Regular embeddings of the stationary tower and Woodin's Σ^2_2 maximality theorem

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Abstract

We present Woodin's proof that if there exists a measurable Woodin cardinal δ, then there is a forcing extension satisfying all Σ^2_2 sentences ϕ such that $CH+\phi$ holds in a forcing extension of V by a partial order in V_δ. We also use some of the techniques from this proof to show that if there exists a stationary limit of stationary limits of Woodin cardinals, then in a homogeneous forcing extension there is an elementary embedding $j\colon V \to M$ with critical point ω_1^V such that M is countably closed in the forcing extension.

1 Introduction

Woodin's Σ^2_1 absoluteness theorem (see [6]) says that if δ is a measurable Woodin cardinal and ϕ is a Σ^2_1 sentence which can be forced by a partial order in V_δ, then ϕ holds in every forcing extension by a partial order in V_δ which satisfies the Continuum Hypothesis. A longstanding open question (due to Steel) is whether this result extends to Σ^2_2 sentences and Jensen's principle $\diamondsuit$, that is, is there a large cardinal concept such that whenever δ is such a cardinal and ϕ is a Σ^2_2 sentence such that $\phi + CH$ can be forced by a partial order in V_δ, then ϕ holds in every forcing extension by a partial order in V_δ which satisfies $\diamondsuit$? This paper presents a theorem of Woodin in this area, saying that if δ is a measurable Woodin cardinal, then there is a forcing extension satisfying all Σ^2_2 sentences ϕ such that $CH + \phi$ holds in a forcing extension of V by a partial order in V_δ. We present this result in a slightly extended form, adding predicates for universally Baire sets of reals.

Before presenting Woodin's proof, we use some of the techniques from the proof to show that if there exists a stationary limit of stationary limits of Woodin

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cardinals, then there is a homogeneous partial order which forces that there is an elementary embedding $j\colon V \to M$ with critical point ω_1^V such that M is countably closed in the forcing extension. The existence of such a partial order has applications in the study of Woodin's *core model induction* (see [8]). For instance, Steel has shown that CH plus the existence of such a partial order implies that the Axiom of Determinacy holds in $L(\mathbb{R})$ and stronger models such as $L(\mathbb{R}^{\#})$, $L(\mathbb{R}^{\#\#})$, etc. The previous consistency strength upper bound for the existence of such a partial order was a superstrong cardinal (see [4] for definitions of the large cardinals used in this paper, and [6] for background on the stationary tower). This work came after we learned Woodin's proof, but since it is simpler we present it first.

1.1 Terminology and background

We say that two partial orders are *forcing-equivalent* if the regular open algebras they generate are isomorphic, and that a partial order P is *homogeneous* if for every pair of conditions p, q in P there are conditions $p' \leq p$ and $q' \leq q$ such that the restrictions of P below p' and q' are forcing-equivalent. If P is a homogeneous partial order, then the theory (with parameters from the ground model) of every P-extension is the same, and thus computable in the ground model. We make key use of a standard forcing fact due to McAloon (Lemma 26.7 of [3] and Theorem A.0.7 of [6]), where for any cardinal γ and any set X, $Coll(\gamma, X)$ is the partial order consisting of partial functions from γ to X of cardinality less than γ, ordered by inclusion.

Theorem 1.1. *Any separative partial order P such that forcing with P makes P countable is forcing-equivalent to $Coll(\omega, P)$.*

A *regular embedding* of a Boolean algebra A into a Boolean algebra B is a map $\pi\colon A \to B$ which preserves order and which maps maximal antichains of A to maximal antichains of B. Given partial order P and Q, we say that P *regularly embeds* into Q if the regular open algebra of P regularly embeds into the regular open algebra of Q. We make use of the following classical fact.

Theorem 1.2. *If A and B are complete Boolean algebras and there is a B-name τ for a V-generic filter $G \subset A$ such that $[\![\check{a} \in \tau]\!]_B$ is nonzero for every element of A, then A regularly embeds into B.*

Proof. For each $a \in A$, let $\pi(a) = [\![\check{a} \in \tau]\!]_B$. □

The following facts appear in the appendix of [6].

Theorem 1.3. *If P regularly embeds into Q, then there is a P-name τ for a partial order such that Q and $P * \tau$ are forcing-isomorphic.*

Theorem 1.4. *If M is a model of ZFC, δ is a limit ordinal of M and x, y are sets such that $\{x, y\}$ exists in a generic extension of M by a partial order in V_δ^M, then x exists in a generic extension of $M[y]$ by a partial order in $V_\delta^{M[y]}$.*

We refer the reader to [6] for the definition of the stationary tower $\mathbb{Q}_{<\delta}$, as well as its basic properties. We will use the following standard facts. For the first of these, see Fact 2.7.3 of [6] and the paragraph which follows it. For the second, see Theorem 2.7.7 of [6]. For the third, see Lemma 2.7.14 of [6] and the paragraph which precedes it.

Theorem 1.5. *If δ is a strongly inaccessible cardinal, then forcing with $\mathbb{Q}_{<\delta}$ makes every ordinal less than δ countable.*

Theorem 1.6. *If δ is a Woodin cardinal, $G \subset \mathbb{Q}_{<\delta}$ is a V-generic filter and $j \colon V \to M$ is the associated elementary embedding, then M is closed under ω-sequences in $V[G]$.*

Theorem 1.7. *If γ is a Woodin cardinal then there is a stationary set a_γ consisting of countable subsets of $V_{\gamma+1}$ such that for every strongly inaccessible cardinal $\eta > \gamma$, the inclusion map regularly embeds $\mathbb{Q}_{<\gamma}$ into the restriction of $\mathbb{Q}_{<\eta}$ to conditions $b \leq a_\gamma$.*

We will use the notation a_γ for the condition referred to in the statement of Theorem 1.7. Theorem 1.7 has a converse: for η and γ as in the statement of the theorem, a_γ is in the generic filter for $\mathbb{Q}_{<\eta}$ if and only if the restriction of the generic filter to $\mathbb{Q}_{<\gamma}$ is generic (see Lemma 2.7.16 of [6]).

2 Slow clubs

Suppose that M is a model of ZF, and let δ be an ordinal in M. An *M-slow club* through δ is a club $D \subset \delta$ with the property that for each limit element β of D, D intersects every club subset of β in M. When β has cofinality ω in the model containing D, the intersection requirement in the notion of slow club is nontrivial. Given a set (or class) of ordinals S, we say that a limit ordinal γ is *1-S-Mahlo* if $S \cap \gamma$ is a stationary subset of γ, and, for any positive $n \in \omega$, γ is *$(n+1)$-S-Mahlo* if the set of n-S-Mahlo ordinals in S below γ is stationary. If D is an M-slow club contained in a set S in M, then every limit point of D is 1-S-Mahlo in M. For any stationary set S consisting of limit ordinals, the set of $\gamma \in S$ which are not 1-S-Mahlo is also stationary, since for any club $C \subset \sup(S)$ consisting of limit ordinals, the first limit point of C in S is such a γ. This puts some limitations on methods for adding slow clubs.

2.1 Definition. Suppose that δ is a limit ordinal and S is a subset of δ. We let $SC(\delta, S)$ be the partial order consisting of triples (c, e, f) such that

- c is a finite subset of S;
- e is a finite set of closed, bounded intervals of δ disjoint from c;
- f is a regressive function whose domain is the set of $\alpha \in c$ which are not 1-S-Mahlo;
- $(f(\alpha), \alpha) \cap c = \emptyset$ for each $\alpha \in dom(f)$.

Given (c, e, f), (b, d, g) in $SC(\delta, S)$, $(c, e, f) \leq (b, d, g)$ if $b \subset c$, $d \subset e$ and $g \subset f$.

The partial order $SC(\delta, S)$ has cardinality δ. Fact 2.2 below shows that if S is cofinal in δ and $G \subset SC(\delta, S)$ is a V-generic filter, then

$$C_G = \bigcup \{c \mid (c, e, f) \in G\}$$

is an unbounded subset of δ (we call C_G the *generic club added by* $SC(\delta, S)$). Fact 2.3 shows that C_G is closed. Together they show that C_G is a V-slow club subset of δ when S is cofinal in δ; moreover, they show that for each limit element β of C_G, $C_G \cap \beta$ intersects every cofinal subset of $\beta \cap S$ in the ground model. By Fact 2.2 and the definition of $SC(\delta, S)$, for each $\gamma \in C_G$, γ is a limit point of C_G if and only if γ is 1-S-Mahlo in V.

