AN Ω-LOGIC PRIMER

JOAN BAGARIA, NEUS CASTELLS, AND PAUL LARSON

Abstract. In [12], Hugh Woodin introduced Ω-logic, an approach to truth in the universe of sets inspired by recent work in large cardinals. Expository accounts of Ω-logic appear in [13, 14, 1, 15, 16, 17]. In this paper we present proofs of some elementary facts about Ω-logic, relative to the published literature, leading up to the generic invariance of Ω-logic and the Ω-conjecture.

Introduction

One family of results in modern set theory, called *absoluteness results*, shows that the existence of certain large cardinals implies that the truth values of certain sentences cannot be changed by forcing[1]. Another family of results shows that large cardinals imply that certain definable sets of reals satisfy certain regularity properties, which in turn implies the existence of models satisfying other large cardinal properties. Results of the first type suggest a logic in which statements are said to be valid if they hold in every forcing extension. With some technical modifications, this is Woodin's Ω-logic, which first appeared in [12]. Results of the second type suggest that there should be a sort of internal characterization of validity in Ω-logic. Woodin has proposed such a characterization, and the conjecture that it succeeds is called the Ω-conjecture. Several expository papers on Ω-logic and the Ω-conjecture have been published [1, 13, 14, 15, 16, 17]. Here we briefly discuss the technical background of Ω-logic, and prove some of the basic theorems in this area.

This paper assumes a basic knowledge of Set Theory, including constructibility and forcing. All undefined notions can be found in [4].

1. $\models_\Omega$

1.1. **Preliminaries.**

Key words and phrases. Ω-logic – Woodin cardinals – A-closed sets – universally Baire sets – Ω-conjecture.

The first author was partially supported by the research projects BFM2002-03236 of the *Ministerio de Ciencia y Tecnología*, and 2002SGR 00126 of the *Generalitat de Catalunya*. The third author was partially supported by NSF Grant DMS-0401603. This paper was written during the third author's stay at the *Centre de Recerca Matemàtica* (CRM), whose support under a Mobility Fellowship of the *Ministerio de Educación, Cultura y Deportes* is gratefully acknowledged. It was finally completed during the first and third authors' stay at the Institute for Mathematical Sciences, National University of Singapore, in July 2005.

[1]Throughout this paper, by "forcing" we mean "set forcing".

Given a complete Boolean algebra $\mathbb{B}$ in V, we can define the *Boolean-valued model* $V^{\mathbb{B}}$ by recursion on the class of ordinals On:

$$V_0^{\mathbb{B}} = \emptyset$$
$$V_\lambda^{\mathbb{B}} = \bigcup_{\beta<\lambda} V_\beta^{\mathbb{B}}, \text{ if } \lambda \text{ is a limit ordinal}$$
$$V_{\alpha+1}^{\mathbb{B}} = \{f\colon X \to \mathbb{B} \mid X \subseteq V_\alpha^{\mathbb{B}}\},$$

Then, $V^{\mathbb{B}} = \bigcup_{\alpha \in On} V_\alpha^{\mathbb{B}}$. The elements of $V^{\mathbb{B}}$ are called $\mathbb{B}$-*names*. Every element x of V has a *standard* $\mathbb{B}$-*name* $\check{x}$, defined inductively by: $\check{\emptyset} = \emptyset$, and $\check{x} : \{\check{y} : y \in x\} \to \{\mathbf{1}^{\mathbb{B}}\}$.

For each $x \in V^{\mathbb{B}}$, let $\rho(x) = \min\{\alpha \in On \mid x \in V_{\alpha+1}^{\mathbb{B}}\}$, the *rank* of x in $V^{\mathbb{B}}$.

Given φ, a formula of the language of set theory with parameters in $V^{\mathbb{B}}$, we say that φ is *true* in $V^{\mathbb{B}}$ if its Boolean-value is $1^{\mathbb{B}}$, i.e.,

$$V^{\mathbb{B}} \models \varphi \quad \text{iff} \quad [\![\varphi]\!]^{\mathbb{B}} = 1^{\mathbb{B}},$$

where $[\![\cdot]\!]^{\mathbb{B}}$ is defined by induction on pairs $(\rho(x), \rho(y))$, under the canonical well-ordering of pairs of ordinals, and the complexity of formulas (see [4]). $V^{\mathbb{B}}$ can be thought of as constructed by iterating the $\mathbb{B}$-valued power-set operation. Modulo the equivalence relation given by $[\![x = y]\!]^{\mathbb{B}} = 1$, $V_\alpha^{\mathbb{B}}$ is precisely V_α in the sense of the Boolean-valued model $V^{\mathbb{B}}$ (see [4]):

Proposition 1.1. *For every ordinal α, and every complete Boolean algebra $\mathbb{B}$, $V_\alpha^{\mathbb{B}} \equiv (V_{\check{\alpha}})^{V^{\mathbb{B}}}$, i.e., for every $x \in V^{\mathbb{B}}$,*

$$(\exists y \in V_\alpha^{\mathbb{B}}\, [\![x = y]\!]^{\mathbb{B}} = 1) \quad \textit{iff} \quad [\![x \in V_{\check{\alpha}}]\!]^{\mathbb{B}} = 1^{\mathbb{B}}.$$

Corollary 1.2. *For every ordinal α, and every complete Boolean algebra $\mathbb{B}$,*

$$V_\alpha^{\mathbb{B}} \models \varphi \quad \textit{iff} \quad V^{\mathbb{B}} \models \text{“}V_{\check{\alpha}} \models \varphi\text{”}.$$

Notation:

i) If $\mathbb{P}$ is a partial ordering, then we write $V^{\mathbb{P}}$ for $V^{\mathbb{B}}$, where $\mathbb{B} = r.o.(\mathbb{P})$ is the *regular open completion* of $\mathbb{P}$ (see [4]).
ii) Given M a model of set theory, we will write M_α for $(V_\alpha)^M$ and $M_\alpha^{\mathbb{B}}$ for $(V_\alpha^{\mathbb{B}})^M = (V_\alpha)^{M^{\mathbb{B}}}$.
iii) *Sent* will denote the set of sentences in the first-order language of set theory.
iv) $T \cup \{\varphi\}$ will always be a set of sentences in the language of set theory, usually extending ZFC.
v) We will write c.t.m. for *countable transitive* $\in$-*model.*
vi) We will write c.B.a. for *complete Boolean algebra.*
vii) For $A \subseteq \mathbb{R}$, we write $L(A, \mathbb{R})$ for $L(\{A\} \cup \mathbb{R})$, the smallest transitive model of ZF that contains all the ordinals, A, and all the reals.

As usual, a *real number* will be an element of the Baire space $\mathcal{N} = (\omega^\omega, \tau)$, where τ is the product topology, with the discrete topology on ω. Thus, the set $\mathbb{R}$ of real numbers is the set of all functions from ω into ω. Throughout this paper, we often talk in terms of generic filters instead of Boolean-valued models. Each way of talking can be routinely reinterpreted in the other.

Let $\mathbb{P}$ be a forcing notion. We say that $\dot{x}$ is a *simple* $\mathbb{P}$*-name* for a real number if:

i) The elements of $\dot{x}$ have the form $((n, \check{m}), p)$ with $p \in \mathbb{P}$ and $n, m \in \omega$, so that $p \Vdash_{\mathbb{P}} \dot{x}(\check{n}) = \check{m}$.
ii) For all $n \in \omega$, $\{p \in \mathbb{P} \mid \exists m \text{ such that } ((n, \check{m}), p) \in \dot{x}\}$ is a maximal antichain of $\mathbb{P}$.

For any forcing notion $\mathbb{P}$ and for all $\mathbb{P}$-names τ for a real, there exists a simple $\mathbb{P}$-name $\dot{x}$ such that $\Vdash_{\mathbb{P}} \tau = \dot{x}$. Hence, any $\mathbb{P}$-generic filter will interpret these two names in the same way.

Let $WF := \{x \in \omega^\omega \mid E_x \text{ is well-founded}\}$, where given $x \in \omega^\omega$, $E_x := \{(n,m) \in \omega \times \omega \mid x(\Gamma(n,m)) = 0\}$, with Γ some fixed recursive bijection between $\omega \times \omega$ and ω. Recall that WF is a complete Π^1_1 set (see [4]).

let T be a theory whose models naturally contain a submodel N of Peano Arithmetic. A model M of T is an *ω-model* if N^M is standard, i.e., it is isomorphic to ω. In this case, we naturally identify M with its isomorphic copy M' in which $N^{M'}$ is ω.

Stationary Tower Forcing, introduced by Woodin in the 1980's, will be used to prove some important facts about Ω-logic:

Definition 1.3. (cf.[6]) (Stationary Tower Forcing)

i) A set $a \neq \emptyset$ is *stationary* if for any function $F\colon [\cup a]^{<\omega} \to \cup a$, there exists $b \in a$ such that $F"[b]^{<\omega} \subseteq b$.
ii) Given a strongly inaccessible cardinal κ, we define the *Stationary Tower Forcing* notion: its set of conditions is

$$\mathbb{P}_{<\kappa} = \{a \in V_\kappa : a \text{ is stationary}\},$$

and the order is defined by:

$$a \leq b \text{ iff } \cup b \subseteq \cup a \text{ and } \{Z \cap (\cup b) \mid Z \in a\} \subseteq b.$$

Fact 1.4. Given $\gamma < \delta$ strongly inaccessible, $a = \mathcal{P}_{\omega_1}(V_\gamma) \in \mathbb{P}_{<\delta}$.

Proof: Given $F\colon [V_\gamma]^{<\omega} \to V_\gamma$, let $x \in [V_\gamma]^{<\omega}$ and let:

$$A_0 = x, \quad A_{n+1} = A_n \cup \{F(y) : y \in [A_n]^{<\omega}\}$$

Let $b = \bigcup_{n\in\omega} A_n$. So, $b \in \mathcal{P}_{\omega_1}(V_\gamma)$ and $F"[b]^{<\omega} \subseteq b$. □

Recall the large-cardinal notion of a *Woodin cardinal*:

Definition 1.5. ([10]) A cardinal δ is a *Woodin cardinal* if for every function $f\colon \delta \to \delta$ there exists $\kappa < \delta$ with $f"\kappa \subseteq \kappa$, and an elementary embedding $j\colon V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subseteq M$.

Theorem 1.6. (*cf.* [6]) *Suppose that δ is a Woodin cardinal and that $G \subseteq \mathbb{P}_{<\delta}$ is a V-generic filter. Then in $V[G]$ there is an elementary embedding $j\colon V \to M$, with M transitive, such that $V[G] \vDash M^{<\delta} \subseteq M$ and $j(\delta) = \delta$. Moreover, for all $a \in \mathbb{P}_{<\delta}$, $a \in G$ iff $j" \cup a \in j(a)$.*

1.2. **Definition of $\models_\Omega$ and invariance under forcing.**

Definition 1.7. ([17]) For $T \cup \{\varphi\} \subseteq Sent$, let

$$T \models_\Omega \varphi$$

if for all c.B.a. $\mathbb{B}$, and for all ordinals α, if $V_\alpha^{\mathbb{B}} \models T$ then $V_\alpha^{\mathbb{B}} \models \varphi$.

If $T \models_\Omega \varphi$, we say that φ is *Ω_T-valid*, or that φ is *Ω-valid from T*.

Observe that the complexity of the relation $T \models_\Omega \varphi$ is at most Π_2. Indeed, $T \models_\Omega \varphi$ iff

$$\forall \mathbb{B} \forall \alpha (\mathbb{B} \text{ a c.B.a.} \wedge \alpha \in On \rightarrow (V_\alpha^{\mathbb{B}} \models T \rightarrow V_\alpha^{\mathbb{B}} \models \varphi))$$

The displayed formula is Π_2, since to be a c.B.a. is Π_1 and the class function $\alpha \mapsto V_\alpha^{\mathbb{B}}$ is Δ_2 definable (i.e., both Σ_2 and Π_2 definable) in V with $\mathbb{B}$ as a parameter.

Clearly, if $T \models \varphi$ then $T \models_\Omega \varphi$. Observe, however, that the converse is not true. Indeed, we can easily find Ω_{ZFC}-valid sentences that are undecidable in first-order logic from ZFC, i.e., sentences φ such that $ZFC \not\models \varphi$ and $ZFC \not\models \neg\varphi$. For example, $CON(ZFC)$: For all $\alpha \in On$ and all c.B.a. $\mathbb{B}$, if $V_\alpha^{\mathbb{B}} \models ZFC$, then since $V_\alpha^{\mathbb{B}}$ is a standard model of ZFC, we have $V_\alpha^{\mathbb{B}} \models CON(ZFC)$.

Under large cardinals, the relation $\models_\Omega$ is absolute under forcing extensions:

Theorem 1.8. ([17]) *Suppose that there exists a proper class of Woodin cardinals. If $T \cup \{\varphi\} \subseteq Sent$, then for every forcing notion $\mathbb{P}$,*

$$T \models_\Omega \varphi \quad \text{iff} \quad V^{\mathbb{P}} \models \text{“}T \models_\Omega \varphi\text{”}$$

Proof: $\Rightarrow$) Let $\mathbb{P}$ be a poset. Suppose $\check{\beta}, \dot{\mathbb{Q}} \in V^{\mathbb{P}}$ are such that $V^{\mathbb{P}} \models$ “$V_{\check{\beta}}^{\dot{\mathbb{Q}}} \models T$”. By Corollary 1.2, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ “$V_{\check{\beta}} \models T$”. By hypothesis, $V^{\mathbb{P} * \dot{\mathbb{Q}}} \models$ “$V_{\check{\beta}} \models \varphi$”, and hence $V^{\mathbb{P}} \models$ “$V_{\check{\beta}}^{\dot{\mathbb{Q}}} \models \varphi$”.

