Ordering finite sets

Paul Larson

February 8, 2017

This note extracts combinatorial facts from Leo Harrington's construction of models of size $\aleph_1$ for counterexamples to Vaught's Conjecture, and the fact (essentially due to Hjorth) that his construction cannot be used to produce models of cardinality $\aleph_2$.

Definition 0.1. *Given a set X, a subset C of $[X]^{<\omega}$ is* $\subseteq$-cofinal *if every member of $[X]^{<\omega}$ is contained in a member of C.*

Definition 0.2. *We say that a wellordered sequence $\langle a_\alpha : \alpha < \beta\rangle$ is $\subseteq$-nondecreasing if there is no pair $\alpha < \delta$ below β such that $a_\delta \subseteq a_\alpha$.*

Definition 0.3. *Given $n \in \omega$ and a wellordered sequence $\overline{a} = \langle a_\alpha : \alpha < \beta\rangle$ of finite sets, we say that $\overline{a}$ is n-good if it is $\subseteq$-nondecreasing and if for each $\gamma < \beta$, for all $s \in [\gamma+1]^{n+1}$ there is a $t \in [\gamma+1]^n$ such that each member of $\{a_\alpha : \alpha \in s\}$ is contained in a member of $\{a_\alpha : \alpha \in t\}$. A witness s to a wellordering not being n-good is an n-bad set*

Clearly, $s \in [\gamma+1]^{n+1}$ could equivalently be replaced with $s \in [\gamma]^{<\omega}$ in the definition of n-good. Another equivalent definition is the following : a wellordered $\subseteq$-nondecreasing sequence $\overline{a} = \langle a_\alpha : \alpha < \beta\rangle$ of finite sets is n-good if for each $s \in [\beta]^{n+1}$ such that the sets a_α ($\alpha \in s$) are pairwise $\subseteq$-incomparable, there is a $\gamma < \max(s)$ such that $|\{\alpha \in s \cap \max(s) : a_\alpha \subseteq a_\gamma\}| \geq 2$.

If $m < n \in \omega$ and $\overline{a}$ is m-good, then it is n-good. Note that n-bad sets have size $n+1$.

Proposition 0.1. *There exists a $\subseteq$-nondecreasing sequence $\overline{a} = \langle a_\alpha : \alpha < \aleph_\omega\rangle$ such that*

- *for each $n \in \omega$ and each nonzero $\beta < \omega_{n+1}$, $\{a_\alpha : \alpha < \omega_n \cdot \beta\}$ is a $\subseteq$-cofinal subset of $[\gamma]^{<\omega}$, where $\gamma = \omega_n + \beta - 1$ if $\beta < \omega$, and $\gamma = \omega_n + \beta$ otherwise;*
- *for each $n \in \omega$, $\overline{a} \upharpoonright (\omega_n + \omega_{n-1} + \cdots + \omega)$ is $(n+1)$-good.*

Proof. For each $n \in \omega$, let a_n be n. Given $n \in \omega$ and nonzero $\beta < \omega_{n+1}$, we show how to extend from $\langle a_\alpha : \alpha < \omega_n \cdot \beta\rangle$ to $\langle a_\alpha : \alpha < \omega_n \cdot (\beta+1)\rangle$, assuming that

- $\{a_\alpha : \alpha < \omega_n\}$ is a $\subseteq$-cofinal subset of $[\omega_n]^{<\omega}$;
- for some ordinal γ, $\{a_\alpha : \alpha < \omega_n \cdot \beta\}$ is a $\subseteq$-cofinal $\subseteq$-nondecreasing subset of $[\gamma]^{<\omega}$ (necessarily of cardinality $\aleph_n$).

Fix a bijection $f_\gamma \colon \omega_n \to \gamma$, and for each $\alpha < \omega_n$, let $a_{(\omega_n \cdot \beta)+\alpha} = f[a_\alpha] \cup \{\gamma\}$. Then $\{a_\alpha : \alpha < \omega_n \cdot (\beta+1)\}$ is a $\subseteq$-cofinal $\subseteq$-nondecreasing subset of $[\gamma+1]^{<\omega}$.

The first conclusion of the claim can now be verified by induction. For the second, observe first that $\overline{a} \upharpoonright \omega$ is 1-good. Next, observe that if $\langle a_\alpha : \alpha < \omega_n \cdot \beta\rangle$ and $\langle a_\alpha : \alpha < \omega_n \cdot (\beta+1)\rangle$ are as in the construction above, and $\delta \leq \omega_n$ and $m, p \in \omega$ are such that $\langle a_\alpha : \alpha < \delta \rangle$ is m-good and $\langle a_\alpha : \alpha < \omega_n \cdot \beta \rangle$ is p-good, then $\langle a_\alpha : \alpha < (\omega_n \cdot \beta) + \delta \rangle$ is $\max\{m+1, p\}$-good, since any putative $(m+1)$-bad set not contained in $\omega_n \cdot \beta$ can contain at most one element of $\omega_n \cdot \beta$, as $\{a_\alpha : \alpha < \omega_n \cdot \beta\}$ is $\subseteq$-directed. Arguing by induction, it follows from this that for each $n \in \omega$, $\overline{a} \restriction \omega_n$ is $(n+1)$-good, and moreover, that the second conclusion of the claim holds. □

The following proposition shows why the proof of Proposition 0.1 doesn't show that initial segments of the form

$$\overline{a} \restriction (\omega_n + \omega_{n-1} + \cdots + \omega + 1)$$

are $(n+1)$-good.

Proposition 0.2. *There is no* 1*-good sequence of length* $\omega + 1$*.*

The condition of $\subseteq$-cofinality does not appear to be needed for the corresponding impossibility result, which can be proved by induction on $n \in \omega$.

Proposition 0.3. *Every* $\subseteq$*-nondecreasing wellordered sequence of finite sets of length* $\omega_n + \omega_{n-1} + \cdots \omega + 1$ *has an* $(n+1)$*-bad set.*

Proof. We have seen this already for $n = 0$, so assume it for a given $n \in \omega$. Let

$$\langle a_\alpha : \alpha < \omega_n + \omega_{n-1} + \cdots \omega + 1 \rangle$$

be a $\subseteq$-nondecreasing sequence of finite sets. Let $s \subseteq [\omega_n, \omega_n + \omega_{n-1} + \cdots \omega + 1)$ be an n-bad set for $\overline{b} = \langle a_\alpha : \omega_n \leq \alpha < \omega_n + \omega_{n-1} + \cdots \omega + 1 \rangle$, and let $\eta < \omega_n$ be such that a_η is not contained in any member of $\overline{b}$. Then $\{\eta\} \cup s$ is an $(n+1)$-bad set. □