2.2 Fact. Let (c, e, f) be a condition in $SC(\delta, S)$ and let γ be any element of

$$S \setminus (\bigcup e \cup \bigcup \{(f(\alpha), \alpha) : \alpha \in dom(f)\}).$$

If γ is 1-S-Mahlo, then $(c \cup \{\gamma\}, e, f) \in SC(\delta, S)$ and $(c \cup \{\gamma\}, e, f) \leq (c, e, f)$. If γ is not 1-S-Mahlo, then $(c \cup \{\gamma\}, e, f \cup \{(\gamma, \max(c \cap \gamma))\}) \in SC(\delta, S)$ and

$$(c \cup \{\gamma\}, e, f \cup \{(\gamma, \max(c \cap \gamma))\}) \leq (c, e, f).$$

2.3 Fact. If (c, e, f) is a condition in $SC(\delta, S)$ and $\gamma \in \delta \setminus c$ is a limit ordinal, then

$$(c, e \cup \{[\max(c \cap \gamma) + 1, \gamma]\}, f) \leq (c, e, f).$$

Fact 2.4 below shows that the forcing $SC(\delta, S)$ factors at each 1-S-Mahlo ordinal in S below δ. We will use this fact to demonstrate the homogeneity of various forcings considered in this paper. It also shows that if δ is a regular cardinal and 2-S-Mahlo, then $SC(\delta, S)$ preserves the regularity of δ, since, in this case, for every dense $D \subset SC(\delta, S)$ there will be club many $\gamma < \delta$ such that $D \cap SC(\gamma, S \cap \gamma)$ is dense in $SC(\gamma, S \cap \gamma)$.

2.4 Fact. For any condition $(c, e, f) \in SC(\delta, S)$, and any 1-$S$-Mahlo $\alpha \in c$, the partial order $SC(\delta, S)$ below (c, e, f) is isomorphic to the partial order

$$SC(\alpha, S \cap \alpha) \times SC(\delta, S \setminus (\alpha + 1))$$

below the condition

$$((c \cap \alpha, \{I \in e \mid I \subset \alpha\}, f \cap \alpha^{\alpha}), (c \setminus (\alpha + 1), \{I \in e \mid I \cap \alpha = \emptyset\}, f \setminus \alpha^{\alpha})).$$

Lemma 2.5 below shows that when δ is a regular cardinal and 2-S-Mahlo, every set of ordinals of cardinality less than δ in the $SC(\delta, S)$-extension is added by an initial segment of the partial order. It follows that forcing with $SC(\delta, S)$ makes CH hold when δ is strongly inaccessible and 2-S-Mahlo, since Lemma 2.6 implies that δ is the ω_1 of such an extension.

Lemma 2.5. *Suppose that δ is a regular cardinal, $S \subset \delta$ and δ is 2-S-Mahlo. Let $G \subset SC(\delta, S)$ be V-generic. Then for every element x of $[Ord]^{<\delta}$ in $V[G]$, there exists a limit member γ of C_G such that $G \cap SC(\gamma, S \cap \gamma)$ is V-generic for $SC(\gamma, S \cap \gamma)$, and $x \in V[G \cap SC(\gamma, S \cap \gamma)]$.*

Proof. Fix $\xi < \delta$ and let τ_α ($\alpha < \xi$) be $SC(\delta, S)$-names for ordinals. For each $\alpha < \xi$, let T_α be the set of pairs (p, β) such that $p \in SC(\delta, S)$ and $p \Vdash \tau_\alpha = \check{\beta}$. Let $q = (c, e, f)$ be a condition in $SC(\delta, S)$. Let θ be a regular cardinal greater than 2^δ and let Z be an elementary submodel of $H(\theta)$ such that

$$\{\delta, S, q, \langle T_\alpha : \alpha < \xi \rangle\} \in Z,$$

$Z \cap \delta \in S$ and $Z \cap \delta$ is 1-S-Mahlo. Let $\gamma = Z \cap \delta$. Then $(c \cup \{\gamma\}, e, f) \leq (c, e, f)$, and, by Lemma 2.4, $(c \cup \{\gamma\}, e, f)$ forces that the restriction of the generic filter to $SC(\gamma, S \cap \gamma)$ will be generic. Furthermore, for each $\alpha < \xi$,

$$\{p \in SC(\gamma, S \cap \gamma) \mid \exists \beta \, (\check{\beta}, p) \in T_\alpha\}$$

is predense in $SC(\gamma, S \cap \gamma)$ below (c, e, f). The lemma then follows by Fact 2.4. □

It follows from Lemma 2.5 that if δ is a regular cardinal and 2-S-Mahlo, then δ has uncountable cofinality in the $SC(\delta, S)$ extension. The following lemma is a sort of converse. Applying Theorem 1.1, it also shows that in many cases $SC(\gamma, S)$ is forcing-equivalent to $Coll(\omega, \gamma)$. It follows that $SC(\delta, S)$ makes δ countable if S consists of regular cardinals and δ is a limit of 1-S-Mahlo ordinals, but not 2-S-Mahlo.

Lemma 2.6. *Let γ be an ordinal, let S be a cofinal subset of γ, and suppose that γ is not a limit of 1-S-Mahlo members of S. Then forcing with $SC(\gamma, S)$ makes $cof(\gamma)^V$ countable.*

Proof. Let β be the supremum of the 1-S-Mahlo members of S below δ (let $\beta = 0$ if this set is empty), and let $\{T_\alpha : \alpha < cof(\gamma)\}$ be a partition of S into cofinal sets. The generic club given by $SC(\gamma, S)$ will have ordertype ω in the interval (β, γ), and will intersect each T_α, inducing a surjection from ω onto $cof(\gamma)^V$. □

The following lemma gives a homogeneity property of $SC(\delta, S)$ for suitable δ and S.

Lemma 2.7. *Suppose that δ is a cardinal, and that S is a set of regular cardinals below δ such that δ is a limit of 1-S-Mahlo members of S. Let p and q be conditions in $SC(\delta, S)$. Then there exist conditions $p' \leq p$ and $q' \leq q$ such that the restrictions of $SC(\delta, S)$ below p' and q' are forcing-equivalent.*

Proof. Let $p = (b, d, g)$ and $q = (c, e, f)$. Let $\gamma \in S$ be 1-S-Mahlo but not a limit of 1-S-Mahlo ordinals, such that γ is larger than every member of $b \cup c \cup \bigcup d \cup \bigcup e$.

Let $p' = (b \cup \{\gamma\}, d, g)$ and let $q' = (c \cup \{\gamma\}, e, f)$. Then $SC(\delta, S)$ below the condition p' is isomorphic to

$$SC(\gamma, S \cap \gamma) \times SC(\delta, S \setminus (\gamma + 1))$$

below the condition

$$((b, d, g), (\emptyset, \emptyset, \emptyset))$$

and $SC(\delta, S)$ below the condition q' is isomorphic to

$$SC(\gamma, S \cap \gamma) \times SC(\delta, S \setminus (\gamma + 1))$$

below the condition

$$((c, e, f), (\emptyset, \emptyset, \emptyset)).$$

By Lemma 2.6, $SC(\gamma, S \cap \gamma)$ below (b, d, g) and $SC(\gamma, S \cap \gamma)$ below (c, e, f) are both forcing-equivalent to $Coll(\omega, \gamma)$. □

3 Slow clubs and the stationary tower

Given $n \in \omega$ and a cardinal δ, we say that δ is n-Mahlo-Woodin if it is n-W-Mahlo, where W denotes the class of Woodin cardinals. Recall that a stationary limit of regular cardinals is regular, so a stationary limit of Woodin cardinals is Woodin. The hypotheses of Theorem 3.1 below imply that ω_1^V is a 2-Mahlo-Woodin cardinal in M.

Our main application of slow clubs is the construction of $\mathbb{Q}^M_{<\delta}$-generic filters for suitable inner models M.