$\Leftarrow$) Suppose $V^{\mathbb{P}} \models$ “$T \models_\Omega \varphi$”. Let $\mathbb{Q}$ be a forcing notion and $\alpha \in On$. Suppose that $V_\alpha^{\mathbb{Q}} \models T$ and G is a V-generic filter for $\mathbb{Q}$. Let $\kappa = |TC(\mathbb{P})|$, and let $\delta > \kappa, \alpha$ be a Woodin cardinal. Let

$$a = \{X \mid X \prec H_{\kappa^+} \text{ and } X \text{ countable}\}.$$

Notice that, by Fact 1.4, $a \in \mathbb{P}_{<\delta}^{V[G]}$. Let $\mathbb{P}_{<\delta}^{V[G]}(a)$ be the forcing $\mathbb{P}_{<\delta}^{V[G]}$ restricted to a.

Let $I \subseteq \mathbb{P}_{<\delta}^{V[G]}(a)$ be a $V[G]$-generic filter. In $V[G][I]$ there is an elementary embedding $j \colon V[G] \to M$ with M transitive such that:

i) $V[G][I] \models M^{<\delta} \subseteq M$,

ii) $(H_{\kappa^+})^V$ is countable in M and $j(\alpha) < \delta$. (See [6].)

$\mathbb{P} \in M$ and the set of dense subsets of $\mathbb{P}$ in V is a countable set in M, so in M there exists a V-generic filter $J \subseteq \mathbb{P}$. Then $V[J] \subseteq V[G][I]$ and for some poset $\mathbb{S} \in V[J]$, there is a $V[J]$-generic $K \subseteq \mathbb{S}$ such that $V[G][I] = V[J][K]$. Since by hypothesis, $V_\alpha^{\mathbb{Q}} \models T$, $V_\alpha^{V[G]} \models T$. Then

$$(V_{j(\alpha)})^M = (V_{j(\alpha)})^{V[G][I]} = (V_{j(\alpha)})^{V[J][K]} \models T.$$

Since $V^{\mathbb{P}} \vDash$ "$T \vDash_\Omega \varphi$", $(V_{j(\alpha)})^{V[J][K]} \vDash \varphi$. So $(V_{j(\alpha)})^{M} \vDash \varphi$, and therefore ${V_\alpha}^{V[G]} \vDash \varphi$. Thus, $V_\alpha^{\mathbb{Q}} \models \varphi$. $\square$

1.3. **Some properties of $\vDash_\Omega$.**

Lemma 1.9. *For every recursively enumerable (r.e.) set $T \cup \{\varphi\} \subseteq Sent$, the following are equivalent:*

i) $T \vDash_\Omega \varphi$.
ii) $\emptyset \vDash_\Omega$ "$T \vDash_\Omega \varphi$".

(Note that since T is r.e., "$T \vDash_\Omega \varphi$" can be written as a sentence in Sent. So, ii) makes sense.)

Proof: i) $\Rightarrow$ ii) Let $\alpha \in On$ and $\mathbb{B}$ a c.B.a. Suppose $\beta < \alpha$, and $\dot{\mathbb{Q}}$ is a c.B.a. in $V_\alpha^{\mathbb{B}}$ such that $V_\alpha^{\mathbb{B}} \vDash$ "$V_{\check{\beta}}^{\dot{\mathbb{Q}}} \vDash T$". Then $V_\beta^{\mathbb{B}*\dot{\mathbb{Q}}} \vDash T$. By i), $V_\beta^{\mathbb{B}*\dot{\mathbb{Q}}} \vDash \varphi$, and hence $V_\alpha^{\mathbb{B}} \vDash$ "$V_{\check{\beta}}^{\dot{\mathbb{Q}}} \vDash \varphi$".

ii) $\Rightarrow$ i) Suppose $\alpha \in On$, $\mathbb{B}$ is a c.B.a., and $V_\alpha^{\mathbb{B}} \models T$. Fix $\beta > \alpha$, β a limit ordinal. Since T is r.e., if $V_\beta^{\mathbb{B}} \models$ "$\psi \in T$", then $\psi \in T$, and therefore $V_\alpha^{\mathbb{B}} \models \psi$. Thus, $V_\beta^{\mathbb{B}} \models$ "$V_{\check{\alpha}} \models T$". By *ii)*, $V_\beta^{\mathbb{B}} \models$ "$T \models_\Omega \varphi$". Hence, $V_\beta^{\mathbb{B}} \models$ "$V_{\check{\alpha}} \models_\Omega \varphi$", and we have $V_\alpha^{\mathbb{B}} \models \varphi$. $\square$

Remarks 1.10. Suppose that ZFC is consistent. For iv) suppose, moreover, that it is consistent with ZFC that $V_\alpha^{\mathbb{B}} \models ZFC$, for some ordinal α and some c.B.a. $\mathbb{B}$. Then,

i) If φ is absolute for transitive sets, then $ZFC \vdash (\varphi \to \emptyset \vDash_\Omega \varphi)$.
ii) For some $\varphi \in Sent$, $ZFC \nvdash (\varphi \to (\emptyset \vDash_\Omega \varphi))$.
iii) For some $\varphi \in Sent$, $ZFC \nvdash ((ZFC \vDash_\Omega \varphi) \to \varphi)$.
iv) For some $\varphi \in Sent$, $ZFC \nvdash ((ZFC \vDash_\Omega$ "$ZFC \vDash_\Omega \varphi$"$) \to (ZFC \vDash_\Omega \varphi))$.

Proof: i) is clear. ii) holds for every sentence φ that can be forced to be true and false, for example CH.

iii) Let φ="$\exists\beta(V_\beta \vDash ZFC)$". Let M be a model of ZFC. If for every α and every $\mathbb{B}$ in M, $M_\alpha^{\mathbb{B}} \not\models ZFC$ (call this Case 1), then $M \vDash$ "$ZFC \vDash_\Omega \varphi$" $+ \neg\varphi$. Otherwise, let β be the least such that $M_\beta^{\mathbb{B}} \models ZFC$, for some $\mathbb{B}$. Then $M_\beta^{\mathbb{B}}$ is a model of ZFC, call it N, and has the property that for every α and every c.B.a. $\mathbb{C}$, $N_\alpha^{\mathbb{C}} \not\models ZFC$. So, we are back to Case 1.

iv) Consider the sentence φ="$\exists\beta\exists\gamma(\beta < \gamma \wedge V_\beta \vDash ZFC \wedge V_\gamma \vDash ZFC)$". Let M be a model of ZFC such that $M \models \exists\alpha\exists\mathbb{B}(V_\alpha^{\mathbb{B}} \models ZFC)$. If for every α and every c.B.a. $\mathbb{B}$, $M_\alpha^{\mathbb{B}} \not\models \varphi$ (call this Case 1), then $M \vDash (ZFC \vDash_\Omega$ "$ZFC \vDash_\Omega \varphi$"$) + \neg(ZFC \vDash_\Omega \varphi)$.

If for some α and $\mathbb{B}$, $M_\alpha^{\mathbb{B}} \models \varphi$, then let γ be the least ordinal such that $M_\gamma^{\mathbb{B}} \vDash ZFC + \exists\beta(V_\beta^{\mathbb{B}} \vDash ZFC)$. Let N be $M_\gamma^{\mathbb{B}}$. Then N has the property that for every α and every $\mathbb{C}$, $N_\alpha^{\mathbb{C}} \not\models \varphi$, and so we are back to Case 1. $\square$

Theorem 1.11 (Non-Compactness of $\vDash_\Omega$)**.** *There is $T \cup \{\varphi\} \subseteq Sent$ such that $T \vDash_\Omega \varphi$, but for all finite $S \subseteq T$, $S \nvDash_\Omega \varphi$.*

Proof: Let φ_0 be the sentence asserting: *There is a largest limit ordinal.*

For each $n \in \omega$, $n > 0$, let φ_n be the sentence asserting: *If α is the largest limit ordinal, then $\alpha + n$ exists.*

Finally, let φ be the sentence that asserts: *Every ordinal has a successor.*

Let $T = \{\varphi_n : n \in \omega\}$.

Then, $T \models_\Omega \varphi$. But if $S \subseteq T$ is finite, then $S \not\models_\Omega \varphi$. $\square$

With a bit more work we can show that Compactness of $\models_\Omega$ also fails for $T = ZFC$. Indeed, recall that by Gödel's Diagonalization, for each formula $\psi(x)$, with x the only free variable and ranging over natural numbers, there is a sentence φ such that $ZFC \vdash (\varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner))$, where $\ulcorner\varphi\urcorner$ is the term denoting the Gödel code of φ.

Theorem 1.12. *If ZFC is consistent, then there is a sentence φ such that $ZFC \models_\Omega \varphi$ but for all finite $S \subseteq ZFC$, $S \not\models_\Omega \varphi$.*

Proof: Let $\psi(x)$ be the formula:

$$x \text{ Gödel-codes a sentence } \varphi_x \wedge \forall S(S \textit{ a finite subset of } ZFC \rightarrow S \not\models_\Omega \varphi_x)$$

By Gödel's Diagonalization, there is a sentence φ such that $ZFC \vdash (\varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner))$. Let $T \subseteq ZFC$ be finite such that $T \vdash (\varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner))$. Let θ be the conjunction of the set of sentences of T. Then, $\emptyset \vdash \theta \rightarrow (\varphi \leftrightarrow \psi(\ulcorner\varphi\urcorner))$.

Claim. $ZFC \models_\Omega \varphi$.

Proof of Claim: Suppose not. Pick α and $\mathbb{B}$ such that $V_\alpha^\mathbb{B} \models ZFC + \neg\varphi$. So, there is $S \in V_\alpha^\mathbb{B}$ a finite set of sentences of ZFC such that $V_\alpha^\mathbb{B} \models$ "$S \models_\Omega \varphi$". Since $V_\alpha^\mathbb{B} \models ZFC$, by reflection, let $\beta < \alpha$ such that $V_\beta^\mathbb{B} \models S + \neg\varphi$. But since $V_\alpha^\mathbb{B} \models$ "$S \models_\Omega \varphi$", and $V_\beta^\mathbb{B} \models S$, we obtain $V_\beta^\mathbb{B} \models \varphi$, a contradiction. $\square$

Claim. If $S \subseteq ZFC$ is finite then $S \not\models_\Omega \varphi$.

Proof of Claim: Suppose there is $S \subseteq ZFC$ finite such that $S \models_\Omega \varphi$. By Lemma 1.9, $\emptyset \models_\Omega$ "$S \models_\Omega \varphi$". Let $\mathbb{B}$ be a c.B.a.. Since $ZFC \vdash \theta + S$ and $V^\mathbb{B} \models ZFC$, by reflection, let α be such that $V_\alpha^\mathbb{B} \models \theta + S$. Since $\emptyset \models_\Omega$ "$S \models_\Omega \varphi$", $V_\alpha^\mathbb{B} \models$ "$S \models_\Omega \varphi$", i.e., $V_\alpha^\mathbb{B} \models (\exists S)(S \text{ finite and } S \models_\Omega \varphi)$. Hence $V_\alpha^\mathbb{B} \models \neg\psi(\ulcorner\varphi\urcorner)$. But since $V_\alpha^\mathbb{B} \models \theta$, $V_\alpha^\mathbb{B} \models \neg\varphi$, contradicting the assumption that $S \models_\Omega \varphi$. $\square$

2. $\vdash_\Omega$

In order to define the Ω-provability relation $\vdash_\Omega$ (Definition 2.29), the syntactic relation associated to $\models_\Omega$, also introduced by W. H. Woodin, we need to recall some notions that will play an essential part in the definition. Along the way we will also prove some useful facts about these notions.

2.1. **Universally Baire sets of reals.**

The universally Baire sets of reals play the role of Ω-proofs in Ω-logic.

Recall that for an ordinal λ, a *tree* on $\omega \times \lambda$ is a set $T \subseteq \omega^{<\omega} \times \lambda^{<\omega}$ such that for all pairs $(s, t) \in T$, $lh(s) = lh(t)$ and $(s{\restriction}i, t{\restriction}i) \in T$ for each $i \in lh(s) \in \omega$. Given a tree on $\omega \times \lambda$, $p[T] = \{x \in \omega^\omega \mid \exists f \in \lambda^\omega (x, f) \in [T]\}$ is the *projection* of T, where $[T] = \{(x, f) \in \omega^\omega \times \lambda^\omega \mid \forall n \in \omega (x{\restriction}n, f{\restriction}n) \in T\}$.

Definition 2.1. ([2])

i) For a given cardinal κ, a set of reals A is *κ-universally Baire* (κ-uB) if there exist trees T and S on $\omega^{<\omega} \times \lambda^{<\omega}$, λ some ordinal, such that $A = p[T]$ and $p[T] = \omega^\omega \setminus p[S]$ in any forcing extension by a partial order of cardinality less than κ. We say that the trees T and S witness that A is κ-uB.

ii) $A \subseteq \mathbb{R}$ is *universally Baire* (uB) if it is κ-uB for each cardinal κ.

Proposition 2.2. ([2]). *For $A \subseteq \mathbb{R}$, the following are equivalent:*

i) *A is universally Baire.*

ii) *For every compact Hausdorff space X and every continuous function $f\colon X \to \mathbb{R}$, the set $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$ has the property of Baire, i.e., there exists an open set $O \subseteq X$ such that the symmetric difference $f^{-1}(A) \triangle O$ is meager.*

iii) *For every notion of forcing $\mathbb{P}$ there exist trees T and S on $\omega \times 2^{|\mathbb{P}|}$ such that $A = p[T] = \omega^\omega \setminus p[S]$ and $V^{\mathbb{P}} \models p[T] = \omega^\omega \setminus p[S]$. We say that the trees T and S witness that A is uB for $\mathbb{P}$.*

The following is a special case of the well-known fact that the well-foundedness of a given tree is absolute to all models of ZFC with the same ordinals.

Proposition 2.3. *Let T and S be trees on $\omega \times \kappa$, for some ordinal κ. Suppose that $p[T] \cap p[S] = \emptyset$. Then, in any forcing extension $V[G]$ we also have that $p[T]^{V[G]} \cap p[S]^{V[G]} = \emptyset$.*

Proof: Towards a contradiction, suppose that $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$, τ is a $\mathbb{P}$-name for a real, and $p \Vdash \tau \in p[T] \cap p[S]$.

