit stages (when for some $\beta < \omega$, $f \restriction \beta$ has been decided and $f(\beta)$ has not, and $\beta$ is forced by $f \restriction \beta$ to be a limit member of the realization of $\tau$) extending so that $f(\beta) = \alpha$ if and only if $\beta \in S$. Then the run of $\mathcal{SG}$ corresponding to $f^{-1}(\alpha)$ is winning for *unsplit*, since the corresponding set $B^\alpha$ is nonstationary.

If there exist no such $\alpha$, $S$, $T$, there is a function $h$ on $\omega_1$ such that each $h(\alpha) \in \{A^\alpha, B^\alpha\}$ and the forcing to shoot a club through the set of $\beta$ such that $f(\beta) = \alpha \Rightarrow \beta \in h(\alpha)$ preserves stationary subsets of the ground model. Then Martin's Maximum applied to the corresponding two step forcing produces a run of $\mathcal{SG}$ (the run for any $f^{-1}(\alpha)$ which is stationary) where *split* plays by $\Sigma$ and loses. $\square$

Theorem 2.1 leads to the following question.

**2.2 Question.** Does the Proper Forcing Axiom imply that $\mathcal{SG}$ is not determined?

The following question is also interesting. The consistency of the $\aleph_1$-density of $NS_{\omega_1}$ (relative to the consistency of $\mathrm{AD}^{L(\mathbb{R})}$) is shown in [13].

**2.3 Question.** Does the $\aleph_1$-density of $NS_{\omega_1}$ decide the determinacy of $\mathcal{SG}$?

## 3 MLO games

The second-order Monadic Logic of Order (MLO) is an extension of first-order logic with logical constants $=$, $\in$ and $\subset$ and a binary symbol $<$ as the only non-logical constant, allowing quantification over subsets of the domain. Every ordinal is a model for MLO, interpreting $<$ as $\in$.

Given an ordinal $\alpha$, an MLO game of length $\alpha$ is determined by an MLO formula $\phi$ with two free variables for subsets of the domain. In such a game, two players each build a subset of $\alpha$, and the winner is determined by whether these two sets satisfy the formula in $\alpha$.

Büchi and Landweber [1] proved the determinacy of all MLO games of length $\omega$. Recently, Shomrat [12] extended this result to games of length less than $\omega^\omega$, and Rabinovich [10] extended it further to all MLO games of countable length. The stationary set splitting game is an example of an MLO game of length $\omega_1$ whose determinacy is independent of ZFC.

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