Theorem 3.1. *Suppose that M is a model of ZFC and $D \subset \omega_1^V$ is an M-slow club contained in the Woodin cardinals of M. Then there exists an M-generic filter for $\mathbb{Q}^M_{<\omega_1^V}$ containing any given condition.*

Proof of Theorem 3.1. Let p be a condition in $\mathbb{Q}^M_{<\omega_1^V}$. Removing an initial segment of D if necessary, we may assume that $p \in \mathbb{Q}^M_{<\gamma_0}$, where γ_0 is the least element of D. For each $\gamma \in D$, let $\mathcal{G}_\gamma$ be the set of g such that

- g is an M-generic filter for $\mathbb{Q}^M_{<\gamma}$ containing p;
- for all $\eta \in D \cap \gamma$, $g \cap V^M_\eta$ is M-generic for $\mathbb{Q}^M_{<\eta}$.

Since ω_1^V is a strongly inaccessible cardinal in M, $\mathcal{G}_{\gamma_0}$ is nonempty.

Let T be the tree on $\bigcup_{\gamma \in D} \mathcal{G}_\gamma$ ordered by: $g \geq h$ whenever $g \in \mathcal{G}_\gamma$ and $h \in \mathcal{G}_\eta$, for some γ, η in D, and $g \cap V^M_\eta = h$. Theorems 1.3 and 1.7 (and the facts that ω_1^V is strongly inaccessible in M and D consists of Woodin cardinals of M) implies that every member of $\mathcal{G}$ has proper extensions in T. The theorem follows from the fact that T is countably closed, and the fact that the union of each uncountable branch through T is an M-generic filter for $\mathbb{Q}^M_{<\omega_1^V}$.

To see that T is countably closed, note that if γ is a limit point of D, then each predense subset of $\mathbb{Q}^M_{<\gamma}$ in M has predense intersection with $\mathbb{Q}^M_{<\eta}$ for club many $\eta < \gamma$, relative to the set of strongly inaccessible cardinals below γ, and thus with $\mathbb{Q}^M_{<\eta}$ for some $\eta \in \gamma \cap D$. It follows that if g is a subset of $\mathbb{Q}^M_{<\gamma}$ such that $g \cap \mathbb{Q}^M_{<\eta}$ is an M-generic filter for all $\eta \in \gamma \cap D$, then g is also an M-generic filter. Similarly, each predense subset of $\mathbb{Q}^M_{<\omega_1^V}$ in M has predense intersection with $\mathbb{Q}^M_{<\eta}$ for (relative) club many $\eta < \omega_1^V$, and thus with $\mathbb{Q}^M_{<\eta}$ for some $\eta \in D$. It follows that if G is a subset of $\mathbb{Q}^M_{<\omega_1^V}$ such that $G \cap \mathbb{Q}^M_{<\eta}$ is an M-generic filter for all $\eta \in D$, then G is also an M-generic filter. □

It follows from Theorem 3.1 that $\mathbb{Q}_{<\delta}$ regularly embeds into any forcing which collapses δ to be ω_1 and adds a V-slow club through the Woodin cardinals below δ. The results of the previous section show that that $SC(\delta, W)$ is such a forcing when W is the set of Woodin cardinals below a 2-Mahlo-Woodin cardinal δ.

The proof of Theorem 3.2 below is a modification of the proof of Theorem 3.1. The new element of the proof is the use of Theorem 1.4 to make sure that the reals of $V[G]$ are all in $V[H]$. As in the proof of Theorem 3.2, the construction follows the generic club G exactly in order to use the fact that G is V-slow. So for each real of $V[G]$ the construction has to wait until it reaches a sufficiently large element of W. We work with a tail of the Woodin cardinals below δ in order to apply the theorem in the proof of Theorem 3.4. We use the notion of *nice* names from [5] (see page 208), simply to restrict to a sufficiently large set-sized collection of names.

Theorem 3.2. *Let δ be a 2-Mahlo-Woodin cardinal, let χ be an element of δ, let W denote the Woodin cardinals of V in the interval (χ, δ), and let $G \subset SC(\delta, W)$ be a V-generic filter. Then there exists in $V[G]$ a V-generic filter $H \subset \mathbb{Q}^V_{<\delta}$, containing any given condition, such that $V[H]$ contains the reals of $V[G]$.*

Proof. Let p be a condition in $\mathbb{Q}^V_{<\delta}$ and let γ_0 be the least $\gamma \in W$ with $p \in \mathbb{Q}_{<\gamma}$. Let W^0_1 be the set of 1-Mahlo-Woodin cardinals in (γ_0, δ) which are not limits of 1-Mahlo-Woodin cardinals. Let $\langle \tau_\xi : \xi < \delta \rangle$ be a listing in V of all nice $SC(\gamma, W \cap \gamma)$-names for reals, for all 1-Mahlo-Woodin $\gamma < \delta$.

For each $\alpha < \beta$ in W^0_1, let $N_{\alpha,\beta}$ be the set of nice $SC(\beta, W \cap \beta)$-names σ for which it is forced that if α and β are in C_G, then the realization of σ is a V-generic filter $h \subset \mathbb{Q}_{<\alpha}$ such that

- $h \cap \mathbb{Q}_{<\gamma_0}$ is a V-generic filter containing p.
- $h \cap \mathbb{Q}_{<\gamma}$ is V-generic for all $\gamma \in (C_G \cap \alpha) \setminus \gamma_0$.

Fix (suppressed) wellorders of the sets $N_{\alpha,\beta}$.

Let C^* be the set of limit points of C_G. Working in $V[G]$, recursively define a sequence $\langle h_\alpha : \alpha \in C^* \setminus (\gamma_0 + 1) \rangle$ such that

- $h_{\min(C^* \setminus (\gamma_0+1))} = \emptyset$;

- if γ is a limit element of $C^* \setminus (\gamma_0+1)$, α is the least element of C^* greater than γ and β is the least element of C^* greater than α, then h_α is the realization by G of the least element of $N_{\alpha,\beta}$ whose realization h extends h_γ and has the realization of τ_ξ in $V[h]$, where $\xi < \delta$ is least such that
  - τ_ξ is an $SC(\eta, W \cap \eta)$-name for a real, for some $\eta \in C^* \cap (\gamma+1)$, and
  - the realization of τ_ξ by h_γ is not in $V[h_\gamma]$,

  if such an ξ exists, otherwise h_α is the realization of the least element of $N_{\alpha,\beta}$ whose realization h extends h_γ;
- if γ is not a limit element of $C^* \setminus (\gamma_0 + 1)$, α is the least element of C^* greater than γ and β is the least element of C^* greater than α, then h_α is the realization of the least element of $N_{\alpha,\beta}$ which extends h_γ;
- if α is a limit element of $C^* \setminus (\gamma_0 + 1)$, then $h_\alpha = \bigcup_{\beta\in\alpha\cap C^*} h_\beta$.

It follows from this construction that whenever γ is a limit element of the set $C^* \setminus (\gamma_0+1)$, $h_\gamma \in V[G \cap SC(\gamma, W\cap\gamma)]$. Let $H = \bigcup\{h_\alpha : \alpha \in C^* \setminus (\gamma_0+1)\}$. Let E be the set of $\xi < \delta$ such that τ_ξ is an $SC(\eta, W \cap \eta)$-name, for some $\xi \in C^*$. By Lemma 2.5, every real in $V[G]$ is the realization of τ_ξ for some $\xi \in E$. If ξ were the least $\zeta \in E$ such that the realization of τ_ζ were not in $V[H]$, then, since ξ is countable in $V[H]$ and δ is uncountable, there would be some limit element γ of $C^* \setminus (\gamma_0 + 1)$ such that ξ is the least $\zeta < \delta$ such that

- τ_ζ is an $SC(\eta, W \cap \eta)$-name for a real, for some $\eta \in C^* \cap (\gamma + 1)$, and
- the realization of τ_ζ by h_γ is not in $V[h_\gamma]$.

Then the realization of τ_ξ is in $V[h_\alpha]$ by the construction above (and Theorem 1.4), where α is the least element of C^* above γ. □

Theorem 3.3 below is the main original result of this paper.