Let $N \prec H(\lambda)$, λ a large enough regular cardinal, N countable and such that $p, \mathbb{P}, \tau, T, S \in N$. Let M be the transitive collapse of N, and let $\bar{p}, \bar{\mathbb{P}}, \bar{\tau}, \bar{T}$ and $\bar{S}$ be the transitive collapses of $p, \mathbb{P}, \tau, T$ and S, respectively. Thus, in M we have

$$\bar{p} \Vdash_{\bar{\mathbb{P}}} \bar{\tau} \in p[\bar{T}] \cap p[\bar{S}].$$

Let g be $\bar{\mathbb{P}}$-generic over M with $\bar{p} \in g$. So, in $M[g]$, we have

$$\bar{\tau}[g] \in p[\bar{T}] \cap p[\bar{S}].$$

Notice that $p[T \cap N] \subseteq p[T]$ and $p[S \cap N] \subseteq p[S]$. Moreover, $\bar{T} \cong T \cap N$ and $\bar{S} \cong S \cap N$. Hence, since the transitive collapse is the identity on natural numbers, $p[\bar{T}] \subseteq p[T]$ and $p[\bar{S}] \subseteq p[S]$, contradicting the fact that $p[T]$ and $p[S]$ are disjoint. □

Corollary 2.4. *Let T, T' and S be trees on $\omega \times \kappa$, for some ordinal κ. Suppose that $p[T] = p[T']$ and $p[S] = \omega^\omega \setminus p[T]$. If in $V[G]$, $p[S]^{V[G]} = \omega^\omega \setminus p[T]^{V[G]}$, then $p[T']^{V[G]} \subseteq p[T]^{V[G]}$.*

Remark 2.5. In general, under the same assumptions as in the Corollary 2.4, we cannot conclude that $p[T']^{V[G]} = p[T]^{V[G]}$. For instance, one can easily construct trees S and T on $\omega \times \omega$ such that $p[S]$ is the set of reals that take the value 0 infinitely often on the even elements of ω, and $p[T]$ is the set of reals that take the value 0 finitely often on the even elements of ω, and such that S and T will project to the sets with these definitions (and thus to complements) in all forcing extensions. Furthermore, if $\{x_\alpha : \alpha < 2^\omega\}$ is the set of reals (in the ground model) that take the value 0 finitely often on the even elements of ω, and T' is the tree consisting all pairs (a, b) where b is a finite constant sequence with some fixed value $\alpha < 2^\omega$ and a is $x_\alpha \restriction |b|$, then

$p[T] = p[T']$ in the ground model, but $p[T] \neq p[T']$ in any forcing extension that adds a real.

By Corollary 2.4, if $A \subseteq \mathbb{R}$ is κ-uB in a model N of ZFC, witnessed by trees T and S, and $N[G]$ is an extension of N by a forcing notion of cardinality less than κ, then $A_G := p[T]^{N[G]}$ is equal to the set of reals in $N[G]$ which are in the projection (in $N[G]$) of some tree in N witnessing that A is κ-uB. Therefore, given $A \subseteq \mathbb{R}$ a uB set, A has a *canonical interpretation* A_G in any set forcing extension $V[G]$ of V, namely:

$$A_G = \bigcup \{p[T]^{V[G]} \mid T \in V \text{ and } A = p[T]^V\}.$$

Thus, if $\mathbb{P}$ is a forcing notion and A is uB for $\mathbb{P}$, witnessed by trees T, S, and also by trees T', S', then in any $\mathbb{P}$-generic extension $V[G]$, $p[T] = p[T'] = A_G$.

Remark 2.6. It is clear from Proposition 2.2 (iii) that a set $A \subseteq \mathbb{R}$ is uB iff for every c.B.a. $\mathbb{B}$, $V^{\mathbb{B}} \models$ "$A_{\dot{G}}$ is uB".

Theorem 2.7. ([2]) *i) Every analytic set, and therefore every coanalytic set, is universally Baire.*

ii) Every $\mathbf{\Sigma}^1_2$ set of reals is uB iff for every set x, $x^\sharp$ exists.

2.2. *A*-closed models.

Let us now define the notion of *A-closed set*, which will be also fundamental for the definition of the Ω-provability relation $\vdash_\Omega$.

Definition 2.8. ([12]) Given a uB set $A \subseteq \mathbb{R}$, a transitive $\in$-model M of (a fragment of) *ZFC* is *A-closed* if for all posets $\mathbb{P} \in M$ and all V-generic filters $G \subseteq \mathbb{P}$,

$$V[G] \models M[G] \cap A_G \in M[G]$$

(i.e., $\Vdash_{\mathbb{P}}$ "$M[\dot{G}] \cap A_{\dot{G}} \in M[\dot{G}]$", where $\dot{G}$ is the standard $\mathbb{P}$-name for the generic filter).

Woodin has given several other definitions of A-closure, but the next proposition shows they are equivalent.

Proposition 2.9. *Given a uB set $A \subseteq \mathbb{R}$ and a transitive model M of ZFC, the following are equivalent:*

a) *M is A-closed.*

b) *For all infinite $\gamma \in M \cap On$, for all $G \subseteq Coll(\omega, \gamma)$ V-generic,*

$$V[G] \models M[G] \cap A_G \in M[G].$$

c) *For all posets $\mathbb{P} \in M$ and all $\tau \in M^{\mathbb{P}}$, $\{p \in \mathbb{P} \mid p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}\} \in M$.*

d) *For all infinite $\gamma \in M \cap On$ and all $\tau \in M^{Coll(\omega,\gamma)}$,*

$$\{p \in Coll(\omega, \gamma) \mid p \Vdash^V_{Coll(\omega,\gamma)} \tau \in A_{\dot{G}}\} \in M.$$

e) *For all posets $\mathbb{P} \in M$,*

$$\{(\tau, p) \mid \tau \in M \text{ a simple } \mathbb{P}\text{-name for a real}, p \in \mathbb{P} \text{ and } p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}\} \in M.$$

f) *For all posets $\mathbb{P}_\gamma = Coll(\omega, \gamma)$, with $\gamma \in M \cap On$ infinite,*

$$\{(\tau, p) \mid \tau \in M \text{ a simple } \mathbb{P}_\gamma\text{-name for a real}, p \in \mathbb{P}_\gamma \text{ and } p \Vdash^V_{\mathbb{P}_\gamma} \tau \in A_{\dot{G}}\} \in M.$$

Proof: Observe that the implications (a)⇒ (b), (c)⇒ (d) and (e)⇒ (f) are immediate.

(b)⇒ (d): Fix $\gamma \in M \cap On$. Since $M \models ZFC$ and M is transitive, $Coll(\omega,\gamma) \in M$. Let $\tau \in M^{Coll(\omega,\gamma)}$. By (b), there exist $p \in Coll(\omega,\gamma)$ and $\sigma_0 \in M^{Coll(\omega,\gamma)}$ such that $p \Vdash^V_{Coll(\omega,\gamma)} M[\dot{G}] \cap A_{\dot{G}} = \sigma_0$. Since $Coll(\omega,\gamma)$ is homogeneous, we can replace σ_0 with a $Coll(\omega,\gamma)$-name σ in M such that every condition in $Coll(\omega,\gamma)$ forces (in V) that $M[\dot{G}] \cap A_{\dot{G}} = \sigma$. Thus, for every $q \in Coll(\omega,\gamma)$,

$$q \Vdash^V_{Coll(\omega,\gamma)} \tau \in \sigma \text{ iff } q \Vdash^V_{Coll(\omega,\gamma)} \tau \in A_{\dot{G}}.$$

Hence, since $\{Coll(\omega,\gamma), \tau, \sigma\} \subseteq M$ and M is transitive, by absoluteness,

$$\begin{aligned}\{p \in \mathbb{P} \mid p \Vdash^V_{Coll(\omega,\gamma)} \tau \in A_{\dot{G}}\} &= \{p \in \mathbb{P} \mid p \Vdash^V_{Coll(\omega,\gamma)} \tau \in \sigma\} \\ &= \{p \in \mathbb{P} \mid p \Vdash^M_{Coll(\omega,\gamma)} \tau \in \sigma\} \in M.\end{aligned}$$

(d)⇒ (c): Fix a poset $\mathbb{P}$ in M and $\tau \in M^{\mathbb{P}}$. We may assume that τ is a simple $\mathbb{P}$-name for a real. Let $\gamma = |\mathbb{P}|^M$, and let τ^* be the simple $\mathbb{P} \times Coll(\omega,\gamma)$-name defined by letting $(\check{(m,n)}, \langle p,q\rangle) \in \tau^*$ if and only if $(\check{(m,n)}, p)$ is in τ. Then since $\mathbb{P} \times Coll(\omega,\gamma)$ has a dense set isomorphic to $Coll(\omega,\gamma)$, by (d), $\{(p,q) \in \mathbb{P} \times Coll(\omega,\gamma) \mid (p,q) \Vdash^V_{\mathbb{P}\times Coll(\omega,\gamma)} \tau^* \in A_{\dot{G}}\} \in M$. Since for all $(p,q) \in \mathbb{P}\times Coll(\omega,\gamma)$, $(p,q) \Vdash^V_{\mathbb{P}\times Coll(\omega,\gamma)} \tau^* \in A_{\dot{G}}$ if and only if $p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}$, the conclusion of (c) follows.

(e) ⇒ (a) (similarly for (f)⇒ (b)): Fix a poset $\mathbb{P} \in M$ and suppose $G \subseteq \mathbb{P}$ is V-generic. Let

$$\sigma = \{(\tau, p) \mid \tau \in M \text{ a simple } \mathbb{P}\text{-name for a real}, p \in \mathbb{P} \text{ and } p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}\}.$$

By (e), $\sigma \in M$. Hence $\sigma \in M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$ and $i_G[\sigma] \in M[G]$.

Claim. $i_G[\sigma] = A_G \cap M[G]$.

Proof of Claim: Suppose $r \in i_G[\sigma]$. Let $p \in G \subseteq \mathbb{P}$ be such that $(\dot{r}, p) \in \sigma$ and $i_G[\dot{r}] = r$. Thus $\dot{r}$ is a simple $\mathbb{P}$-name in M for a real and $p \Vdash^V_{\mathbb{P}} \dot{r} \in A_{\dot{G}}$. Hence $r \in A_G \cap M[G]$.

Suppose now $r \in A_G \cap M[G]$. Let $p \in G$ and $\dot{r} \in M^{\mathbb{P}}$ be such that $p \Vdash^V_{\mathbb{P}} \dot{r} \in A_{\dot{G}}$. Let τ be a simple $\mathbb{P}$-name for a real in M such that $p \Vdash^V_{\mathbb{P}} \tau = \dot{r}$. Then $(\tau, p) \in \sigma$ and therefore $r \in i_G[\sigma]$. □

(d) ⇒ (f) : Fix $\gamma \in M \cap On$. Let $\mathbb{P} = Coll(\omega,\gamma)$ and $\mathbb{P}' = Coll(\omega, 2^{|\gamma|})$. Let $\langle \tau_\alpha \mid \alpha < 2^{|\gamma|}\rangle \in M$ be an enumeration of all the simple $\mathbb{P}$-names in M for reals. Let $\pi \colon \mathbb{P} \times \mathbb{P}' \to \mathbb{P}'$ be an order-preserving bijection. Define a simple $\mathbb{P} \times \mathbb{P}'$-name σ as follows:

$$\sigma = \{(\check{(i,j)}, (p,q)) \mid \exists \alpha < 2^{|\gamma|} \text{ such that } q(0) = \alpha \text{ and } (\check{(i,j)}, p) \in \tau_\alpha\}$$

Let σ^* be the simple $\mathbb{P}'$-name $\{(\check{(i,j)}, \pi(p,q)) \mid (\check{(i,j)}, (p,q)) \in \sigma\}$.
By (d), $X = \{q \in \mathbb{P}' \mid q \Vdash^V_{\mathbb{P}'} \sigma^* \in A_{\dot{G}}\} \in M$.
Hence,

$$\begin{aligned}Z =& \{(p,q) \in \mathbb{P} \times \mathbb{P}' \mid \pi(p,q) \in X\} = \{(p,q) \in \mathbb{P} \times \mathbb{P}' \mid \pi(p,q) \Vdash^V_{\mathbb{P}'} \sigma^* \in A_{\dot{G}}\} \\ =& \{(p,q) \in \mathbb{P} \times \mathbb{P}' \mid (p,q) \Vdash^V_{\mathbb{P}\times\mathbb{P}'} \sigma \in A_{\dot{G}\times\dot{H}}\} \in M.\end{aligned}$$

Let

$$Y = \{(\tau, p) \mid \exists\alpha < 2^{|\gamma|} \text{ such that } \tau = \tau_\alpha \text{ and } (p, (0,\alpha)) \in Z\}.$$

Since $Z \in M$, $Y \in M$. For $\tau \in M^{\mathbb{P}}$, let $\bar{\tau}$ be the corresponding $\mathbb{P} \times \mathbb{P}'$-name which depends only on the first coordinate. In particular, for each $\alpha < 2^{|\gamma|}$, since $\tau_\alpha \in M^{\mathbb{P}}$, for all $(p, q) \in \mathbb{P} \times \mathbb{P}'$,

$$p \Vdash^V_{\mathbb{P}} (\check{i,j}) \in \tau_\alpha \text{ iff } (p,q) \Vdash^V_{\mathbb{P}\times\mathbb{P}'} (\check{i,j}) \in \bar{\tau}_\alpha.$$

Claim. For each $\alpha < 2^{|\gamma|}$, for all $p \in \mathbb{P}$, $(p, (0,\alpha)) \Vdash^V_{\mathbb{P}\times\mathbb{P}'} \sigma = \bar{\tau}_\alpha$.

Proof of Claim: Let $G = G_1 \times G_2 \subseteq \mathbb{P}\times\mathbb{P}'$ be V-generic such that $(p, (0,\alpha)) \in G$. We check that $i_G[\sigma] = i_G[\bar{\tau}_\alpha]$: If $(i,j) \in i_G[\sigma]$, then for some $(r,s) \in G$, $((\check{i,j}), (r,s)) \in \sigma$, $s(0) = \beta$ for some $\beta < 2^{|\gamma|}$ and $r \Vdash^V_{\mathbb{P}} (\check{i,j}) \in \tau_\beta$. Since $(r,s), (p,(0,\alpha)) \in G$, $\alpha = \beta$ and $(i,j) \in i_G[\bar{\tau}_\alpha]$.

