Definable Combinatorics at the First Uncountable Cardinal

William Chan

University of North Texas

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Consequences of the Axiom of Choice

Example

Let X be any set. By the axiom of choice, AC, every set can be wellordered so there is an ordinal κ and a bijection $\Phi : X \to \kappa$. For each $\alpha < \kappa$, let $X_\alpha = \Phi^{-1}[\{\alpha\}]$. Thus $X = \bigcup_{\alpha<\kappa} X_\alpha$ where $|X_\alpha| = 1$.

The axiom of choice implies the class of cardinality of sets is wellorderable, and each cardinality has a canonical ordinal representative. Successor cardinals are regular.

The Axiom of Determinacy

The axiom of determinacy, AD, states that in every two player integer game, one of the two players has a winning strategy. Under AD, many combinatorial questions becomes an analysis of definability.

AD is incompatible with AC. Martin showed the first singular cardinal of determinacy is ω_3 with $\text{cof}(\omega_3) = \omega_2$. The first four infinite regular cardinals under AD are ω, ω_1, ω_2, and $\delta^1_3 = \omega_{\omega+1}$.

Cardinalities are no longer wellordered, and the study of the cardinal structure below familiar sets becomes very interesting and difficult.

The Axiom of Determinacy

Woodin showed under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$, there are only five uncountable cardinals under $|[\omega_1]^\omega|$ with the structure given below.

$$\begin{tikzcd}
 & {[\omega_1]^\omega} & \\
 & \mathbb{R} \times \omega_1 \arrow[u] & \\
 & \mathbb{R} \sqcup \omega_1 \arrow[u] & \\
\mathbb{R} \arrow[ur] & & \omega_1 \arrow[ul]
\end{tikzcd}$$

$\mathsf{ZF} + \mathsf{AD}_{\frac{1}{2}\mathbb{R}}$

Chan-Jackson

$\prod_{\mathscr{D}_J} \omega_1 / \mu_J \dashrightarrow \mathbb{R} \times \omega_1$

$L(\mathbb{R}) \models \mathsf{AD}$

$$\begin{tikzcd}
 & \mathbb{R} \times \omega_1 & \\
W_2^J \arrow[ur, dashed] & & W_2^J \sqcup \omega_1 \arrow[ul, dashed] \\
\mathbb{R} \arrow[u] & & \mathbb{R} \sqcup \omega_1 \arrow[u] \\
 & \omega_1 \arrow[ur] &
\end{tikzcd}$$

The Axiom of Determinacy

AD Chan-Jackson-Trang

$$\begin{tikzcd}
 & & & & & & {[\omega_2]^{\omega_2}} \\
 & \mathbb{R} \sqcup \omega_2 \arrow[r] & \mathbb{R} \times \omega_2 \arrow[r] & {[\omega_2]^{\omega}} \arrow[r] & {[\omega_2]^{<\omega_1}} \arrow[r] & {[\omega_2]^{\omega_1}} \arrow[r] & {[\omega_2]^{<\omega_2}} \arrow[u] \\
\omega_2 \arrow[ur] & & & & & & \\
 & \mathbb{R} \sqcup \omega_1 \arrow[r] \arrow[uu] & \mathbb{R} \times \omega_1 \arrow[r] \arrow[uu] & {[\omega_1]^{\omega}} \arrow[r] \arrow[uu] & {[\omega_1]^{<\omega_1}} \arrow[r] \arrow[uu] & {[\omega_1]^{\omega_1}} \arrow[uu] & \\
\omega_1 \arrow[uu] \arrow[ur] & \mathbb{R} \arrow[u] \arrow[rr, red] & & \color{red}{S_1} \arrow[ur, red] & & &
\end{tikzcd}$$

Woodin AD^+

Wellordered Partitions

$|\mathscr{P}(\omega)| = |[\omega]^\omega| = |\mathbb{R}|$. Under AD, the meager ideal has full wellordered additivity so wellordered union of meager sets are meager.

Fact

Assume AD. Suppose $\kappa \in \mathrm{ON}$ and $\langle X_\alpha : \alpha < \kappa \rangle$ is such that $\bigcup_{\alpha<\kappa} X_\alpha = \mathscr{P}(\omega)$. Then there is an α so that X_α is nonmeager and in particular $|X_\alpha| = |\mathscr{P}(\omega)|$.

Question (Zapletal for $\kappa = \omega_1$)

Assume $\mathrm{ZF} + \mathrm{AD}$. If $\kappa \in \mathrm{ON}$ and $\langle X_\alpha : \alpha < \kappa \rangle$ is such that $\bigcup_{\alpha<\kappa} X_\alpha = \mathscr{P}(\omega_1)$, then is there some $\alpha < \kappa$ so that $|X_\alpha| = |\mathscr{P}(\omega_1)|$?