Theorem 3.3. *Suppose that δ is a 2-Mahlo-Woodin cardinal, and let W denote the set of Woodin cardinals below δ. Then the partial order $SC(\delta, W)$ is homogeneous, and in the extension by this partial order there is an elementary embedding from V into a model M which is closed under ω-sequences in the forcing extension.*

Proof. The partial order $SC(\delta, W)$ is homogeneous by Lemma 2.7. By Lemma 2.5, every countable set of ordinals in any forcing extension of V by $SC(\delta, W)$ is in a model of the form $V[x]$ for some real in the extension. By Lemma 3.2, in any $SC(\delta, W)$ extension there is a V-generic filter $H \subset \mathbb{Q}^V_{<\delta}$ such that $V[H]$ contains all the reals of the $SC(\delta, W)$-extension, and therefore all initial segments of the $SC(\delta, W)$-generic filter. By Theorem 1.6, the image model M of the embedding induced by H is ω-closed in $V[H]$, which is ω-closed in the $SC(\delta, W)$-extension, which means that M is ω-closed in this extension. □

Theorem 3.2 has the following additional corollary.

Theorem 3.4. *Let δ be a 2-Mahlo-Woodin cardinal, let W denote the Woodin cardinals of V below δ, and let $G \subset SC(\delta, W)$ be a V-generic filter. Then every Σ^2_1 sentence which can be forced by a partial order in V_δ holds in $V[G]$.*

Proof. Let P be a partial order in V_δ forcing a Σ^2_1 sentence $\exists X \subset \mathbb{R}\, \phi(X)$, where all quantifiers in ϕ range over the reals. Let $\kappa \in C_G$ be 1-Mahlo-Woodin and not a limit of 1-Mahlo-Woodin cardinals. Then by Lemma 2.6 and Fact 2.4 there is a V-generic filter $h \subset P$ in $V[G \cap SC(\kappa, W \cap \kappa)]$. Let X be a set of reals satisfying ϕ in $V[h]$. By [2], the set of Woodin cardinals in the interval (κ, δ) is the same in $V[h]$, V and $V[G \cap SC(\kappa, W \cap \kappa)]$. By Fact 2.4, $C_G \setminus (\kappa + 1)$ is a $V[h]$-generic club for $SC(\delta, W \setminus (\kappa + 1))$. By Theorem 3.2 there exists in $V[G]$ a $V[h]$-generic filter $H \subset \mathbb{Q}^{V[h]}_{<\delta}$ such that $V[h][H]$ contains the reals of $V[G]$. Letting $j : V[h] \to M$ be the embedding induced by H, it follows from Theorem 1.6 that $j(X)$ satisfies ϕ in $V[G]$. □

4 Σ^2_2 maximality

In this final section we give a proof of Woodin's Σ^2_2 maximality theorem. The theorem is presented in various forms in Corollaries 4.11, 4.12 and 4.14. Most of the work goes into the proof of Theorem 4.10, which is a variant of the proof of Woodin's Σ^2_1 absoluteness theorem. The proof of Theorem 4.10 in turn requires setting up some machinery. We start by discussing symmetric extensions.

Given a strong limit cardinal δ of a ZFC model M, we take a δ-*symmetric extension* of M to be the least model $M(\mathbb{R}^*)$ of ZF containing M and a set of reals $\mathbb{R}^*$ with the properties that

- $M(\mathbb{R}^*) \cap \mathbb{R} = \mathbb{R}^*$;
- every member of $\mathbb{R}^*$ is generic over M by a forcing in V^M_δ;
- the supremum of $\{\omega_1^{L[x]} : x \in \mathbb{R}^*\}$ is δ.

We refer the reader to [3, 6] for more general definitions of *symmetric extension*. We typically denote a symmetric extension of a model M by $M(\mathbb{R}^*)$, where $\mathbb{R}^*$ is understood to be the reals of the extension. We note the following facts about δ-symmetric extensions, for a strong limit cardinal δ: (1) any two δ-symmetric extensions of M are elementarily equivalent (even with parameters from M); (2) if $M(\mathbb{R}^*)$ is a δ-symmetric extension of M and P is a partial order in V^M_δ then $M(\mathbb{R}^*)$ is a δ-symmetric extension of an extension of M by P.

The following is Theorem 3.1.6 in [6].

Theorem 4.1. *If δ is a Woodin limit of Woodin cardinals and $G \subset \mathbb{Q}_{<\delta}$ is a V-generic filter, then $V(\mathbb{R}^{V[G]})$ is a δ-symmetric extension of V.*

Whenever δ is a strongly inaccessible cardinal and G is V-generic for the partial order $Coll(\omega, {<}\delta)$, $V(\mathbb{R}^{V[G]})$ is a δ-symmetric extension of V. Fact 2.4 and Lemmas 2.5 and 2.6 show that the same is true for $SC(\delta, S)$, when δ is a strongly inaccessible and 2-S-Mahlo, and S is a set of regular cardinals.

Given a model M of ZF, an ordinal $\delta \in M$ and $S \subset \delta$ in M, let $SL(M,\delta,S)$ be the partial order consisting of all M-generic filters for partial orders of the form $SC(\gamma, S\cap\gamma)^M$, where $\gamma \in S$ is 1-S-Mahlo in M, ordered by end-extension. When $g \in SL(M,\delta,S)$ is an M-generic filter for $SC(\gamma, S\cap\gamma)^M$, we say that the *length* of g is γ. Since filters for $SC(\delta,S)$ are uniquely determined by their corresponding club sets, we sometimes identify a condition g in $SL(M,\delta,S)$ with the set $C_g \cup \{\sup(C_g)\}$; so each condition in $SL(M,\delta,S)$ can be identified with a closed, bounded subset of S.

The partial order $SL(M,\delta,S)$ is not ω-closed. However, it is a tree ordering, so if the set of 1-S-Mahlo $\gamma \in S$ is cofinal in δ and δ is the ω_1 of some $SL(M,\delta,S)$-extension, then there are no new countable sequences of ordinals in this extension.

We let $Add(1,\delta)$ denote the forcing which adds a subset of δ by initial segments. The following lemma follows from Theorem 1.1, Fact 2.4, Lemmas 2.6 and 2.5, and genericity.

Lemma 4.2. *Suppose that*

- *δ is a regular uncountable cardinal;*
- *S is a set of regular cardinals below δ and δ is 2-S-Mahlo;*
- *$V(\mathbb{R}^*)$ is a δ-symmetric extension of V;*

Then

- *if D is a $V(\mathbb{R}^*)$-generic club for $SL(V,\delta,S)$, then*
  - *D is V-generic for $SC(\delta,S)$,*
  - *$\mathbb{R}^* \subset V[D]$,*
  - *$V(\mathbb{R}^*)[D] = V[D]$;*
- *if (D,B) is $V(\mathbb{R}^*)$-generic for $SL(V,\delta,S) \times Add(1,\delta)$, then*
  - *B is $V[D]$-generic for $Add(1,\delta)$,*
  - *$V(\mathbb{R}^*)[D][B] = V[D][B]$,*
  - *$V[D][B]$ is a generic extension of V by the partial order*

$$SC(\delta,S) * Add(1,\delta).$$

- *forcing with $SL(V,\delta,S)$ over $V(\mathbb{R}^*)$ does not collapse δ.*

Proof. To see that D is V-generic for $SC(\delta,S)$, let E be a dense subset of $SC(\delta,S)$ in V and let g be a condition in $SL(V,\delta,S)$. Let γ be the length of g. By Fact 2.4, $SC(\delta,S)$ below $(\{\gamma\},\emptyset,\emptyset)$ is isomorphic to

$$SC(\gamma, S\cap\gamma) \times SC(\delta, S \setminus (\gamma+1)),$$

and we can let E' be the image of E (below $(\{\gamma\},\emptyset,\emptyset)$) in this product. Since g is a generic filter for $SC(\gamma, S\cap\gamma)$, there is a condition (p,q) in E' with $p \in g$. Let

$\eta > \gamma$ be 1-S-Mahlo in V with $q \in SC(\eta, S \cap (\gamma, \eta))$, and let h be a $V[g]$-generic filter for $SC(\eta, S \cap (\gamma, \eta))$ with $q \in h$. Then the preimage of (g, h) in $SC(\delta, S)$ is a condition in $SL(V, \delta, S)$ extending g meeting E. By genericity, then, D is V-generic for $SC(V, \delta, S)$.