If $(i,j) \in i_G[\bar{\tau}_\alpha]$, let $(r,s) \leq (p,(0,\alpha))$ in G be such that $(r,s) \Vdash^V_{\mathbb{P}\times\mathbb{P}'} (\check{i,j}) \in \bar{\tau}_\alpha$. Then $r \Vdash^V_{\mathbb{P}} (\check{i,j}) \in \tau_\alpha$. Moreover, since $s \leq (0,\alpha)$, $s(0) = \alpha$. Hence, $((\check{i,j}), (r,(0,\alpha))) \in \sigma$ and $(r,(0,\alpha)) \Vdash^V_{\mathbb{P}\times\mathbb{P}'} (\check{i,j}) \in \sigma$. Since $(r,(0,\alpha)) \geq (r,s)$, $(r,(0,\alpha)) \in G$ and $(i,j) \in i_G[\sigma]$. □

Moreover, given $p \in \mathbb{P}$, and τ a simple $\mathbb{P}$-name in M,

$$\begin{aligned}(\tau,p) \in Y \text{ iff } & \exists\alpha < 2^{|\gamma|} \text{ such that } \tau = \tau_\alpha \text{ and } (p,(0,\alpha)) \Vdash^V_{\mathbb{P}\times\mathbb{P}'} \sigma \in A_{\dot{G}\times\dot{H}} \\ \text{iff } & \exists\alpha < 2^{|\gamma|} \text{ such that } \tau = \tau_\alpha \text{ and } p \Vdash^V_{\mathbb{P}} \tau_\alpha \in A_{\dot{G}} \\ \text{iff } & p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}.\end{aligned}$$

Hence,

$$Y = \{(\tau,p) \mid \tau \in M \text{ a simple } \mathbb{P}\text{-name for a real}, p \in \mathbb{P} \text{ and } p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}\}.$$

(f) $\Rightarrow$ (e) : Fix $\mathbb{P} \in M$. Let $\gamma = |\mathbb{P}|^M$ and $\mathbb{P}_\gamma = Coll(\omega,\gamma)$. Let $X =$

$$\{(\tau,p) \mid \tau \in M \text{ a simple } \mathbb{P}_\gamma\text{-name for a real, } p \in \mathbb{P}_\gamma \text{ and } p \Vdash^V_{\mathbb{P}_\gamma} \tau \in A_{\dot{G}}\}.$$

By f), $X \in M$. In M, let e be a complete embedding of $\mathbb{P}$ into $Coll(\omega,\gamma)$. As before, e extends naturally to an embedding $e^* \colon M^{\mathbb{P}} \to M^{Coll(\omega,\gamma)}$ in M. Let

$$Y = \{(\tau,p) \mid \tau \in M \text{ a simple } \mathbb{P}\text{-name for a real, } p \in \mathbb{P} \text{ and } p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}\}.$$

So,

$$Y = \{(\tau,p) \mid \tau \in M \text{ a simple } \mathbb{P}\text{-name for a real, } p \in \mathbb{P} \text{ and } (e^*(\tau), e(p)) \in X\}.$$

Thus, $Y \in M$. □

For M countable, the notion of A-closure has a simpler formulation, as shown in Proposition 2.11 below.

Lemma 2.10. *Suppose $A \subseteq \mathbb{R}$ is uB and M is an A-closed c.t.m. of ZFC. Let α be such that M is countable and A-closed in V_α. Suppose $X \prec V_\alpha$ is countable with $\{M, A, S, T\} \in X$, where T and S are trees witnessing that A is ω_1-uB, and N is the transitive collapse of X. Then, for every forcing notion $\mathbb{P} \in M$ and every N-generic filter $g \subseteq \mathbb{P}$, $M[g] \cap A \in M[g]$.*

Proof: Let π be the transitive collapsing function on X. So, $N = \pi(X)$. Let $\pi(S) = \bar{S}$ and $\pi(T) = \bar{T}$. Observe that $\pi(M) = M$ and $\pi(A) = A \cap X = A \cap N$. Fix $g \subseteq \mathbb{P} \in M$ N-generic. Since $p[\bar{T}] \subseteq p[T] = A$, writing $(A_g)^{N[g]}$ for $(\pi(A)_g)^{N[g]}$, we have:

$$(A_g)^{N[g]} = (p[\bar{T}])^{N[g]} \subseteq N[g] \cap A$$

and since $p[\bar{S}] \subseteq p[S] = \omega^\omega \setminus A$,

$$N[g] \cap A \subseteq (p[\bar{T}])^{N[g]}.$$

Hence $(A_g)^{N[g]} = N[g] \cap A$. Since M is A-closed in N, $M[g] \cap (A_g)^{N[g]} \in M[g]$. Hence, $M[g] \cap A = M[g] \cap N[g] \cap A = M[g] \cap (A_g)^{N[g]} \in M[g]$. □

If M is a countable transitive model and $\mathbb{P}$ is a partial order in M, we say that a set $\mathcal{G}$ of M-generic filters $g \subset \mathbb{P}$ is *comeager* if there exists a countable set $\mathcal{D}$ of dense subsets of $\mathbb{P}$ (not necessarily in M) such that $\mathcal{G}$ contains the set of M-generic filters that intersect every member of $\mathcal{D}$.

Notice that if $\mathcal{G}$ is comeager, then its complement in the set of all M-generic filters is not comeager. For suppose $\mathcal{D}$ and $\mathcal{D}'$ witness the comeagerness of $\mathcal{G}$ and its complement, respectively. Then, since $\mathcal{D} \cup \mathcal{D}'$ is countable, there is an M-generic filter G that intersects all dense sets in $\mathcal{D} \cup \mathcal{D}'$. But then G would belong to both $\mathcal{G}$ and its complement, which is impossible.

The following provides, in the case of a c.t.m. M, yet another characterization of M being A-closed, in addition to Proposition 2.9.

Proposition 2.11. *Given A a uB set and M a c.t.m. of ZFC, the following are equivalent:*

i) *M is A-closed.*
ii) *for all $\mathbb{P} \in M$, the set of M-generic filters $g \subset \mathbb{P}$ such that*

$$M[g] \cap A \in M[g]$$

is comeager.

Proof: *i)* $\Rightarrow$ *ii)* Let $\mathbb{P} \in M$. Let N be as in Lemma 2.10. Since N is countable, there are countably many dense sets of $\mathbb{P}$ in N. Let $\mathcal{D} = \{D_i : i \in \omega\}$ be this collection. Let $g \subseteq \mathbb{P}$ be an $(M \cup \mathcal{D})$-generic filter. Since g intersects each dense set in N, g is N-generic and by Lemma 2.10, $M[g] \cap A \in M[g]$.

ii) $\Rightarrow$ *i)* Let $\mathbb{P} \in M$. Towards a contradiction, let $p \in \mathbb{P}$ be such that $p \Vdash_{\mathbb{P}} M[\dot{G}] \cap A_{\dot{G}} \notin M[\dot{G}]$. By ii), let $\mathcal{D} = \{D_i : i \in \omega\}$ be a collection of dense subsets of $\mathbb{P}$ such that for all $(M \cup \mathcal{D})$-generic g, $M[g] \cap A \in M[g]$. Let V_α, α a large-enough uncountable regular cardinal, be such that $M, A, \mathcal{D} \in V_\alpha$. Let T, S be trees witnessing that A is ω_1-uB in V_α. Let $X \prec V_\alpha$ be countable with $\{\mathcal{D}, M, A, T, S\} \in X$ and let N be the transitive collapse of X. Let g be N-generic such that $p \in g$. By elementarity, $p \Vdash^N_{\mathbb{P}} M[\dot{G}] \cap A_{\dot{G}} \notin M[\dot{G}]$. Hence, $M[g] \cap A = M[g] \cap (A_g)^{N[g]} \notin M[g]$. But this contradicts ii), since g is $(M \cup \mathcal{D})$-generic. □

Corollary 2.12. *If M is a c.t.m. of ZFC and A is a uB set, then "M is A-closed" is correctly computed in $L(A, \mathbb{R})$.*

Proof: The next sentence is true in V iff it is true in $L(A,\mathbb{R})$ and says that M is A-closed:

$$\varphi(A,M) :=(\forall \mathbb{P}\in M)(\exists \langle D_i : i\in\omega\rangle)[D_i\subseteq \mathbb{P} \text{ dense} \wedge (\forall g)(g\subseteq \mathbb{P})((g \text{ a filter } M\text{-generic} \wedge (\forall i\in\omega)(g\cap D_i\neq\emptyset))\rightarrow M[g]\cap A\in M[g])]. \qquad \square$$

The following alternate form of Proposition 2.11 is sometimes useful.

Lemma 2.13. *Given a uB set $A\subseteq\mathbb{R}$, M a c.t.m. of ZFC, $\mathbb{P}\in M$ a poset, $p\in\mathbb{P}$, and τ a $\mathbb{P}$-name in M for a real, the following are equivalent:*

i) $p\Vdash^V \tau\in A_{\dot G}$.
ii) *The set of M-generic filters $g\subseteq\mathbb{P}$ such that $p\in g$ and $i_g[\tau]\in A$ is comeager.*

Proof: i) $\Rightarrow$ ii) Let T,S be witnesses for A being ω_1-uB, $A=p[T]$, $\omega^\omega\setminus A = p[S]$. There exists $\dot z$ such that for all $i\in\omega$, $p\Vdash^V_{\mathbb{P}} (\tau\restriction i, \dot z\restriction i)\in\check T$.
Let $\{D_i \mid i<\omega\}$ be such that D_i decides $\dot z(i)$, $i\in\omega$, i.e.,

$$D_i=\{q\in\mathbb{P}\mid q\Vdash^V \text{“}\dot z(i)=k\text{”, for some } k\}.$$

For all i, D_i is a dense subset of $\mathbb{P}$. Then if $g\subseteq\mathbb{P}$ is M-generic with $p\in g$ and $g\cap D_i\neq\emptyset$ for every $i\in\omega$, g decides $\dot z(i)$ and for all $i\in\omega$, $(i_g[\tau]\restriction i, i_g[\dot z]\restriction i)\in T$. So $i_g[\tau]\in p[T]=A$.

ii) $\Rightarrow$ i) Let V_α, α a large enough uncountable cardinal, be such that ii) holds in V_α. Let T,S be trees witnessing A is ω_1-uB in V_α. Let $X\prec V_\alpha$ be countable with $\{M,A,T,S\}\in X$ and let N be the transitive collapse of X. Observe that $\pi(A)=A\cap N$ and $\pi(M)=M$, hence $\pi(\mathbb{P})=\mathbb{P}$ and $\pi(p)=p$. Let $\pi(S)=\bar S$ and $\pi(T)=\bar T$. By elementarity, there is in N a collection $\{D_i : i\in\omega\}$ of dense subsets of $\mathbb{P}$ such that for all M-generic filters $g\subseteq\mathbb{P}$, if $p\in g$ and $g\cap D_i\neq\emptyset$ for all $i\in\omega$, then $i_g[\tau]\in A\cap N$. Pick any G N-generic with $p\in G$. Since $G\cap D_i\neq\emptyset$ for all i and G is M-generic, by Lemma 2.10, $i_G[\tau]\in A\cap M[G]=(A_G)^{N[G]}\cap M[G]$, so $N[G]\models i_G[\tau]\in A_G$. Since G was an arbitrary N-generic filter containing p, $p\Vdash^N \tau\in A_{\dot G}$. By elementarity, $p\Vdash^V\tau\in A_{\dot G}$. $\square$

For a c.t.m. M, being A-closed is preserved by *most* generic extensions, i.e., by a comeager set of M-generic filters, for any partial order in M.

Proposition 2.14. *For every uB set A, if M is an A-closed c.t.m. and $\mathbb{P}$ is a partial order in M, then the set of M-generic filters $g\subset\mathbb{P}$ such that $M[g]$ is A-closed is comeager.*

Proof: By Proposition 2.11, for each $\mathbb{P}$-name τ in M for a partial order there is a countable set $\mathcal{E}_\tau$ of dense subsets of $\mathbb{P}*\tau$ such that for every $(M\cup\mathcal{E}_\tau)$-generic forcing extension N of M by $\mathbb{P}*\tau$, $N\cap A\in N$. For each $\mathbb{P}$-name σ for a condition in τ and each $E\in\mathcal{E}_\tau$ there is a dense set $D(\tau,E,\sigma)$ of conditions $p\in\mathbb{P}$ for which there is some $\mathbb{P}$-name ρ for a condition in τ such that $(p,\rho)\in E$ and $p\Vdash_{\mathbb{P}}\rho\leq_\tau\sigma$. Let $\mathcal{D}$ be the set of all such sets $D(\tau,E,\sigma)$.