The original question for ω_1 was solved by Chan and Jackson while studying the continuity of functions of the form $\Phi : [\omega_1]^{\omega_1} \to \omega_1$ and the club uniformization.

Wellordered Unions

$\mathscr{P}(\omega_1)$ is the next natural candidate after $\mathscr{P}(\omega)$ for a set which cannot be decomposed into subsets of smaller cardinality. Many of its smaller subsets such as $[\omega_1]^\omega$ and $[\omega_1]^{<\omega_1}$ can be decomposed into an ω_1-union of sets of size $|\mathbb{R}|$.

For instance, $[\omega_1]^{<\omega_1} = \bigcup_{\alpha \leq \beta < \omega_1} [\beta]^\alpha$ and $|[\beta]^\alpha| = |\mathbb{R}|$.

Wellordered Partitions of $\mathscr{P}(\omega_1)$

Theorem (Chan)

Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. Let $\kappa \in \mathrm{ON}$. Let $\Phi : [\omega_1]^{\omega_1} \to \kappa$. Then there exists a $\delta < \kappa$ so that $|\Phi^{-1}[\{\delta\}]| = |[\omega_1]^{\omega_1}|$.

Now to prove this theorem. Suppose this is not true. Let κ be minimal so that there is a $\Phi : [\omega_1]^{\omega_1} \to \kappa$ so that for all $\alpha < \kappa$, $|\Phi^{-1}[\{\alpha\}]| < |[\omega_1]^{\omega_1}|$. By minimality, one may assume that Φ is onto κ and κ is a regular cardinal. Since Φ is onto, an application of the Moschovakis coding lemma shows that $\kappa < \Theta$. Minimality also gives the following key observation:

(Key Observation) For all $\delta < \kappa$,
$|\Phi^{-1}[\delta]| = |\{f \in [\omega_1]^{\omega_1} : \Phi(f) < \delta\}| < |[\omega_1]^{\omega_1}|$.

Partition Relations

Fact (Martin, AD)

For all $\varepsilon \leq \omega_1$, $\omega_1 \to_* (\omega_1)^{\varepsilon}_2$: For all $P : [\omega_1]^{\varepsilon}_* \to 2$, there exists a club $C \subseteq \omega_1$ and an $i \in 2$ so that for all $f \in [C]^{\varepsilon}_*$, $P(f) = i$.

Let μ_{ω_1} be the filter on $[\omega_1]^{\omega_1}_*$ defined by $X \in \mu_{\omega_1}$ if and only if there is a club $C \subseteq \omega_1$ so that $[C]^{\omega_1}_* \subseteq X$. The partition property implies μ_{ω_1} is a countable complete ultrafilter, which is called the strong partition measure.

To show the partition property, Martin established a good coding system for ${}^{\omega_1}\omega_1$: Solovay showed that there is a formula $\mathfrak{T}$ in the language $\{\dot{\in}, \dot{E}\}$ so that for all $A \subseteq \omega_1 \times \omega_1$, there is some $x \in \mathbb{R}$ so that $(\alpha, \beta) \in A$ if and only if $L[x] \models \mathfrak{T}(\beta, \alpha)$. Let $\mathfrak{t}$ be a Skolem term for $\mathfrak{T}$ (on the first coordinate).

Partition Relations

For $\beta, \gamma < \omega_1$, let $x \in \mathrm{GC}_{\beta,\gamma}$ if and only if x satisfies the syntactic conditions of being a sharp, β belongs to the wellfounded part of $\Gamma(x, \beta + \omega)$, and $\gamma = \mathfrak{t}^{\Gamma(x,\beta+\omega)}(\beta)$. $\mathrm{GC}_{\beta,\gamma}$ is Δ^1_1. Let decode : $\mathbb{R} \to \mathscr{P}(\omega_1 \times \omega_1)$ be defined by decode$(x)(\beta, \gamma)$ if and only if $x \in \mathrm{GC}_{\beta,\gamma}$.

For any $f : \omega_1 \to \omega_1$, Solovay's result shows that there is some $x \in \mathbb{R}$ so that $(\alpha, \beta) \in f$ if and only if $L[x] \models \mathfrak{T}(\beta, \alpha)$. Then one has that decode$(x^\sharp) = f$.

Let $\mathrm{GC}_\beta = \bigcup_{\gamma < \omega_1} \mathrm{GC}_{\beta,\gamma}$. Let $\mathrm{GC} = \bigcap_{\beta < \omega_1} \mathrm{GC}_\beta$. One can check that GC_β has the key boundedness property that if $A \subseteq \mathrm{GC}_\beta$ is Σ^1_1, then there is some $\delta < \omega_1$ so that $A \subseteq \bigcup_{\gamma < \delta} \mathrm{GC}_{\beta,\gamma}$.