To see that $\mathbb{R}^* \subset V[D]$, fix $x \in \mathbb{R}^*$ and let g be a condition in $SL(V, \delta, S)$. Let γ be the length of g. By Fact 2.4, $SC(\delta, S)$ below $(\{\gamma\}, \emptyset, \emptyset)$ is isomorphic to $SC(\gamma, S \cap \gamma) \times SC(\delta, S \setminus (\gamma + 1))$. Let $\eta < \delta$ be the least 1-S-Mahlo cardinal in S such that the pair $\{g, x\}$ is V-generic for a partial order of cardinality η. Let h be a $V[g]$-generic filter for $SC(\eta, S \cap (\gamma, \eta))$ with $x \in V[g][h]$. Then the preimage of (g, h) in $SC(\delta, S)$ is a condition g' in $SL(V, \delta, S)$ extending g with $x \in V[g']$. By genericity, then, $\mathbb{R}^* \subset V[D]$.

To see that B is $V[D]$-generic for $Add(1, \delta)$, let (g, a) be a condition in $SL(V, \delta, S) \times Add(1, \delta)$, and let τ be an $SC(\delta, S)$-name for a dense subset of $Add(1, \delta)$. By the V-genericity of D, and Lemma 2.5, whenever D^* is $V(\mathbb{R}^*)$-generic for $SL(V, \delta, S)$, every real in $V[D^*]$ is in $V[D^* \cap \eta]$ for some $\eta < \delta$. Therefore, there is a condition g' below g in $SL(V, \delta, S)$ such that $a \in V[g']$ and such that some extension b of a in $V[g']$ is forced by some condition in g' to be in the realization of τ. Then (g', b) is below (g, a), and the $V[D]$-genericity of B follows by the $V(\mathbb{R}^*)$-genericity of (D, B).

By Lemma 2.5 and the V-genericity of D for $SC(\delta, S)$, forcing with $SL(V, \delta, S)$ over $V(\mathbb{R}^*)$ does not collapse δ. This in turn implies that $V(\mathbb{R}^*)[D] = V[D]$, and that forcing with $SL(V, \delta, S) \times Add(1, \delta)$ over $V(\mathbb{R}^*)$ does not collapse δ.

$\square$

Lemmas 4.3 and 4.4 give homogeneity properties for partial orders of the form $SL(V, \delta, S)$. We will use these facts in different contexts. In some sense, the proof of Lemma 4.3 is more important that the statement of the lemma itself. Lemma 4.3 uses Corollary 26.10 of [3], which (for our purposes) says that if γ is a regular cardinal, $G \subset Coll(\omega, \gamma)$ is a V-generic filter, and $x \in V[G]$ is subset of V such that γ is uncountable in $V[x]$, then there exists a $V[x]$-generic filter $H \subset Coll(\omega, \gamma)$ such that $V[G] = V[x][H]$. (This is very similar to our Theorems 1.1 and 1.3, but not quite the same.)

Lemma 4.3. *Suppose that*

- *M is a model of ZFC;*
- *$\delta \leq \omega_1^V$ is an ordinal;*
- *$\mathcal{P}(\alpha)^M$ is countable for each $\alpha < \delta$;*
- *$S \subset \delta$ is a set of regular cardinals in M;*
- *δ is a limit of 1-S-Mahlo ordinals in M.*

Then $SL(M, \delta, S)$ is homogeneous.

Proof. Let p, q be conditions in $SL(M, \delta, S)$ of length γ_p and γ_q, respectively. Let γ be the least 1-S-Mahlo cardinal of M above both γ_p and γ_q such that the pair $\{p, q\}$ is M-generic for a partial order in V_γ^M. Since $SC(\gamma, S \setminus (\gamma_p + 1))$ and $SC(\gamma, S \setminus (\gamma_q + 1))$ are both forcing-equivalent to $Coll(\omega, \gamma)$, there exist by Corollary 26.10 of [3] and Lemma 2.4 conditions $p' \leq p$ and $q' \leq q$ of length γ such that $M[p'] = M[q']$. Then the restrictions of the partial order $SL(M, \delta, S)$ below the conditions p' and q' are isomorphic. □

Since $SL(V, \delta, W) \times Add(1, \delta)$ is homogeneous (in the context of Lemma 4.3), Lemma 4.4 shows that the $SL(V, \delta, W) \times Add(1, \delta)$-extension of $V(\mathbb{R}^*)$ is elementarily equivalent to the same extension defined over any forcing extension of V by a partial order in V_δ. An analogous version of the lemma for the partial order $SC(\delta, W) * Add(1, \delta)$ follows from the existence of a 2-Mahlo-Woodin cardinal. We will apply the lemma in an even stronger context.

Lemma 4.4. *Suppose that*

- *δ is a strongly inaccessible limit of 1-Mahlo-Woodin cardinals;*
- *$V(\mathbb{R}^*)$ is a δ-symmetric extension of V;*
- *P, Q are partial orders in V_δ;*
- *$g \subset P$ and $h \subset Q$ are V-generic filters in $V(\mathbb{R}^*)$;*
- *W_g is the set of Woodin cardinals of $V[g]$ below δ;*
- *W_h is the set of Woodin cardinals of $V[h]$ below δ;*
- *p is a condition in $SL(V[g], \delta, W_g)$;*
- *q is a condition in $SL(V[h], \delta, W_h)$.*

Then there exist conditions $p' \leq p$ and $q' \leq q$ such that $SL(V[g], \delta, W_g)$ below p' and $SL(V[h], \delta, W_h)$ below q' are isomorphic.

Proof. Let γ_p and γ_q be the respective lengths of p and q. Let γ be the least 1-S-Mahlo cardinal of V above both γ_p and γ_q such that the set $\{p, q, g, h\}$ is V-generic for a partial order in V_γ. Since $SC(\gamma, W_g \setminus (\gamma_p + 1))^{V[g]}$ and $SC(\gamma, W[h] \setminus (\gamma_q + 1))^{V[h]}$ are both forcing-equivalent to $Coll(\omega, \gamma)$ in their respective models, there exist by Corollary 26.10 of [3] and Lemma 2.4 conditions $p' \leq p$ in $SL(V[g], \delta, W_g)$ and $q' \leq q$ in $SL(V[h], \delta, W_h)$ of length γ such that $V[g][p'] = V[h][q']$. Then since $W_g \setminus \gamma = W_h \setminus \gamma$, p' and q' are as desired. □

Very roughly (i.e., suppressing a few issues for a moment), the last remaining tool we need to develop for the proof of Theorem 4.10 is the ability to use a V-slow club to construct a $V(\mathbb{R}^*)$-generic filter for a partial order of the form $SL(V, \delta, S) \times Add(1, \delta)$ such that the corresponding extension contains a given V-generic filter for $\mathbb{Q}_{<\delta}$. To do this there needs to be a name in $V(\mathbb{R}^*)$ which gives rise to the desired filter. The filters $g_{(d,b)}$ defined below are initial segments of the realization of this name.

If $V(\mathbb{R}^*)$ is a δ-symmetric extension of V and B is $V(\mathbb{R}^*)$-generic for $Add(1,\delta)$, then, considering consecutive ω-sequences from δ and membership (or not) in B, B lists all the members of $\mathbb{R}^*$. We fix a recursive coding of elements of $H(\omega_1)$ by subsets of ω, and consider elements of $H(\omega_1)$ coded by consecutive ω-sequences from B in this fashion.