Now suppose that $M[g]$ is a $\mathcal{D}$-generic extension of M by $\mathbb{P}$. Let $\mathbb{Q}$ be a poset in $M[g]$. Then $\mathbb{Q}=i_g[\tau]$ for some $\mathbb{P}$-name $\tau\in M$. Since g is $\mathcal{D}$-generic, for each $E\in\mathcal{E}_\tau$, the set $E^*=\{i_g[\rho] : \exists p\in g \text{ such that } (p,\rho)\in E\}$ is dense in $\mathbb{Q}$. Let $\mathcal{E}'$ be the set of these E^*'s, and let $h\subset\mathbb{Q}$ be a $(M[g]\cup\mathcal{E}')$-generic filter. Then

$$g*h=\{(p,\sigma)\in\mathbb{P}*\tau : p\in g \text{ and } i_g[\sigma]\in h\}$$

is an $(M \cup \mathcal{E}_\tau)$-generic filter, and so $M[g][h] \cap A \in M[g][h]$. □

Let ZFC^* be a finite fragment of ZFC. Proposition 2.18 below shows that for any uB set A, there is an A-closed c.t.m. M which is a model of ZFC^*. But first let us prove the following:

Lemma 2.15. *If $A \subseteq \mathbb{R}$ is uB and κ is such that $V_\kappa \models ZFC$, then A is uB in V_κ.*

Proof: Let us see that for each poset $\mathbb{P}$ in V_κ there are trees $T, S \in V_\kappa$ such that $p[T] = A$ and $p[S] = \omega^\omega \setminus A$, and for all $\mathbb{P}$-generic filters G over V_κ, $V_\kappa[G] \models p[T] = \omega^\omega \setminus p[S]$. So fix $\mathbb{P} \in V_\kappa$ and suppose S, T witness A is uB for $\mathbb{P}$ in V. Let τ be a $\mathbb{P}$-name in V_κ for the set of reals of the $\mathbb{P}$-extension. Let θ be a large-enough regular cardinal such that $S, T \in H(\theta)$. Take $X \prec H(\theta)$ such that $|X| < \kappa$ and $\{S, T\} \cup \tau \cup A \subseteq X$. Let M be the image of X by the transitive collapse π. Then $\pi(S), \pi(T) \in V_\kappa$ and they witness the universal Baireness of A for $\mathbb{P}$ in V_κ, since $p[T] = p[\pi(T)]$ and $p[S] = p[\pi(S)]$. □

The notion of strong A-closure defined below is not standard. However, as we shall see in Section 2.5 below, the syntactic relation for Ω-logic (Definition 2.29) would not change if strong A-closure is used in place of A-closure.

Definition 2.16. Given $A \subseteq \mathbb{R}$, a transitive $\in$-model M of (a fragment of) ZFC is *strongly A-closed* if for all posets $\mathbb{P} \in M$ and all M-generic $G \subseteq \mathbb{P}$, $M[G] \cap A \in M[G]$.

Notice that by Lemma 2.11, for c.t.m.'s, if A is a uB set, then strong A-closure implies A-closure. Note also that if M is strongly A-closed, $\mathbb{P} \in M$, and $G \subseteq \mathbb{P}$ is M-generic, then $M[G]$ is also strongly A-closed.

Example 2.17. Let M be a c.t.m. of ZFC and let A be a uB set such that M is not A-closed. Then if c is a Cohen real over M, then M is $(\{c\} \times A)$-closed but not strongly $(\{c\} \times A)$-closed. Furthermore, $M[c]$ is not $(\{c\} \times A)$-closed.

Proposition 2.18. *Suppose $A \subseteq \mathbb{R}$ is uB, and κ is such that $V_\kappa \models ZFC$. Then every forcing extension of the transitive collapse of any countable elementary submodel of V_κ containing A is strongly A-closed. In particular, the transitive collapse of any countable elementary submodel of V_κ containing A is A-closed.*

Proof: By Lemma 2.15, A is uB in V_κ. Let $X \prec V_\kappa$ be countable such that $A \in X$. Let M be the image of X by the transitive collapse π. We want to see that any forcing extension of M is strongly A-closed. It suffices to see that M is strongly A-closed. Let $\mathbb{P} \in M$ and let $g \subseteq \mathbb{P}$ be an M-generic filter.

Let S and T be trees in X witnessing the universal Baireness of A for $\pi^{-1}(\mathbb{P})$. Then $\pi(S) = \bar{S}$ and $\pi(T) = \bar{T}$ are trees in M witnessing the universal Baireness of $A \cap M$ for $\mathbb{P}$. If σ is a $\mathbb{P}$-name for a real in M, in $M[g]$, $i_g[\sigma]$ is in $p[\bar{S}]$ or in $p[\bar{T}]$ and not in both, by elementarity of the collapsing map. Thus, since $p[\bar{S}] \subseteq p[S]$ and $p[\bar{T}] \subseteq p[T]$,

$$i_g[\sigma] \in A \text{ iff } i_g[\sigma] \in (p[\bar{T}])^{M[g]}.$$

Hence, $M[g] \cap A = (p[\bar{T}])^{M[g]} \in M[g]$, and M is strongly A-closed. □

Recall the following result of Woodin:

Theorem 2.19 (cf.[7]). *Suppose there is a proper class of Woodin cardinals. Then for every uB set of reals A and every forcing notion $\mathbb{P}$, if $G \subseteq \mathbb{P}$ is a V-generic filter, then in $V[G]$ there is an elementary embedding from $L(A, \mathbb{R}^V)$ into $L(A_G, \mathbb{R}^{V[G]})$ sending A to A_G.*

Corollary 2.20. *Suppose there is a proper class of Woodin cardinals. Then for every uB set of reals A and every forcing notion $\mathbb{P}$, if $G \subseteq \mathbb{P}$ is V-generic, then in $V[G]$, for every formula $\varphi(x,y)$ and every $r \in \mathbb{R}^V$,*

$$L(A, \mathbb{R}^V) \models \varphi(A, r) \quad \textit{iff} \quad L(A_G, R^{V[G]}) \models \varphi(A_G, r).$$

In particular, if $\varphi(x,y)$ is the formula that defines A-closure, as in Corollary 2.12, it follows that a c.t.m. M is A-closed iff for every (some) generic extension $V[G]$ of V, M is A_G-closed in $V[G]$.

The notion of A-closed model makes sense even for non-well-founded ω-models, i.e., given a uB set $A \subseteq \mathbb{R}$, an ω-model M of (a fragment of) ZFC is *A-closed* if for all posets $\mathbb{P} \in M$, for all $G \subseteq \mathbb{P}$ V-generic,

$$V[G] \models M[G] \cap A_G \in M[G]$$

i.e., $\Vdash_{\mathbb{P}}$ "$M[\dot{G}] \cap A_{\dot{G}} \in M[\dot{G}]$", where $\dot{G}$ is the standard $\mathbb{P}$-name for the generic filter.

However, let us see that the notion of A-closed set is a natural generalization of wellfoundedness.

Lemma 2.21. *Let ZFC^* be ZF minus the Powerset axiom. Suppose N is an ω-model of ZFC^* such that $WF \cap N \in N$. Then for all $x \in \omega^\omega \cap N$, $x \in WF$ iff $x \in WF^N$.*

Proof: $\Rightarrow$) By the downward absoluteness of Π^1_1 formulas for ω-models.
$\Leftarrow$) Suppose $x \in \omega^\omega \cap N$, $x \in WF^N$ and $x \notin WF$. For each n, let $E_x{\restriction}n\{m|mE_xn\}$, and let x_n be a real coding $E_x{\restriction}n$. Since $N \models$ "E_x is wellfounded" and $WF \cap N \in N$, there is a $n_0 \in \omega$ such that $x_{n_0} \notin WF$ but for all mE_xn_0, $x_m \in WF$. Since $E_x{\restriction}n_0$ is illfounded, there is an mE_xn_0 such that $E_x{\restriction}m$ is illfounded, giving a contradiction. □

Lemma 2.22. *Every ω-model of ZFC which is WF-closed is well-founded.*

Proof: Suppose (M,E) is a non well-founded WF-closed ω-model of ZFC. Let γ be an "ordinal" of M which is illfounded in V, let G be M-generic for a partial order in M making γ countable and let x be a real in $M[G]$ coding a wellordering of ω of ordertype γ. Then $x \in WF^{M[G]} \setminus WF$, which by Lemma 2.21 implies that $M[G] \cap WF \notin M[G]$. Since M is WF-closed, by the previous Lemma, $x \notin WF^{M[G]}$. So $E_x \in M[G]$ and is not well-founded. Hence $M[G] \not\models$ "Foundation", contradicting the fact that $M \models$ "Foundation" and $M[G]$ is a forcing extension of M. □

Theorem 2.23. *For every ω-model of ZFC, (M,E), the following are equivalent:*

i) *(M,E) is well-founded.*
ii) *(M,E) is A-closed for each Π^1_1 set A.*

Proof: i) $\Rightarrow$ *ii)* Suppose (M, E) is an ω-model of ZFC which is well-founded. Fix $A \subseteq \mathbb{R}$ a Π^1_1 set. Let $\mathbb{P} \in M$ and let H be a $\mathbb{P}$-generic over V.

Let $(N, \in)$ be the transitive collapse of (M, E), and let $G = \pi[H]$. Since $\pi(\mathbb{P}) \in N$, G is $\pi(\mathbb{P})$-generic over V and N is transitive, G is $\pi(\mathbb{P})$-generic over N. Since Π^1_1 sets are absolute for transitive models of ZFC and A is Π^1_1, in $V[G]$, $A^{N[G]} = N[G] \cap A = N[G] \cap A \cap V[G] = N[G] \cap A^{V[G]}$. And since $A^{V[G]} = A_G$,

$$A^{N[G]} = N[G] \cap A_G \in N[G].$$

Since M is an ω-model, the transitive collapse π is the identity on the reals and therefore,

$$A^{M[H]} = M[H] \cap A_H \in M[H].$$

ii) $\Rightarrow$ *i)* Suppose (M, E) is A-closed for each Π^1_1 set. Then it is WF-closed, since WF is Π^1_1. So by Lemma 2.22, (M, E) is well-founded. □

2.3. *AD*$^+$.

Definition 2.24. (cf.[12]) A set $A \subseteq \mathbb{R}$ is ∞-*Borel* if for some $S \cup \{\alpha\} \subseteq On$ and some formula with two free variables $\varphi(x, y)$,

$$A = \{y \in \mathbb{R} \mid L_\alpha[S, y] \vDash \varphi(S, y)\}.$$

Assuming $AD + DC$, a set of reals A is ∞-Borel iff $A \in L(S, \mathbb{R})$, for some $S \subseteq Ord$ (cf. [12]).

Definition 2.25. Θ is the least ordinal α which is not the range of any function $\pi \colon \mathbb{R} \to \alpha$. So, if the reals can be well ordered, then $\Theta = (2^\omega)^+$.

Recall that $DC_\mathbb{R}$ is the statement:

$$\forall R(R \subseteq \omega^\omega \times \omega^\omega \wedge \forall x \in \omega^\omega \exists y \in \omega^\omega((x, y) \in R) \to \\ \exists f \in (\omega^\omega)^\omega \forall n \in \omega((f(n), f(n+1)) \in R)).$$

Definition 2.26. (cf.[12]) $(ZF + DC_\mathbb{R})$ AD^+ says:

i) Every set of reals is ∞-*Borel,*
ii) If $\lambda < \Theta$ and $\pi \colon \lambda^\omega \to \omega^\omega$ is a continuous function, where λ has been given the discrete topology, then $\pi^{-1}(A)$ is determined for every $A \subseteq \omega^\omega$.

AD^+ trivially implies AD, and it is not known if AD implies AD^+. Woodin has shown that if $L(\mathbb{R}) \models AD$, then $L(\mathbb{R}) \models AD^+$.

AD^+ is absolute for inner models containing all the reals:

Theorem 2.27. (cf.[12])(AD^+) *For any transitive inner model M of ZF with $\mathbb{R} \subseteq M$, $M \vDash AD^+$.*

Theorem 2.28. ([12]) *If there exists a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is uB then:*

1) $L(A, \mathbb{R}) \models AD^+$,
2) *Every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is uB.*

2.4. **Definition of $\vdash_\Omega$ and invariance under forcing.**

Note that the following are equivalent:

i) For all A-closed c.t.m. M of ZFC, all $\alpha \in M \cap On$, and all $\mathbb{B}$ such that $M \models$ "$\mathbb{B}$ is a c.B.a", if $M_\alpha^{\mathbb{B}} \models T$, then $M_\alpha^{\mathbb{B}} \models \varphi$.

ii) For all A-closed c.t.m. M of ZFC, and for all $\alpha \in M \cap On$, if $M_\alpha \models T$, then $M_\alpha \models \varphi$.

Proof: ii) $\Rightarrow$ *i)* Let M be an A-closed c.t.m. of ZFC, $\alpha \in M \cap On$, and let $\mathbb{B}$ be such that $M \models$ "$\mathbb{B}$ is a c.B.a". Suppose $M_\alpha^{\mathbb{B}} \models T$ and, towards a contradiction, suppose that, in M, for some $b \in \mathbb{B}$, $b \Vdash$ "$M[\dot{g}]_\alpha \models \neg\varphi$", where $\dot{g}$ is the standard name for the generic filter. By Proposition 2.14, there is g $\mathbb{B}$-generic over M such that $b \in g$ and $M[g]$ is A-closed. We have $M[g]_\alpha \models T$. Hence, by ii) $M[g]_\alpha \models \varphi$, contradicting the assumption that b forced $M[\dot{g}]_\alpha \models \neg\varphi$. □

Definition 2.29. ([17]) For $T \cup \{\varphi\} \subseteq Sent$, we write $T \vdash_\Omega \varphi$ if there exists a uB set $A \subseteq \mathbb{R}$ such that:

1) $L(A, \mathbb{R}) \models AD^+$,
2) Every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is uB,
3) For all A-closed c.t.m. M of ZFC and for all $\alpha \in M \cap On$, if $M_\alpha \models T$, then $M_\alpha \models \varphi$.

Thus, by Theorem 2.28, if there exists a proper class of Woodin cardinals, $T \vdash_\Omega \varphi$ iff there exists a uB set $A \subseteq \mathbb{R}$ such that 3) above holds.

Notice that, by the equivalence of i) and ii) above, if T is recursive, then point 3) of the last definition can be written as:

3') For all A-closed c.t.m. M of ZFC, $M \vDash$ "$T \vDash_\Omega \varphi$".

By Theorem 2.28, if there exists a proper class of Woodin cardinals, or if just $L(\mathbb{R}) \models AD$ and every set of reals in $L(\mathbb{R})$ is uB, then for every $T \cup \{\varphi\} \subseteq Sent$, $T \vdash \varphi$ implies $T \vdash_\Omega \varphi$. However, as we would expect, the converse does not hold: Let M be a c.t.m. of ZFC and let $\alpha \in M \cap On$ be such that $M_\alpha \vDash ZFC$. Since M_α is a standard model, $M_\alpha \vDash CON(ZFC)$. This shows $ZFC \vdash_\Omega CON(ZFC)$.

