The number of models of a fixed Scott rank, for a counterexample to the analytic Vaught conjecture

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Abstract

We show that if $\mathcal{A}$ is a counterexample to the analytic Vaught conjecture with only countably many models of Scott rank ω_1, then there exist a $\gamma \leq \omega$ and a club $C \subseteq \omega_1$ such that $\mathcal{A}$ has exactly γ many models of Scott rank α, for each $\alpha \in C$.

Throughout this note τ represents a countable relational vocabulary. The set of τ-structures with domain ω is naturally seen as a Polish space X_τ, where a basic open set is given by the set of structures in which $R(i_0, \ldots, i_{n-1})$ holds, for R an n-ary relation symbol from τ and $i_0, \ldots, i_{n-1} \in \omega$ (see Section 11.3 of [1], for instance). Given a sentence $\phi \in \mathcal{L}_{\aleph_1,\aleph_0}(\tau)$, the set of models of ϕ with domain ω is a Borel subset of X_τ. By a theorem of Lopez-Escobar [4], every Borel subset of X_τ which is closed under isomporphism is also the set of models (with domain ω) of some $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$ sentence.

We call the following (false) statement the *analytic Vaught conjecture*: for every countable relational vocabulary τ, every analytic subset of X_τ which is closed under isomorphism and contains uncountably many nonisomorphic structures contains a perfect set of nonisomorphic structures. Steel [11] presents two counterexamples to this statement, one due to H. Friedman and the other to K. Kunen.

Given a τ-structure M, we let $\mathrm{SP}_\alpha(M)$ denote the Scott process of M of length α, as defined in [3] (this is essentially the same as the standard definition appearing in [9, 5, 6]; we assume some familiarity with [3] in the arguments below, but expect that familiarity with the classical Scott analysis will suffice). Scott's Isomorphism Theorem [9] (rephrased) says that if α is a (necessarily countable) ordinal and M and N are countable τ-structures of Scott rank at most α, then M and N are isomorphic if and only if $\mathrm{SP}_{\alpha+1}(M) = \mathrm{SP}_{\alpha+1}(N)$.

Given a set $\mathcal{A} \subseteq X_\tau$, we let $\mathcal{A}^*$ denote the class of (ground model, but possibly uncountable) τ-structures M which are isomorphic to an element of the reinterpretation of $\mathcal{A}$ in any (equivalently, every, by Σ^1_1-absoluteness) outer model in which M is countable. If $\mathcal{A}$ is the set of τ-structures on ω satisfying a sentence ϕ of $\mathcal{L}_{\aleph_1,\aleph_0}(\tau)$, then $\mathcal{A}^*$ as defined above is simply the class of models of ϕ.

For an ordinal α, we let $\mathrm{SP}_\alpha(\mathcal{A})$ denote the set of the Scott processes of length α for structures in $\mathcal{A}^*$. If $\mathcal{A}$ is a counterexample to the analytic Vaught conjecture, then $|\mathrm{SP}_\alpha(\mathcal{A})| \leq |\alpha|$ (for $\alpha < \omega_1$ this follows

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by an induction argument using the Perfect Set Property for analytic sets; considering of a forcing extension via $\mathrm{Col}(\omega, \alpha)$ completes the argument for $\alpha \geq \omega_1$).

We also let $\mathcal{A}_\alpha$ denote respectively the class of structures $\mathcal{A}^*$ of Scott rank α. The following well-known fact (slightly restated here) appears as Corollary 10.2 in [3].

Fact 0.1. *Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Let M be a member of the reinterpreted version of $\mathcal{A}$ in a forcing extension of V, and let α be an ordinal. Then $\mathrm{SP}_\alpha(M) \in L[x]$.*

The proof of Fact 0.1 given in [3] shows the following. Similar arguments appear in Section 1 of [2] and Chapter 32 of [7].