Suppose that δ is a limit of Woodin cardinals, and let W denote the set of Woodin cardinals below δ. Given a condition (d,b) in $SL(V,\delta,W)\times Add(1,\delta)$, we define a set $g_{(d,b)}$ and an ordinal $\eta_{(d,b)}$ such that either $g_{(d,b)}=\emptyset$ and $\eta_{(d,b)}=0$ or $g_{(d,b)}$ is a V-generic filter $g_{(d,b)}$ in $\mathbb{Q}^V_{<\eta_{(d,b)}}$ and $\eta_{(d,b)}\in d$. If d is empty, so is $g_{(d,b)}$ (so $\eta_{(d,b)}=0$). Otherwise, $\eta_{(d,b)}$ and $g_{(d,b)}$ are defined as follows. Let $g_0=0$ and $\beta_0=0$, and, for each limit element γ of d, if g_η and β_η are defined for each $\eta\in d\cap\gamma$, then let

$$g_\gamma=\bigcup\{g_\eta:\eta\in d\cap\gamma\}$$

and $\beta_\gamma=\sup\{\beta_\eta:\eta<\gamma\}$. If g_γ is defined for each $\gamma\in d$, then $g_{(d,b)}=g_{\max(d)}$ and $\eta_{(d,b)}=\max(d)$. For each $\gamma\in(d\cup\{0\})\setminus\max(d)$, if g_γ and β_γ are defined, let γ^+ denote the least member of d above γ. Then we choose g_{γ^+} and β_{γ^+} (or $g_{(d,b)}$) in the following way.

- If some consecutive ω-sequence from b above $\gamma\cup\beta_\gamma$ codes a V-generic filter $g\subset\mathbb{Q}^V_{<\gamma^+}$ such that $g\cap\mathbb{Q}^V_{<\gamma}=g_\gamma$, then let g_{γ^+} be the first filter of this type coded by a consecutive ω-sequence from b above $\gamma\cup\beta_\gamma$, and let β_{γ^+} be supremum of the indices of this ω-sequence.

- If there is no such consecutive ω-sequence from b above $\gamma\cup\beta_\gamma$, then let $g_{(d,b)}=g_\gamma$ and $\eta_{(d,b)}=\gamma$, and g_{γ^+} and β_{γ^+} are undefined.

If $(d',b')\le(d,b)$ are conditions in $SL(V,\delta,W)\times Add(1,\delta)$, then $g_{(d,b)}\subset g_{(d',b')}$ (and indeed the construction just given for (d,b) is an initial segment of the construction for (d',b')). The argument given in the proof of Theorem 3.1, using the fact that d is an V-slow club, shows that $g_{(d,b)}$ is either $\emptyset$ or an V-generic filter for $\mathbb{Q}^V_{<\eta_{(d,b)}}$. We say that (d,b) is *complete* if either (d,b) is the empty condition or

$$\eta_{(d,b)}=\sup(b)=\max(d)$$

and every real coded by a consecutive ω-sequence from b is in $V[g_{(d,b)}]$.

The following lemma shows how to extend (d,b) in order to extend $g_{(d,b)}$.

Lemma 4.5. *Suppose that*

- *M is a model of ZFC;*
- *δ is a 2-Mahlo-Woodin cardinal in M;*
- *$\mathbb{R}^*$ is the set of reals of V;*
- *$M(\mathbb{R}^*)$ is a δ-symmetric extension of M;*
- *W is the set of Woodin cardinals of M below δ;*

- (d,b) *is a condition in* $SL(M,\delta,W)\times Add(1,\delta)$*;*
- g *is an* M*-generic filter for* $\mathbb{Q}^M_{<\max(d)}$ *extending* $g_{(d,b)}$ *such that* $a_\gamma\in g$ *for every* $\gamma\in d\setminus\eta_{(d,b)}$*.*

Then there exists a b' *extending* b *such that* $g_{(d,b')}=g$*.*

Proof. Clearly, if $g_{(d,b)}=g$, we can let $b'=b$. Otherwise, $\eta_{(d,b)}\in d\setminus\{max(d)\}$ and there is no consecutive ω-sequence from b above $\eta_{(d,b)}\cup\{\beta_{(d,b)}\}$ coding an M-generic filter $g\subset\mathbb{Q}^M_{<\gamma_0}$ such that $g\cap\mathbb{Q}^M_{<\eta_{(d,b)}}=g_{\eta_{(d,b)}}$, where γ_0 is the least element of d above $\eta_{(d,b)}$. Let the first ω-sequence of b' extending b above $\eta_{(d,b)}$ be a real in $M[g\cap\mathbb{Q}^M_{<\gamma_1}]$ coding $g\cap\mathbb{Q}^M_{<\gamma_0}$, where γ_1 is the least element of d above γ_0. Then $\beta_{\gamma_0}=\sup(b')$.

For each $\gamma\geq\gamma_0$ in d, let the first ω-sequence of b' above $\gamma\cup\beta_\gamma$ be a real in $M[g\cap\mathbb{Q}^M_{<\gamma_2}]$ coding $g\cap\mathbb{Q}^M_{<\gamma_1}$, where γ_1 is the least member of d above γ, and γ_2 is the least member of d above γ_1. Then $\beta_{\gamma_1}=(\gamma\cup\beta_\gamma)+\omega$.

For limit members γ of d above γ_0, β_γ is the supremum of $\{\beta_\eta:\eta<\gamma\}$.

Let these be the only elements of $b'\setminus b$. □

In the context we will be working in, the complete conditions are dense.

Lemma 4.6. *Suppose that* δ *is a 2-Mahlo-Woodin cardinal in a model* M *of ZFC, and* $M(\mathbb{R}^*)$ *is a* δ*-symmetric extension of* M*, where* $\mathbb{R}^*$ *is the set of reals in* V*. Let* (d,b) *be a condition in* $SL(M,\delta,W)\times Add(1,\delta)$*, where* W *is the set of Woodin cardinals of* M *below* δ*. Then there is complete condition* (d',b') *in* $SL(M,\delta,W)\times Add(1,\delta)$ *below* (d,b)*.*

Proof. By Theorem 3.1 and Lemma 4.5, we may assume that $\eta_{(d,b)}=\max(d)$. The set of reals coded by an ω-sequence from b is countable, so there is a real x constructing all such reals. Let $\langle\gamma_i:i\leq\omega\rangle$ be a continuous, increasing sequence of Woodin cardinals of M above $\max(d)\cup\sup(b)$ such that

1. γ_0 is the least Woodin cardinal $\gamma>\max(d)\cup\sup(b)$ such that
   - $a\in\mathbb{Q}^M_{<\gamma}$;
   - the pair $\{g_{(d,b)},x\}$ exists in a generic extension of M by a partial order of cardinality less than γ;
2. γ_ω is the least 1-Mahlo-Woodin cardinal of M greater than γ_0;
3. $d\cup\{\gamma_i:i<\omega\}$ is M-generic for $SC(\gamma_\omega,W\cap\gamma_\omega)$;

Then let $d'=d\cup\{\gamma_i:i<\omega\}$ and let b' be a subset of γ_ω with the property that

- b' end-extends b;
- the first ω-sequence of b' above $\max(d)\cup\sup(b)$ is a real y_0 coding an M-generic filter $g_0\subset\mathbb{Q}_{<\gamma_0}$ such that $g_0\cap V^M_{\eta_{(d,b)}}=g_{(d,b)}$, $a\in g_0$, $x\in M[g_0]$, and y_0 exists in a generic extension of M by a partial order in $V^M_{\gamma_1}$;

- for all $i \in \omega$, the first ω-sequence of b' above γ_i is a real y_{i+1} coding an M-generic filter $g_{i+1} \subset \mathbb{Q}_{<\gamma_{i+1}}$ such that $g_{i+1} \cap V^M_{\gamma_i} = g_i$, $y_i \in M[g_{i+1}]$, and y_{i+1} exists in a generic extension of M by a partial order in $V^M_{\gamma_{i+2}}$;
- all elements of $b' \setminus b$ are of the the form $(\max(d) \cup \sup(b)) + n$ or $\gamma_i + n$, for some i, n in ω.

Then (d', b') is the desired condition. $\square$

If (D, B) is a filter contained in $SL(V, \delta, W) \times Add(1, \delta)$, we let

$$g_{(D,B)} = \bigcup \{g_{(d,b)} \mid (d, b) \in (D, B)\}.$$

Lemma 4.7 follows from Lemmas 4.2, 4.5 and 4.6.