We say that a sentence $\varphi \in Sent$ is *Ω_T-provable* if $T \vdash_\Omega \varphi$. And if A witnesses $T \vdash_\Omega \varphi$, then we say that A is an *Ω_T-proof* of φ, or that A is an Ω-proof of φ from T.

Notice that if A is uB and satisfies 1) and 2) of Definition 2.29, then A *is an Ω_T-proof of φ* iff

$L(A, \mathbb{R}) \vDash \forall M \forall \alpha$ (M is a A-closed c.t.m. of $ZFC \wedge \alpha \in M \cap On \wedge M_\alpha \models T \to M_\alpha \vDash \varphi$).

It is not very difficult to see that the complexity of the relation $T \vdash_\Omega \varphi$ is at most Σ_3.

Remark 2.30. Arguments in [7] essentially show that if AD$^+$ holds then there exist A-closed models of ZFC for every set of reals A.

Lemma 2.31. *Given A, B uB sets, the set $C = A \times B$ is uB, and if M is a C-closed c.t.m., then M is both A-closed and B-closed.*

Proof: Given $\gamma \in M \cap On$, let $\mathbb{P} = Coll(\omega, \gamma)$. For a fixed $\mathbb{P}$-name $\dot{y}$ for an element of $B_{\dot{G}}$,

$$\{(\tau, p) \mid p \in \mathbb{P}, \tau \text{ is a } \mathbb{P}\text{-name for a real number and } p \Vdash^V (\tau, \dot{y}) \in (A \times B)_{\dot{G}}\}$$

$$= \{(\tau, p) \mid p \in \mathbb{P}, \tau \text{ is a } \mathbb{P}\text{-name for a real number and } p \Vdash^V \tau \in A_{\dot{G}}\}.$$

Hence if M is C-closed, this set belongs to M and thus M is A-closed. Symmetrically, the same holds for B. □

Corollary 2.32. *Let $T \cup \{\varphi, \psi\} \subseteq Sent$. Suppose that for every uB set A, $L(A, \mathbb{R}) \models AD^+$ and every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is uB. Suppose $T \vdash_\Omega \psi$ and $T \vdash_\Omega \varphi$. If $T \cup \{\psi, \varphi\} \vdash \theta$, then $T \vdash_\Omega \theta$. Hence,*

i) *If $T \vdash_\Omega \varphi$ and $T \vdash_\Omega \psi$, then $T \vdash_\Omega \varphi \wedge \psi$.*
ii) *If $T \vdash_\Omega \varphi$ and $T \vdash_\Omega \varphi \to \psi$, then $T \vdash_\Omega \psi$.*

Proof: Let A and B be Ω_T-proofs of ψ and φ, respectively. Let us see that $A \times B$ is a Ω_T-proof of θ. Let M be an $A \times B$-closed model. Thus, M is both A-closed and B-closed. Suppose $\alpha \in M \cap On$ and $\mathbb{B} \in M$ are such that $M_\alpha^\mathbb{B} \models T$. Since M is A-closed, $M_\alpha^\mathbb{B} \models \psi$ and since M is B-closed, $M_\alpha^\mathbb{B} \models \varphi$. So, $M_\alpha^\mathbb{B} \models \theta$. □

The notion of Ω-provability differs from the usual notions of provability, e.g., in first-order logic, in that there is no deductive calculus involved. In Ω-logic, the same uB set may witness the Ω-provability of different sentences. For instance, all tautologies have the same *proof* in Ω-logic, namely, $\emptyset$. In spite of this, it is possible to define a notion of *length of proof* in Ω-logic. This can be accomplished in several ways. For instance: for $A \subseteq \mathbb{R}$, let M_A be the model $L_{\kappa_A}(A, \mathbb{R})$, where κ_A is the least admissible ordinal for $(A, \mathbb{R})$, i.e., the least ordinal $\alpha > \omega$ such that $L_\alpha(A, \mathbb{R})$ is a model of Kripke-Platek set theory. The following result is due to Solovay:

Lemma 2.33. *Assume AD. Then for every $A, B \subseteq \mathbb{R}$, either $A \in M_B$ or $B \in M_A$.*

Proof: Consider the two-player game in which both players play integers so that at the end of the game player I has produced x and player II has produced y. Player I wins the game iff $x \in A \leftrightarrow y \in B$. It τ is a winning strategy for player I, then for every real z, $z \in B$ iff $\tau * z \in A$, and so $B \in M_A$. And if σ is a winning strategy for player II, then for every real z, $z \in A$ iff $z * \sigma \notin B$, and so $A \in M_B$. □

Thus, under AD, for $A, B \subseteq \mathbb{R}$, we have $\kappa_A < \kappa_B$ iff $A \in M_B$ and $B \notin M_A$. It follows that $\kappa_A = \kappa_B$ iff $M_A = M_B$.

If A is a uB set of reals that witnesses $T \vdash_\Omega \varphi$, then we can say that κ_A is the *length of the Ω_T-proof A*. Using this notion of length of proof we can find sentences, like the Gödel-Rosser sentences in first-order logic, that are undecidable in Ω-logic. For instance, let $\varphi(A, \theta)$ be the formula:

$$\forall M \forall \alpha((M \text{ is an } A\text{-closed c.t.m. of } ZFC \wedge$$

$$\alpha \in M \cap On \ \wedge \ M_\alpha \models ZFC) \to M_\alpha \models \theta).$$

Using Gödel's diagonalization, let $\theta \in Sent$ be such that:

$$ZFC \vdash \text{“}\theta \leftrightarrow \forall A(\varphi(A, \theta) \to \exists B(\varphi(B, \neg\theta) \wedge \kappa_B < \kappa_A))\text{”}$$

Assuming there is a proper class of Woodin cardinals, we have:

$$ZFC \vdash_\Omega \text{“}\theta \leftrightarrow \forall A(\varphi(A,\theta) \rightarrow \exists B(\varphi(B,\neg\theta) \wedge \kappa_B < \kappa_A))\text{”}$$

Suppose $ZFC \vdash_\Omega \theta$ and C witnesses it. Then

$$ZFC \vdash_\Omega \text{“}\forall A(\varphi(A,\theta) \rightarrow \exists B(\varphi(B,\neg\theta) \wedge \kappa_B < \kappa_A))\text{”}$$

is witnessed by some D. Assuming there is an inaccessible limit of Woodin cardinals, we can find a $C \times D$-closed c.t.m. M of ZFC with a strongly inaccessible cardinal α, such that M satisfies that for every uB set of reals A, AD^+ holds in $L(A,\mathbb{R})$, and every set of reals in $L(A,\mathbb{R})$ is uB (see 2.28). By reflection, let $\alpha \in M \cap On$ be such that $C \cap M \in M_\alpha$, $M_\alpha \models$ “$C \cap M$ is uB”, and

$$M_\alpha \models ZFC + \forall A(A \text{ is uB} \rightarrow L(A,\mathbb{R}) \models AD).$$

Then, $M_\alpha \models \theta$ and

$$M_\alpha \models \text{“}\forall A(\varphi(A,\theta) \rightarrow \exists B(\varphi(B,\neg\theta) \wedge \kappa_B < \kappa_A)).\text{”}$$

Moreover, $M_\alpha \models \varphi(C \cap M, \theta)$. Hence, in M_α there is B such that $\varphi(B,\neg\theta)$ and $\kappa_B < \kappa_{C\cap M}$. But since $M_\alpha \models$ “$L(B, C \cap M, \mathbb{R}) \models AD$”, by Lemma 2.33, we have $M_\alpha \models B \in M_{C\cap M}$. It follows that:

(1) $M_{C\cap M} \models \varphi(C \cap M, \theta)$
(2) $M_{C\cap M} \models \varphi(B, \neg\theta)$.

Let $N \in M_{C\cap M}$ be a c.t.m. of ZFC that is both $C \cap M$-closed and B-closed (see Remark 2.30). Then, for any β, if $N_\beta \models ZFC$, we would have $N_\beta \models \theta \wedge \neg\theta$, which is impossible.

An entirely symmetric argument would yield a contradiction under the assumption that $ZFC \vdash_\Omega \neg\theta$, thereby showing that θ is undecidable from ZFC in Ω-logic.

A much finer notion of length of proof in Ω-logic is provided by the Wadge hierarchy of sets of reals (see [9] and [16]).

We shall now see that the relation $\vdash_\Omega$ is also invariant under forcing. In the proof of this, we will use the following result (see [6], section 3.4).

Theorem 2.34. *Suppose that there exists a proper class of Woodin cardinals, δ is a Woodin cardinal and $j\colon V \rightarrow M[G]$ is an embedding derived from forcing with $\mathbb{P}_{<\delta}$. Then every universally Baire set of reals in $V[G]$ is universally Baire in M.*

Theorem 2.35. ([17]) *Suppose that there exists a proper class of Woodin cardinals. Then for all $\mathbb{P}$,*

$$T \vdash_\Omega \varphi \quad \text{iff} \quad V^{\mathbb{P}} \vDash \text{“}T \vdash_\Omega \varphi\text{”}$$

Proof: $\Rightarrow$) Let A be an Ω_T-proof of φ.

Then $L(A,\mathbb{R}) \vDash \forall M \forall \alpha$ (M is a A-closed c.t.m. of $ZFC \wedge \alpha \in M \cap On \wedge M_\alpha \models T \rightarrow M_\alpha \vDash \varphi$).

Suppose $G \subseteq \mathbb{P}$ is V-generic. By Corollary 2.20, in $V[G]$,

$L(A_G, \mathbb{R}^{V[G]}) \vDash \forall M \forall \alpha$ (M is a A_G-closed c.t.m. of $ZFC \wedge \alpha \in M \cap On \wedge M_\alpha \models T \rightarrow M_\alpha \vDash \varphi$).

Since A is uB, by Remark 2.6, A_G is uB in $V[G]$. Hence, A_G is an Ω_T-proof of φ in $V[G]$.

$\Leftarrow$) Assume $V^{\mathbb{P}} \models$ "$T \vdash_\Omega \varphi$". Let γ be a strongly inaccessible cardinal such that $\mathbb{P} \in V_\gamma$. Pick a Woodin cardinal $\delta > \gamma$. Consider $a = \mathcal{P}_{\omega_1}(V_\gamma) \in \mathbb{P}_{<\delta}$ (see Fact 1.4). Forcing with $\mathbb{P}_{<\delta}$ below a makes V_γ countable, so there is a $\mathbb{P}$-name τ for a partial order such that $\mathbb{P}_{<\delta}(a)$ is forcing-equivalent to $\mathbb{P} * \tau$. Fix $G \subseteq \mathbb{P}_{<\delta}(a)$ V-generic, and let $j \colon V \to M$ be the induced embedding. Then $j(\delta) = \delta$ and $V[G] \models M^{<\delta} \subseteq M$. We have $V[G] = V[H_0][H_1]$, with $H_0 \subseteq \mathbb{P}$, V-generic. Thus, $V[H_0] \models$ "$T \vdash_\Omega \varphi$", witnessed by some uB set A. By the other direction of this theorem, $V[G] \models$ "$T \vdash_\Omega \varphi$", witnessed by A_G. Hence,

$V[G] \models$ "A_G is $uB \wedge \forall N \forall \alpha$ (N is a A_G-closed c.t.m. of $ZFC \wedge \alpha \in N \cup On \wedge N_\alpha \models T \to N_\alpha \models \varphi$)".

By Theorem 2.34, A_G is a uB set in M, and since M is closed under countable sequences,

$M \models$ "$\forall N \forall \alpha$ (N is a A-closed c.t.m. of $ZFC \wedge \alpha \in N \cap On \wedge N_\alpha \models T \to N_\alpha \models \varphi$)". Thus, $M \models$ "$T \vdash_\Omega \varphi$". By applying the induced elementary embedding, we have $V \models$ "$T \vdash_\Omega \varphi$". □

2.5. A-closure vs strong A-closure.

Recall (Definition 2.16) that for $A \subseteq \mathbb{R}$, a transitive $\in$-model M of (a fragment of) ZFC is *strongly A-closed* if for all posets $\mathbb{P} \in M$ and all M-generic $G \subseteq \mathbb{P}$, $M[G] \cap A \in M[G]$.

We shall see that the relation $\vdash_\Omega$ would not change if we were to use strong A-closure in place of A-closure in its definition.

Recall the definition of scale on a set of reals (see [9]):

Definition 2.36. If A is a set of reals, then a *scale* on A is a sequence $\langle \leq_i : i < \omega \rangle$ of prewellorderings of A satisfying the property that whenever $\langle x_i : i < \omega \rangle$ is a sequence contained in A converging to a real x and $f \colon \omega \to \omega$ is a function such that

$$\forall i < \omega \ \forall j \in [f(i), \omega) \ (x_{f(i)} \leq_i x_j \wedge x_j \leq_i x_{f(i)}),$$

then x is in A, and for all $i < \omega$ we have $x \leq_i x_{f(i)}$.

If Γ is a pointclass that is closed under continuous preimages, $A \in \Gamma$, and $\langle \leq_i : i < \omega \rangle$ is a scale on A, then $\langle \leq_i : i < \omega \rangle$ is called a *Γ-scale* if there are sets $X, Y \subset \omega \times \omega^\omega \times \omega^\omega$ in Γ (identifying each integer with the corresponding constant function) such that

$$X = \{(i, x, y) \mid x \leq_i y\} = (\omega \times \omega^\omega \times \omega^\omega) \setminus Y \cap (\omega \times \omega^\omega \times A).$$

We say that Γ has the *scale property* if for every $A \in \Gamma$ there is a Γ-scale on A. If there exists a proper class of Woodin cardinals, then the class of uB sets has the scale property (this fact is due to Steel; see, for instance, Section 3.3 of [6]).

If $\langle \leq_i : i < \omega \rangle$ is a scale on a set of reals A, and for each $i \in \omega$ and $x \in A$ we let $\rho_i(x)$ denote the $\leq_i$-rank of x, then the tree

$$S = \{(s, \sigma) \in \omega^{<\omega} \times Ord^{<\omega} \mid \exists x \in A \ x \restriction |s| = s \wedge \langle \rho_i(x) : i < |s| \rangle = \sigma\}$$

projects to A. We call this *the tree corresponding to the scale*.