Theorem 0.1. *Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Let Y be a countable elementary submodel of $H((2^{\aleph_1})^+)$ with $x \in Y$, let $\delta = Y \cap \omega_1$ and let P be the transitive collapse of Y. Then $\mathrm{SP}_{\delta+1}(\mathcal{A}) = \mathrm{SP}_{\delta+1}(\mathcal{A})^P$.*

Proof. Since $\mathcal{A}$ is a counterexample to the analytic Vaught conjecture, $\mathrm{SP}_\alpha(\mathcal{A})^P$ is a countable set in P, for each $\alpha < \delta$. It follows that $\mathrm{SP}_\alpha(\mathcal{A})^P = \mathrm{SP}_\alpha(\mathcal{A})$ for each such α, since P is correct about the Σ^1_1 statement asserting that some object satisfying the conditions for membership in $\mathrm{SP}_\alpha(\mathcal{A})$ is unequal to all the members of the countable set $\mathrm{SP}_\alpha(\mathcal{A})^P$. Letting g be P-generic for $\mathrm{Col}(\omega, \delta)$ (the partial order of finite partial functions from ω to δ, ordered by inclusion), the same argument applies to show first that $\mathrm{SP}_\delta(\mathcal{A}) = \mathrm{SP}_\delta(\mathcal{A})^{P[g]}$ and then that $\mathrm{SP}_{\delta+1}(\mathcal{A}) = \mathrm{SP}_{\delta+1}(\mathcal{A})^{P[g]}$. However, each member of $\mathrm{SP}_\delta(\mathcal{A})$ in $P[g]$ must be in P, since otherwise there is a $\mathrm{Col}(\omega, \delta)$-name for an element not in P, and one can find perfectly many generic filters for P giving distinct realizations of this name. The same argument again shows that each member of $\mathrm{SP}_{\delta+1}(\mathcal{A})$ in $P[g]$ must be in P. $\square$

Theorem 0.2 below can also be proved using material from [10].

Theorem 0.2. *Suppose that $\mathcal{A}$ is a counterexample to the analytic Vaught Conjecture and $\gamma \in \omega \cup \{\aleph_0\}$ is such that there are up to isomorphism exactly γ many elements of $\mathcal{A}^*$ of Scott rank ω_1. Then for club many $\alpha < \omega_1$ there are exactly γ many models in $\mathcal{A}$ of Scott rank α, up to isomorphism.*

Proof. Let $\mathcal{M} = \{M_n : n \leq \gamma\}$ be pairwise nonisomorphic elements of $\mathcal{A}_{\omega_1}$ such that every element of $\mathcal{A}_{\omega_1}$ is isomorphic to some element of $\mathcal{M}$. Let $\mathcal{Y}$ be the set of countable elementary substructures of $H((2^{\aleph_1})^+)$ containing (as elements) $\mathcal{M}$ and a (fixed) code for $\mathcal{A}$. We show that for each $Y \in \mathcal{Y}$, letting $\mathcal{M}_Y$ be the image of $\mathcal{M}$ under the transitive collapse of Y, every element of $\mathcal{A}_{Y \cap \omega_1}$ is isomorphic to an element of $\mathcal{M}_Y$. As the members of $\mathcal{M}_Y$ will be nonisomorphic, this will establish the theorem.

Fix $Y \in \mathcal{Y}$, let $\alpha = Y \cap \omega_1$ and let P be the transitive collapse of Y. By Theorem 0.1, $\mathrm{SP}_{\alpha+1}(\mathcal{A}) = \mathrm{SP}_{\alpha+1}(\mathcal{A})^P$. Suppose toward a contradiction that there exists an $N \in \mathcal{A}_\alpha \setminus \mathcal{M}_Y$. Then $\mathrm{SP}_{\alpha+1}(N) \in P$. Proposition 5.19 of [3] then implies that the δ-th level of $\mathrm{SP}_{\delta+1}(N)$ amalgamates, as defined in Definition 5.16 of [3]. Since amalgamation is a first order property it is witnessed in P. It follows from Propositon 7.10 of [3] that there is a model of $\mathrm{SP}_{\delta+1}(N)$ in P, contradicting the elementarity of the collapse and the assumed property of $\mathcal{M}$. $\square$

The proofs of Theorems 0.1 and 0.2 can be used to prove the following variation, which we leave to the interested reader : there is fragment T of ZFC such that, if $x \subseteq \omega$ is a code for an analytic class $\mathcal{A}$ of τ-structures then for any transitive model P of T containing x and any ordinal α in P, if $\mathrm{SP}_\alpha(\mathcal{A})$ is countable then $\mathrm{SP}_\alpha(\mathcal{A}) = \mathrm{SP}_\alpha(\mathcal{A})^P$, and, if in addition $\alpha < \omega_2^P$, then every structure in $\mathcal{A}^*$ of Scott rank α is isomorphic to one in P.

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