Lemma 4.7. *Suppose that δ is (in V) a 2-Mahlo-Woodin cardinal. Let $V(\mathbb{R}^*)$ be a δ-symmetric extension of V and let (D, B) be $V(\mathbb{R}^*)$-generic for*

$$SL(V, \delta, W) \times Add(1, \delta).$$

Then $\delta = \omega_1^{V[D][B]}$, $g_{(D,B)}$ is a V-generic filter for $\mathbb{Q}^V_{<\delta}$ and $\mathbb{R}^{V[g_{(D,B)}]} = \mathbb{R}^*$.

Lemma 4.9 below is the main technical lemma for the proof of Theorem 4.10, showing that the genericity requirements for (D, B) don't interfere with the desired result for $g_{(D,B)}$. Lemma 4.9 in turn requires the following technical lemma.

Lemma 4.8. *Suppose that*

- *M is a model of ZFC;*
- *δ is a 2-Mahlo-Woodin cardinal in M;*
- *$\mathbb{R}^*$ is the set of reals in V;*
- *$M(\mathbb{R}^*)$ is a δ-symmetric extension of M;*
- *W is the set of Woodin cardinals of M below δ;*
- *(d, b) is a complete condition in $SL(M, \delta, W) \times Add(1, \delta)$,*
- *a is a condition in $\mathbb{Q}^M_{<\delta}$ below $a_{\eta_{(d,b)}}$.*
- *$\dot{d}$ and $\dot{b}$ are $((\mathbb{Q}_{<\delta} \restriction a)/\mathbb{Q}_{<\eta_{(d,b)}})^{M[g_{(d,b)}]}$-names such that $(\dot{d}, \dot{b})$ is forced to be a complete condition in $SL(M, \delta, W) \times Add(1, \delta)$ such that $g_{(\dot{d},\dot{b})} = g_{(d,b)}$;*

Then there exist continuous, increasing sequences

$$d^* = \langle \gamma_i : i \leq \omega \rangle$$

and

$$d' = \langle \gamma'_i : i \leq \omega \rangle,$$

and sets b^*, b' and g such that

- *$\gamma_0 = \gamma'_0$ is a Woodin cardinal of M greater than $\eta_{(d,b)}$ with $a \in \mathbb{Q}^M_{<\gamma_0}$;*
- *$\gamma_\omega = \gamma'_\omega$ is the least 1-Mahlo-Woodin cardinal of M above γ_1;*
- *g is an M-generic filter contained in $\mathbb{Q}_{<\gamma_0}$ extending $g_{(d,b)}$ with a in g;*
- *g decides all of $\dot{d}$ and $\dot{b}$;*
- *$d \cup d^*$ is M-generic for $SC(\gamma_\omega, W \cap \gamma_\omega)$;*
- *$\dot{d}_g \cup d'$ is M-generic for $SC(\gamma_\omega, W \cap \gamma_\omega)$;*
- *$M[d \cup d^*] = M[\dot{d}_g \cup d^*]$;*
- *$(d \cup d^*, b^*)$ is a complete condition in $SL(M, \delta, W) \times Add(1, \delta)$ below (d, b);*
- *$(\dot{d}_g \cup d', b')$ is a complete condition in $SL(M, \delta, W) \times Add(1, \delta)$ below $(\dot{d}_g, \dot{b}_g)$;*
- *$g_{(d \cup d^*, b^*)} = g_{(\dot{d}_g \cup d', b')}$ extends g.*

Proof. Let γ_0 be the least Woodin cardinal γ of M such that

- $\gamma > \eta_{(d,b)}$;
- $a \in \mathbb{Q}^M_{<\gamma}$;
- the antichains deciding $\dot{d}$ and $\dot{b}$ are all predense in $\mathbb{Q}^M_{<\gamma}$

Let g be an M-generic filter for $\mathbb{Q}^M_{<\gamma_0}$ such that

- $a \in g$;
- $g \cap \mathbb{Q}^M_{<\eta(d,b)} = g_{(d,b)}$;

Let γ_1 be the least $\gamma > \gamma_0$ which is a Woodin cardinal in M such that the pair $\{d, \dot{d}_g\}$ is M-generic for a partial order in V^M_γ. Let γ_ω be the least 1-Mahlo-Woodin cardinal of M above γ_1. As in the proof of Lemma 4.3, there exist sequences $d^* = \langle \gamma_i : i < \omega \rangle$ and $d' = \langle \gamma'_i : i < \omega \rangle$ with supremum γ_ω such that

- $d \cup d^*$ is M-generic for $SC(\gamma_\omega, W \cap \gamma_\omega)$;
- $\dot{d}_g \cup d'$ is M-generic for $SC(\gamma_\omega, W \cap \gamma_\omega)$;
- $M[d \cup d^*] = M[\dot{d}_g \cup d']$.

Let g^* be a generic filter for $\mathbb{Q}^M_{<\gamma_\omega}$ extending g such that $a_\gamma \in g^*$ for every $\gamma \in d^* \cup d'$. Then by Lemma 4.5, there exist b^* and b' such that

$$g_{(d \cup d^*, b^*)} = g_{(\dot{d}_g \cup d', b')} = g^*,$$

as desired. □

Lemma 4.9. *Suppose that*

- δ *is a 2-Mahlo-Woodin cardinal in* V*;*
- $G \subset \mathbb{Q}_{<\delta}$ *is a* V*-generic filter;*
- (d,b) *is a complete condition in* $SL(V,\delta,W) \times Add(1,\delta)$*;*
- $G \cap V_{\eta_{(d,b)}} = g_{(d,b)}$*;*
- $\mathcal{D}$ *is a dense open subset of* $SL(V,\delta,W) \times Add(1,\delta)$ *in* $V(\mathbb{R}^{V[G]})$*.*

Then there exist a complete condition (d',b') *in* $(SL(V,\delta,W) \times Add(1,\delta)) \cap \mathcal{D}$ *extending* (d,b) *such that* $\eta_{(d',b')} > \eta_{(d,b)}$ *and* $G \cap V_{\eta_{(d',b')}} = g_{(d',b')}$*.*

Proof. Let η denote $\eta_{(d,b)}$. If the lemma fails, there exist a condition a in

$$((\mathbb{Q}_{<\delta} \restriction a_\eta)/\mathbb{Q}_{<\eta})^{V[G \cap V_\eta]}$$

(call this forcing Q) and Q-names $\dot{b}$, $\dot{d}$ and $\dot{\mathcal{D}}$ such that a forces over the extension $V[G \cap V_\eta]$ that $\dot{\mathcal{D}}$ is a dense open subset of the partial order

$$SL(V,\delta,W) \times Add(1,\delta)$$

of $V(\mathbb{R}^{V[\dot{G}]})$ and $(\dot{d},\dot{b})$ is a complete element of this partial order such that $G \cap V_\eta = g_{(\dot{d}_G,\dot{b}_G)}$ and such that for no complete condition (d^+,b^+) in

$$(SL(V,\delta,W) \times Add(1,\delta)) \cap \dot{\mathcal{D}}$$

are $(d^+,b^+) \leq (\dot{d},\dot{b})$ and $G \cap V_{\eta_{(d^+,b^+)}} = g_{(d^+,b^+)}$.

Let $d^* = \langle \gamma_i : i \leq \omega \rangle$, $d' = \langle \gamma'_i : i \leq \omega \rangle$, b^*, b' and g be as in Lemma 4.8, with respect to a. Let (D,B) be a $V(\mathbb{R}^{V[G]})$-generic filter for

$$SL(V,\delta,W) \times Add(1,\delta)$$

extending $(d \cup d^*, b^*)$, and let $H = g_{(D,B)}$. Then by Lemma 4.7, $\mathbb{R}^{V[G]} = \mathbb{R}^{V[H]}$. By the choice of a, $\dot{d}$ and $\dot{b}$, there is a dense open subset $\mathcal{D}'$ of the partial order $SL(V,\delta,W) \times Add(1,\delta)$ of $V(\mathbb{R}^{V[G]})$ such that for no complete condition (d^+,b^+) in

$$(SL(V,\delta,W) \times Add(1,\delta)) \cap \mathcal{D}'$$

are $(d^+,b^+) \leq (\dot{d}_g,\dot{b}_g)$ and $H \cap V_{\eta_{(d^+,b^+)}} = g_{(d^+,b^+)}$.