The argument below comes from [11].

Theorem 2.37. *Let A be a universally Baire set of reals and suppose that M is an A-closed c.t.m. of ZFC. Let B denote the complement of A. Let $\langle \leq_i^A : i < \omega \rangle$ be a uB scale on A as witnessed by uB sets X and Y, let $\langle \leq_i^B : i < \omega \rangle$ be a uB scale on B as witnessed by uB sets W and Z, and suppose that M is $X \times Y \times W \times Z$-closed. Then M is strongly A-closed.*

Proof: First note that for any wellfounded model N, if $\{N \cap X, N \cap Y, N \cap A\} \in N$, then $\langle \leq_i^A \cap N : i < \omega \rangle$ is in N and is a scale for $A \cap N$ in N (and similarly, for W, Z and B). Furthermore, if N is $X \times Y \times A$-closed, then for every partial order $\mathbb{P}$ in N there are $\mathbb{P}$-names $\chi_{\mathbb{P}}$, $\upsilon_{\mathbb{P}}$ and $\alpha_{\mathbb{P}}$ such that for comeagerly-many N-generic filters $g \subset \mathbb{P}$, $X \cap N[g] = \chi_g$, $Y \cap N[g] = \upsilon_g$ and $A \cap N[g] = \alpha_g$ (the proof of this is similar to the second parts of the proofs of Lemmas 2.11 and 2.13).

Let γ be an ordinal in M. Since $Coll(\omega, \gamma)$ is homogeneous and M is $X \times Y \times A$-closed, for every pair of conditions p, q in $Coll(\omega, \gamma)$ there exist M-generic filters g_p and g_q contained in $Coll(\omega, \gamma)$ such that $p \in g_p$, $q \in g_q$, $M[g_p] = M[g_q]$,

$$i_{g_p}[\chi_{Coll(\omega,\gamma)}] i_{g_q}[\chi_{Coll(\omega,\gamma)}] = M[g_p] \cap X,$$

$$i_{g_p}[\upsilon_{Coll(\omega,\gamma)}] i_{g_q}[\upsilon_{Coll(\omega,\gamma)}] = M[g_p] \cap Y,$$

and

$$i_{g_p}[\alpha_{Coll(\omega,\gamma)}] i_{g_q}[\alpha_{Coll(\omega,\gamma)}] = M[g_p] \cap A.$$

Therefore, for every pair $(a, b) \in \omega^{<\omega} \times Ord^{<\omega}$, the empty condition in $Coll(\omega, \gamma)$ decides whether (a, b) is in the tree corresponding to the scale associated to $\chi_{Coll(\omega,\gamma)}$ and $\upsilon_{Coll(\omega,\gamma)}$, and therefore the tree T_γ corresponding to this scale in any M-generic extension by $Coll(\omega, \gamma)$ exists already in M. Since there exists a model N such that $\{N \cap A, N \cap X, N \cap Y\} \in N$ and T_γ is the tree of the scale corresponding to $N \cap X$ and $N \cap Y$ in N, $p[T_\gamma]^V \subset A$ (since X and Y define a scale on A). The remarks above apply to B, W and Z, as well, and so there is a tree S_γ in M which projects in V to a subset of B, and furthermore, T_γ and S_γ project to complements in all forcing extensions of M by $Coll(\omega, \gamma)$.

Let $\mathbb{P}$ be a partial order in M. Then $\mathbb{P}$ regularly embeds into some partial order of the form $Coll(\omega, \gamma)$, $\gamma \in On \cap M$. Fixing such a γ, we have that for any $\mathbb{P}$-generic extension N of M, $p[T_\gamma]^N = A \cap N$ and $p[S_\gamma]^N = B \cap N$. $\square$

Let the relation $\vdash_\Omega^-$ be defined as $\vdash_\Omega$ (Definition 2.29) but requiring strong A-closure instead of A-closure. i.e.,

$T \vdash_\Omega^- \varphi$ if there exists a uB set $A \subseteq \mathbb{R}$ such that:

1) $L(A, \mathbb{R}) \models AD^+$,
2) Every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is uB,
3) For all strongly A-closed c.t.m. M of ZFC and for all $\alpha \in M \cap On$, if $M_\alpha \models T$, then $M_\alpha \models \varphi$.

Since for any uB set A and any c.t.m. M strong A-closure implies A-closure (see Lemma 2.11), clearly $T \vdash_\Omega \varphi$ implies $T \vdash_\Omega^- \varphi$.

Now suppose $T \vdash_\Omega^- \varphi$, witnessed by a uB set A. We would like to see that there is a uB set B such that all B-closed models are strongly A-closed. Theorem 2.37 gives us this, under the assumption that the collection of universally Baire sets has the scale property, which, as we mentioned above, it does when there exist proper class many Woodin cardinals. Even without this assumption one can show that such a B exists, though the proof of this is beyond the scope of this paper. Here is a sketch. Note first that M is a strongly A-closed c.t.m. iff $L(A,\mathbb{R}) \models$ "M is a strongly A-closed c.t.m." So, in $L(A,\mathbb{R})$, A satisfies the following predicate $P(X)$ on sets $X \subseteq \mathbb{R}$:

$$\forall M \forall \alpha (M \text{ a strongly } X\text{-closed c.t.m. of ZFC} \wedge$$

$$\alpha \in M \cap On \ \wedge \ M_\alpha \models T \rightarrow M_\alpha \models \varphi).$$

We now apply Woodin's generalizations of the Martin-Steel theorem on scales in $L(\mathbb{R})$ [8] and the Solovay Basis Theorem (see [3]) to the context of AD^+, stated as follows.

Theorem 2.38. *(ZF + $DC_{\mathbb{R}}$) If AD^+ holds and $V L(\mathcal{P}(\mathbb{R}))$ then*

- *the pointclass Σ^2_1 has the scale property,*
- *every true Σ_1-sentence is witnessed by a $\utilde{\Delta}^2_1$ set of reals.*

We may then let B be a $\utilde{\Delta}^2_1$ (in $L(A,\mathbb{R})$) solution to $P(X)$. Note that by (2) above, B is uB and, by Theorem 2.27, it is also a witness to $T \vdash^-_\Omega \varphi$. Since $L(A,\mathbb{R}) \models AD^+$, both B and its complement have $\utilde{\Sigma}^2_1$ scales in $L(A,\mathbb{R})$. Those scales are uB (again, by (2) above). So, as in Theorem 2.37, we can find $C \in L(A,\mathbb{R})$ such that if M is a C-closed c.t.m., then M is strongly B-closed. Thus, C witnesses $T \vdash_\Omega \varphi$.

One can formulate a property which roughly captures the difference between A-closure and strong A-closure. We will call this property A-*completeness*, though that term is not standard.

Definition 2.39. Let A be a set of reals. Let us call a c.t.m. M of ZFC A-*complete* if for every forcing notion $\mathbb{P} \in M$, every name for a real $\tau \in M^{\mathbb{P}}$, and every $p \in \mathbb{P}$:

(1) If for comeagerly-many M-generic $G \subseteq \mathbb{P}$, $p \in G$ implies $i_G[\tau] \in A$, then for every M-generic $G \subseteq \mathbb{P}$, $p \in G$ implies $i_G[\tau] \in A$.
(2) If for comeagerly-many M-generic $G \subseteq \mathbb{P}$, $p \in G$ implies $i_G[\tau] \notin A$, then for every M-generic $G \subseteq \mathbb{P}$, $p \in G$ implies $i_G[\tau] \notin A$.

The conjunction of A-closure and A-completeness implies strong-A-closure.

Lemma 2.40. *Let M be a c.t.m. and A a uB set. If M is both A-closed and A-complete, then it is strongly-A-closed.*

Proof: Fix M and A and suppose M is A-closed and A-complete.
Let

$$\sigma = \{(\tau, p) \mid \tau \in M \text{ a simple } \mathbb{P}\text{-name for a real }, p \in \mathbb{P} \text{ and } p \Vdash^V_{\mathbb{P}} \tau \in A_{\dot{G}}\}.$$

By Proposition 2.9, σ is a $\mathbb{P}$-name that belongs to M.

We claim that for every M-generic $G \subseteq \mathbb{P}$, $i_G[\sigma] = M[G] \cap A$.

So, suppose $G \subseteq \mathbb{P}$ is an M-generic filter. If $\tau \in M$ is a simple $\mathbb{P}$-name for a real and $i_G[\tau] \in A$, then for some $p \in \mathbb{P}$, for a comeager set of M-generic filters g, if $p \in g$, then $i_g[\tau] \in A$. By 2.13, $p \Vdash^V \tau \in A_{\dot{G}}$. Hence, $i_G[\tau] \in i_G[\sigma]$.

Now suppose $i_G[\tau] \in i_G[\sigma]$. So, for some $p \in G$, $p \Vdash^V \tau \in A_{\dot{G}}$. By 2.13, the set of M-generic filters $g \subseteq \mathbb{P}$ such that $p \in g$ and $i_g[\tau] \in A$ is comeager. But since M is A-complete, for all M-generic $g \subseteq \mathbb{P}$ such that $p \in g$, $i_g[\tau] \in A$. In particular, $i_G[\tau] \in A$. □

Strong A-closure does not imply A-completeness, however. To see this, note that if x is a real and $A = \{x\}$, then every c.t.m. M is strongly-A-closed. But if x is Cohen-generic over M, then M is not A-complete, for if $\mathbb{P}$ is the Cohen forcing, and $\tau \in M^{\mathbb{P}}$ is a name for x, then the set $D = \{p \in \mathbb{P} : \ p \Vdash \tau \neq x\}$ is a dense subset of $\mathbb{P}$ (although $D \notin M$!). So, there is a comeager set of $\mathbb{P}$-generic filters over M such that for each G in the set, $i_G[\tau] \neq x$. i.e., $i_G[\tau] \notin A$. But for some M-generic G, $i_G[\tau] = x \in A$.

Similarly, A-completeness does not imply strong A-closure (and so it does not imply A-closure, either). As an example, let M satisfy ZFC + "$0^\sharp$ does not exist," and let $A = 0^\sharp$ (i.e., $\{n \mid n \in 0^\sharp\}$). Then M is clearly not A-closed, since $M[G] \cap A = A$ for all M-generic $G \subseteq \mathbb{P}$, all $\mathbb{P}$. But M is A-complete. To see this, fix $\mathbb{P}$, p, and τ, and suppose that for comeagerly-many M-generic G, if $p \in G$, then $i_G[\tau] \in A$. It follows then that $X = \{n : \exists p' \leq p\ (p' \Vdash \tau = n)\}$ is contained in A, which in turn implies that $i_G[\tau] \in A$ for all M-generic filters $G \subseteq \mathbb{P}$ that contain p.

3. The Ω-conjecture

Definition 3.1.

i) A sentence φ is *Ω_T-satisfiable* if $T \nvDash_\Omega \neg\varphi$, i.e., there exists α and $\mathbb{B}$ such that $V_\alpha^{\mathbb{B}} \models T + \varphi$.

ii) A set of sentences T is *Ω-satisfiable* if there exists a c.B.a. $\mathbb{B}$ and an ordinal α for which $V_\alpha^{\mathbb{B}} \models T$.

iii) A sentence φ is *Ω_T-consistent* if $T \nvdash_\Omega \neg\varphi$, i.e., for all uB set $A \subseteq \mathbb{R}$ satisfying 1) and 2) of Definition 2.29, there exists a countable transitive A-closed set M such that $M \models ZFC$, and there exists $\alpha \in M \cap On$ such that $M_\alpha \models T + \varphi$.

iv) A set of sentences T is *Ω-consistent* if $T \nvdash_\Omega \perp$, where $\perp$ is any contradiction, i.e., if for all $A \subseteq \mathbb{R}$ uB satisfying 1) and 2) of Definition 2.29, there exists a c.t.m. A-closed $M \models ZFC$ and $\alpha \in M$ such that $M_\alpha \models T$.

v) T is *Ω-inconsistent* if it is not Ω-consistent.

Observe that if AD^+ holds in $L(\mathbb{R})$ and every set of reals in $L(\mathbb{R})$ is uB, then every *Ω_T-consistent* sentence is consistent with T.

Fact 3.2. The following are equivalent for a set of sentences T:

i) T is *Ω-consistent.*

ii) $T \nvdash_\Omega \varphi$ for some φ.

iii) $T \nvdash_\Omega \neg\varphi$ for all $\varphi \in T$, i.e., for all $\varphi \in T$, φ is Ω_T-consistent.

Proof: i) ⇒ ii) Trivial.

ii) ⇒ iii) Without loss of generality, we may assume that for some uB set A, 1) and 2) of Definition 2.29 hold. Given such an A, by hypothesis there exist an A-closed c.t.m. M and $\alpha \in M \cap On$ such that $M_\alpha \models T + \neg\varphi$. Since $M_\alpha \models \psi$ for all $\psi \in T$, the same M and α witness that $T \nvdash_\Omega \neg\psi$, for all $\psi \in T$.

iii ⇒ i) W.l.o.g., we may assume 1) and 2) of Definition 2.29 hold for some uB set A. Moreover, we may also assume that $T \neq \emptyset$. So, let $\varphi \in T$. By hypothesis there exist an A-closed c.t.m. M and $\alpha \in M \cap On$ such that $M_\alpha \vDash T + \varphi$. Since $M_\alpha \vDash T + \neg\bot$, the same M and α witness that $T \nvdash_\Omega \bot$. □

Theorem 3.3 (Soundness)**.** ([12]) *Assume there is a proper class of strongly inaccessible cardinals. For every $T \cup \{\varphi\} \in Sent$, $T \vdash_\Omega \varphi$ implies $T \vDash_\Omega \varphi$.*

Proof: Let A be a uB set A witnessing $T \vdash_\Omega \varphi$. Fix α and $\mathbb{B}$, and suppose $V_\alpha^{\mathbb{B}} \models T$. Let $\lambda > \alpha$ be a strongly inaccessible cardinal such that $A, \mathbb{B}, T \in V_\lambda$ and $V_\lambda \models$ "$\mathbb{B}$ is a c.B.a.". Take $X \prec V_\lambda$ countable with $A, \mathbb{B}, T \in X$. Let M be the transitive collapse of X, and let $\bar{\mathbb{B}}$ be the transitive collapse of $\mathbb{B}$. By Lemma 2.18 M is A-closed. Hence, if $M_\alpha^{\bar{\mathbb{B}}} \models T$, then $M_\alpha^{\bar{\mathbb{B}}} \models \varphi$. Since $V_\lambda \models$ "$V_\alpha^{\mathbb{B}} \models T$", by elementarity, $M \models$ "$M_\alpha^{\bar{\mathbb{B}}} \models T$". Hence, $M \models$ "$M_\alpha^{\bar{\mathbb{B}}} \models \varphi$". So, again by elementarity, $V_\lambda \models$ "$V_\alpha^{\mathbb{B}} \models \varphi$". Hence, $V_\alpha^{\mathbb{B}} \models \varphi$. □

The assumption of the existence of a proper class of inaccessible cardinals in the Theorem above is not necessary. However, the proof without this assumption is no longer elementary and would take us beyond the scope of this paper.