Let (D',B') be the filter in $SL(V,\delta,W) \times Add(1,\delta)$ of $V(\mathbb{R}^*)$ formed by replacing $(d \cup d^*, b^*)$ with $(\dot{d}_g \cup d', b')$ (since $M[d \cup d^*] = M[\dot{d}_g \cup d']$, this replacement sends conditions to conditions). Then (D',B') is $V(\mathbb{R}^*)$-generic, and, by the final three conclusions of Lemma 4.8, $g_{(D',B')} = H$. By the genericity of (D',B'), there exists an $\eta' > \eta$ such that the restriction of D' and B' to η' is a complete pair (d^+,b^+) in $\mathcal{D}'$, giving a contradiction. □

Given a model M of ZF and an ordinal δ of M, an *M-fast club* through δ is a club $C \subset \delta$ with the property that for all limit elements β of C, $C \cap \beta$ is eventually contained in every club subset of β in M.

The proof of Theorem 4.10 uses the full stationary tower $\mathbb{P}_{<\kappa}$ (see [6], though we give specific references below for the facts we need).

Theorem 4.10. *Suppose that CH holds, δ is a measurable Woodin cardinal in V, and $\kappa > \delta$ is a Woodin cardinal. Suppose that $V(\mathbb{R}^*)$ is a δ-symmetric extension of V and (D, B) is $V(\mathbb{R}^*)$-generic for $SL(V, \delta, W) \times Add(1, \delta)$ as defined in $V(\mathbb{R}^*)$. Then every Σ^2_2 sentence which holds in V holds in $V[D][B]$.*

Proof. Let $\exists X \subset \mathbb{R}\, \forall Y \subset \mathbb{R}\, \phi(X, Y)$ be a Σ^2_2 sentence which holds in V. Any two models of the form $V[D][B]$ are elementarily equivalent, so it suffices to show that $\exists X \subset \mathbb{R}\, \forall Y \subset \mathbb{R}\, \phi(X, Y)$ holds in some such model.

Let a be a condition in $\mathbb{P}_{<\kappa}$ such that

- a forces that $H \cap V_\delta$ will be V-generic for $\mathbb{Q}_{<\delta}$, where $H \subset \mathbb{P}_{<\kappa}$ is the generic filter;
- a forces that $j(\omega_1) = \delta$, where j is the embedding induced by H;
- a forces that $\mathcal{P}(\delta)^V$ has cardinality $\aleph_1$ in $V[H]$;
- a forces that there exists a V-fast club contained in the 1-Mahlo-Woodin cardinals of V below δ.

The existence of such an a from a measurable Woodin cardinal is shown in [1] and the proof of Theorem 3.2.1 in [6], modulo the fact that any normal measure on δ concentrates on the 1-Mahlo-Woodin cardinals of V below δ (1-Mahlo-Woodin cardinals themselves are not mentioned in [1, 6]). Let $H \subset \mathbb{P}_{<\kappa}$ be a V-generic filter with $a \in H$. Let $j : V \to M$ be the induced embedding. Then $G = H \cap V_\delta$ is V-generic for $\mathbb{Q}_{<\delta}$. Let $j' : V \to M'$ be the embedding induced by G. Let C be the V-fast club added by H. Let ζ be the least strongly inaccessible cardinal of V above δ. Let $\mathbb{R}^*$ be the reals of M. Since CH holds in V and $j(\omega_1) = \delta$, $\mathbb{R}^* = j(\mathbb{R}) = j'(\mathbb{R})$, so $\mathbb{R}^*$ is also the set of reals in M'. Then $V_\zeta[G]$ is in M, and $V_\zeta(\mathbb{R}^*)$ is a symmetric extension of V_ζ.

Since $C \in M$ (see Theorem 2.5.8 of [6]) and C is a V-slow club through the 1-Mahlo-Woodin cardinals of V below δ, M can construct filters in $SL(V, \delta, W) \times Add(1, \delta)$ below any condition and meeting any $\aleph_1$ many dense sets in $V(\mathbb{R}^*)$. Working in M, construct a $V_\zeta(\mathbb{R}^*)$-generic filter (D, B) for $SL(V_\zeta, \delta, W) \times Add(1, \delta)$ such that $g_{(D,B)} = G$. Since measurable Woodin cardinals are 2-Mahlo-Woodin, this can be done by Lemma 4.9, using C to guarantee genericity at limit states, and using the fact that $\mathcal{P}(\delta)^V$ has cardinality $\aleph_1$ in M to ensure genericity of the final filter.

Now let X_0 be a set of reals in V such that $V \models \forall Y \subset \mathbb{R}\, \phi(X_0, Y)$. Then $j'(X_0) = j(X_0)$, so $j(X_0) \in V[G] \subset V[D][B]$, and $M \models \forall Y \subset \mathbb{R}\, \phi(j(X_0), Y)$. Since $\mathcal{P}(\delta)^{V[D][B]} \subset M$, $V[D][B] \models \forall Y \subset \mathbb{R}\, \phi(j(X_0), Y)$, and thus the sentence $\exists X \subset \mathbb{R} \forall Y \subset \mathbb{R}\, \phi(X, Y)$ holds in $V[D][B]$. □

It suffices in the statement of Theorem 4.10 (and the corollaries below) to let δ be a *full* Woodin cardinal (in the terminology of [1]) and let κ be a Woodin cardinal. The full Woodin cardinals constitute a measure one set for any normal measure on a measurable Woodin cardinal.

By Lemma 4.4, we get that all Σ^2_2 sentences holding in any extension by a partial ordering in V_δ satisfying CH hold in $V[D][B]$.

Corollary 4.11. *Suppose that δ is a measurable Woodin cardinal in V, and $\kappa > \delta$ is a Woodin cardinal. Suppose that $V(\mathbb{R}^*)$ is a δ-symmetric extension of V and (D, B) is $V(\mathbb{R}^*)$-generic for $SL(V, \delta, W) \times Add(1, \delta)$ as defined in $V(\mathbb{R}^*)$, where W is the set of Woodin cardinals of V below δ. Then if ϕ is a Σ^2_2 sentence and CH + ϕ holds in a forcing extension of V by a partial order in V_δ, then ϕ holds in $V[D][B]$.*

By Lemma 4.2, we get the following.

Corollary 4.12. *Suppose that δ is a measurable Woodin cardinal in V, and $\kappa > \delta$ is a Woodin cardinal. Let (D, B) be V-generic for $SC(\delta, W) * Add(1, \delta)$, where W is the set of Woodin cardinals of V below δ. Then if ϕ is a Σ^2_2 sentence and CH + ϕ holds in a forcing extension of V by a partial order in V_δ, then ϕ holds in $V[D][B]$.*

Theorem 4.10 and Corollary 4.11 continue to hold when a predicate for a universally Baire set of reals is added to the language. Showing this requires only that, in the proof of Theorem 4.10, if A is universally Baire set of reals in V, then $j(A)$ is equal to the reinterpretation of A in $V[D][B]$. This in turn follows from the following theorem of Steel (proofs appear in [1, 6]).

Theorem 4.13. *Let λ be a strongly inaccessible cardinal and let T be a a λ^+-weakly homogeneous tree. If S is the Martin-Solovay tree for the complement of the projection of T and k is an elementary embedding derived from forcing with $\mathbb{Q}_{<\lambda}$ then the corresponding generic embedding $k \colon V \to M$ satisfies $k(S) = S$.*

Corollary 4.14. *Suppose that δ is a measurable Woodin cardinal, A is set of reals such that A and $\mathbb{R} \setminus A$ are δ^+-weakly homogeneously Suslin and $\kappa > \delta$ is a Woodin cardinal. Suppose that (D, B) is V-generic for $SL(V, \delta, W) * Add(1, \delta)$, where W is the set of Woodin cardinals of V below δ. Then every Σ^2_2-sentence with an additional predicate for A which can be forced to hold by a partial order in V_δ holds in $V[D][B]$.*

It is not possible to add a predicate for NS_{ω_1} to the language in Theorem 4.10. One way to see this is given in [7].

A natural question is whether the forcing $Add(1, \delta)$ is necessary to achieve Σ^2_2-maximality.

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