Thus, if there exists κ such that $V_\kappa \vDash ZFC + \varphi$, then $ZFC \nvdash_\Omega \neg\varphi$. i.e., φ is Ω_{ZFC}-consistent.

Another consequence of Soundness is that for every finite fragment T of ZFC, an Ω_T-provable sentence cannot be made false by forcing over V.

The following equivalence can be proved without using Theorem 3.3.

Fact 3.4. For every $T \subseteq Sent$, the following are equivalent:

i) For all $\varphi \in Sent$, $T \vdash_\Omega \varphi$ implies $T \vDash_\Omega \varphi$.
ii) T is Ω-satisfiable implies T is Ω-consistent.

Proof: i) ⇒ ii) Suppose T is not Ω-consistent, i.e., $T \vdash_\Omega \bot$. By hypothesis, $T \vDash_\Omega \bot$ and so for all c.B.a. $\mathbb{B}$ and for all $\alpha \in On$, $V_\alpha^{\mathbb{B}} \nvDash T$, and therefore T is not Ω-satisfiable.
ii) ⇒ i) Suppose $T \nvDash_\Omega \varphi$. Let $\mathbb{B}$ and α be such that $V_\alpha^{\mathbb{B}} \vDash T$ and $V_\alpha^{\mathbb{B}} \vDash \neg\varphi$. Then $T \cup \{\neg\varphi\}$ is Ω-satisfiable and therefore Ω-consistent. If $T \vdash_\Omega \varphi$, then $T \cup \{\neg\varphi\} \vdash_\Omega \varphi$. But then $T \cup \{\neg\varphi\} \vdash_\Omega \varphi \wedge \neg\varphi$, a contradiction. □

Thus, by Theorem 3.3 and Fact 3.4, if T is Ω-satisfiable then T is Ω-consistent, i.e., if there exist α and $\mathbb{B}$ such that $V_\alpha^{\mathbb{B}} \vDash T$, then for every uB set A there exist an A-closed c.t.m. M of ZFC and α in $On \cap M$ such that $M_\alpha \vDash T$.

Corollary 3.5 (Non-Compactness of $\vdash_\Omega$)**.** *Suppose $L(R) \models AD$ and every set of reals in $L(R)$ is universally Baire. Then there is a sentence φ such that $ZFC \vdash_\Omega \varphi$ and for all $S \subseteq ZFC$ finite, $S \nvdash_\Omega \varphi$.*

Proof: Take the sentence φ of Theorem 1.12. Suppose $ZFC \nvdash_\Omega \varphi$. Then for each uB set A there is an A-closed c.t.m. M and $\alpha \in M \cap On$ such that $M_\alpha \vDash ZFC + \neg\varphi$. With the same argument as in the proof of Theorem 1.12 applied to M_α we arrive to a contradiction.

Suppose now there is S finite such that $S \vdash_\Omega \varphi$. Then by Soundness, $S \vDash_\Omega \varphi$, and this yields a contradiction as in the proof of Theorem 1.12. □

The Ω-*conjecture* says: *If there exists a proper class of Woodin cardinals, then for each sentence of the language of set theory* φ,

$$\emptyset \models_\Omega \varphi \quad \text{iff} \quad \emptyset \vdash_\Omega \varphi.$$

The "if" direction is given by Soundness. So, the Ω-*conjecture* is just Completeness for Ω-logic, i.e., if $\emptyset \models_\Omega \varphi$, then $\emptyset \vdash_\Omega \varphi$, for every $\varphi \in Sent$.

Lemma 3.6. *The following are equivalent:*

i) *For all* $\varphi \in Sent$, $\emptyset \models_\Omega \varphi$ *implies* $\emptyset \vdash_\Omega \varphi$.
ii) *For every r.e. set* $T \cup \{\varphi\} \subseteq Sent$, $T \models_\Omega \varphi$ *implies* $T \vdash_\Omega \varphi$.

Proof: i) $\Rightarrow$ *ii)* Fix T r.e. and φ such that $T \models_\Omega \varphi$. Let $\varphi^* :=$ "$T \models_\Omega \varphi$". By Lemma 1.9, $\emptyset \models_\Omega \varphi^*$, and so by i), $\emptyset \vdash_\Omega \varphi^*$. Hence, there is a uB set A such that for every A-closed c.t.m. $M \models ZFC$, $M \models$ "$\emptyset \models_\Omega \varphi^*$". Then for all $\alpha \in M$, $M_\alpha \models$ "$T \models_\Omega \varphi$". Since $M \models ZFC$, by reflection, $M \models$ "$T \models_\Omega \varphi$". This shows that A witnesses $T \vdash_\Omega \varphi$. $\square$

The Ω-conjecture is absolute under forcing:

Theorem 3.7. *Suppose that there exists a proper class of Woodin cardinals. Then for every c.B.a.* $\mathbb{B}$,

$$V^{\mathbb{B}} \models \Omega\text{-Conjecture} \quad \textit{iff} \quad V \models \Omega\text{-Conjecture}.$$

Proof: By Theorems 1.8 and 2.35, for every c.B.a. $\mathbb{B}$, $\emptyset \models_\Omega \varphi$ if and only if $V^{\mathbb{B}} \models$ "$\emptyset \models_\Omega \varphi$" and $\emptyset \vdash_\Omega \varphi$ if and only if $V^{\mathbb{B}} \models$ "$\emptyset \vdash_\Omega \varphi$". Hence if $V^{\mathbb{B}} \models \Omega$-Conjecture, then $V \models$ "$\emptyset \models_\Omega \varphi$" iff $V^{\mathbb{B}} \models$ "$\emptyset \models_\Omega \varphi$" iff $V^{\mathbb{B}} \models$ "$\emptyset \vdash_\Omega \varphi$" iff $V \models$ "$\emptyset \vdash_\Omega \varphi$". Similarly for the converse. $\square$

Remarks 3.8. i) Assume $L(\mathbb{R}) \models AD^+$ and every set of reals in $L(\mathbb{R})$ is uB. If T is r.e. and $ZFC \models$ "$T \models_\Omega \varphi$", then $T \vdash_\Omega \varphi$, witnessed by $\emptyset$.

ii) Suppose that ZFC + there exists a strongly inaccessible cardinal is consistent. Let $\varphi =$ *"There is a non-constructible real"*. Then,

$$ZFC \not\vdash ((ZFC \models_\Omega \varphi) \rightarrow (ZFC \models \text{"}ZFC \models_\Omega \varphi\text{"})).$$

For suppose $V \models ZFC$ + *"There is a non-constructible real"* + $\exists\alpha(V_\alpha \models ZFC)$. Then $ZFC \models_\Omega \varphi$ holds in V. For if γ is an ordinal and $V_\gamma^{\mathbb{B}} \models ZFC$, then $V_\gamma^{\mathbb{B}} \models \varphi$, since $V_\gamma^{\mathbb{B}}$ contains all the reals of V. But, since ZFC plus the existence of a strongly inaccessible cardinal is consistent, there exists in V a model of ZFC + "there exists a strongly inaccessible cardinal" + $V = L$. This model satisfies $ZFC \not\models_\Omega \phi$.

iii) Suppose that ZFC is consistent. Then, for any sentence φ,

$$ZFC \not\vdash \neg((ZFC \models_\Omega \varphi) \rightarrow (ZFC \models \text{"}ZFC \models_\Omega \varphi\text{"})).$$

Since there is a model of ZFC + "*There are no models of* ZFC".

Recall that:

i) T is Ω-satisfiable iff there exists a c.B.a. $\mathbb{B}$ and an ordinal α such that $V_\alpha^{\mathbb{B}} \models T$.
ii) T is Ω-consistent iff $T \nvdash_\Omega \perp$.

The following gives a restatement of the Ω-conjecture.

Fact 3.9. The following are equivalent for every $T \subseteq Sent$:

i) For all $\varphi \in Sent$, $T \models_\Omega \varphi$ implies $T \vdash_\Omega \varphi$
ii) T is Ω-consistent implies T is Ω-satisfiable.

Proof: i)⇒ii) Suppose T is not Ω-satisfiable. Then for all c.B.a. $\mathbb{B}$ and all α, $V_\alpha^\mathbb{B} \not\models T$. So, for all $\mathbb{B}$ and all α, if $V_\alpha^\mathbb{B} \models T$, then $V_\alpha^\mathbb{B} \models \bot$, vacuously. Hence, $T \models_\Omega \bot$. By hypothesis, $T \vdash_\Omega \bot$, and we have that T is Ω-inconsistent.
ii)⇒i) Suppose $T \nvdash_\Omega \varphi$. Then $T \cup \{\neg\varphi\} \nvdash_\Omega \varphi$, since otherwise $T \vdash_\Omega \neg\varphi \rightarrow \varphi$, and then $T \vdash_\Omega \varphi \vee \varphi$, giving a contradiction. So, $T \cup \{\neg\varphi\}$ is Ω-consistent. Since by hypothesis, $T \cup \{\neg\varphi\}$ is Ω-satisfiable, there are $\mathbb{B}$ and α such that $V_\alpha^\mathbb{B} \models T \cup \{\neg\varphi\}$. Therefore $T \not\models_\Omega \varphi$. □

Finally, we note that it is consistent that Ω-conjecture is true, as Woodin has shown that it holds in fine structural models with a proper class of Woodin cardinals.

References

[1] P. Dehornoy, *Progrès récents sur l'hypothèse du continu (d'après Woodin)*, Séminaire Bourbaki 55ème année, 2002-2003, #915.
[2] Q. Feng, M. Magidor, W.H.Woodin, *Universally Baire Sets of Reals.* Set Theory of the Continuum (H. Judah, W.Just and W.H.Woodin, eds), MSRI Publications, Berkeley, CA, 1989, pp. 203-242, Springer Verlag 1992.
[3] S. Jackson, *Structural consequences of AD*, Handbook of Set Theory, M. Foreman, A. Kanamori and M. Magidor, eds. To appear.
[4] T. Jech, *Set theory*, 3d Edition, Springer, New York, 2003.
[5] A. Kanamori, *The Higher Infinite.Large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer-Verlag. Berlin, 1994.
[6] P.B. Larson, *The Stationary Tower. Notes on a course by W. Hugh Woodin.* University Lecture Series, Vol. 32. American Mathematical Society, Providence, RI. 2004.
[7] P.B. Larson, *Forcing over models of determinacy*, Handbook of Set Theory, M. Foreman, A. Kanamori and M. Magidor, eds. To appear.
[8] D.A. Martin, J.R. Steel, *The extent of scales in $L(\mathbb{R})$*, Cabal seminar 79–81, Lecture Notes in Math. 1019, Springer, Berlin, 1983, 86-96.
[9] Y. N. Moschovakis, *Descriptive Set Theory*, Studies in Logic and the Foundations of Mathematics. Vol. 100. North-Holland Publishing Company. Amsterdam, New York, Oxford, 1980.
[10] S. Shelah, W.H. Woodin, *Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable.* Israel J. of Math. vol. 70, n. 3 (1990), 381-394.
[11] J. Steel, *A theorem of Woodin on mouse sets.* Preprint. July 14, 2004.
[12] W.H. Woodin,*The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal.* DeGruyter Series in Logic and Its Applications, vol. 1, 1999.
[13] W.H.Woodin, *The Continuum Hypothesis.* Proceedings of the Logic Colloquium, 2000. To appear.
[14] W.H.Woodin, *The Ω-Conjecture.* Aspects of Complexity (Kaikoura, 2000). DeGruyter Series in Logic and Its Applications, vol. 4, pages 155-169. DeGruyter, Berlin, 2001.
[15] W.H.Woodin, *The Continuum Hypothesis, I.* Notices Amer. Math. Soc., 48(6):567-576, 2001.
[16] W.H.Woodin, *The Continuum Hypothesis, II.* Notices Amer. Math. Soc., 48(7):681-690, 2001; 49(1):46, 2002.
[17] W.H.Woodin, *Set theory after Russell; The journey back to Eden.* In *One Hundred Years of Russell's Paradox*, edited by Godehard Link. DeGruyter Series in Logic and Its Applications, vol. 6, pages 29-48.

Centre de Recerca Matemàtica (CRM). Apartat 50, E-08193 Bellaterra (Barcelona), Spain.

ICREA (Institució Catalana de Recerca i Estudis Avançats) and Departament de Lògica, Història i Filosofia de la Ciència. Universitat de Barcelona. Baldiri Reixac, s/n. 08028 Barcelona, Spain. bagaria@ub.edu

Departament de Lògica, Història i Filosofia de la Ciència. Universitat de Barcelona. Baldiri Reixac, s/n. 08028 Barcelona, Spain. ncastema@mat.ub.edu

Centre de Recerca Matemàtica (CRM). Apartat 50, E-08193 Bellaterra (Barcelona), Spain.

Department of Mathematics and Statistics. Miami University. Oxford, Ohio 45056, USA. larsonpb@muohio.edu