$(\Pi^1_1, \text{decode}, \text{GC}_{\beta,\gamma} : \beta, \gamma < \omega_1)$ is called a good coding system for ${}^{\omega_1}\omega_1$. The good coding system is devised so that in certain games where both players happen to play elements of GC through a particular club, Σ^1_1-boundedness can be used to show the player who wins controls which payoff set that the joint function belongs to but the losing player control the identity of the joint function. This idea is used to prove the partition relation as well as the following uniformization.

Theorem (Almost Everywhere Good Code Uniformization; AD)

Suppose $R \subseteq [\omega_1]^{\omega_1}_* \times \mathbb{R}$. Then there is a club $C \subseteq \omega_1$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \text{GC}$ so that $\text{decode}(x) \in [C]^{\omega_1}$ and $\text{block}(\text{decode}(x)) \in \text{dom}(R)$, $R(\text{block}(\text{decode}(x)), \Xi(x))$, where if $f \in {}^{\omega_1}\omega_1$, then $\text{block}(f)(\alpha) = \sup\{f(\omega \cdot \alpha + n) : n \in \omega\}$.

Regular Cardinal Greater than $\delta^1_3 = \omega_{\omega+1}$

Now suppose κ is a regular cardinal greater than or equal to $\delta^1_3 = \omega_{\omega+1}$. Assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, Steel showed that there is a $P \subseteq \mathbb{R}$ and a surjective map $\pi : P \to \kappa$ which is ω_ω-Suslin bounding, meaning if $Q \subseteq P$ is ω_ω-Suslin, then there is a $\delta < \kappa$ so that $\pi[Q] \subseteq \delta$.

Define a relation $R \subseteq [\omega_1]^{\omega_1}_* \times P$ by $R(f, x)$ if and only if $\pi(x) = \Phi(f)$. By the almost everywhere good code uniformization, there is a club $C \subseteq \omega_1$ and a Lipschitz function $\Xi : \mathbb{R} \to \mathbb{R}$ so that for all $x \in \mathrm{GC}$, if $\mathrm{decode}(x) \in [C]^{\omega_1}$, then $R(\mathrm{block}(\mathrm{decode}(x)), \Xi(x))$.

Solovay showed that from a certain game, one can find a real z which code a club $\mathfrak{C}_z \subseteq C$ in a particular explicit way. The set $\mathrm{GC}(\mathfrak{C}_z)$ consisting of those $x \in \mathrm{GC}$ so that $\mathrm{decode}(x) \in [\mathfrak{C}_z]^{\omega_1}$ is Π^1_2. (The complexity essentially comes from saying x is a sharp of a real.)

Thus $\Xi[\mathrm{GC}(\mathfrak{C}_z)] \subseteq P$ is Σ^1_3 and therefore ω_ω-Suslin. Thus since π satisfies ω_ω-Suslin bounding, there is a $\delta < \kappa$ so that $\pi[\Xi[\mathrm{GC}(\mathfrak{C}_z)]] \subseteq \delta$. One can find an uncountable set $A \subseteq \omega_1$ so that for all $f \in [A]^{\omega_1}$, there is an $x \in \mathrm{GC}(\mathfrak{C}_z)$ so that $\mathrm{block}(\mathrm{decode}(x)) = f$. Hence $\Phi(f) = \Phi(\mathrm{block}(\mathrm{decode}(x)) = \pi(x) < \delta$. One has shown that $|\Phi^{-1}[\delta]| = |[\omega_1]^{\omega_1}|$. This contradicts the key observation from earlier

(Key Observation) For all $\delta < \kappa$, $|\Phi^{-1}[\delta]| = |\{f \in [\omega_1]^{\omega_1} : \Phi(f) < \delta\}| < |[\omega_1]^{\omega_1}|$.

Now it remains to analyze the countable regular cardinals, ω_1, and ω_2.

If $\Phi : [\omega_1]^{\omega_1} \to \kappa$ where κ is finite or ω, then the countable additivity of μ_{ω_1} implies that there is a $\delta < \kappa$ and a club C so that $\Phi(f) = \delta$ for all $f \in [C]^{\omega_1}_*$. Since $|[C]^{\omega_1}_*| = |[\omega_1]^{\omega_1}|$, this implies $|\Phi^{-1}[\{\delta\}]| = |[\omega_1]^{\omega_1}|$ which violates the assumption that Φ is a counterexample to the theorem.

The ω_1 Case

This case is immediate from the following continuity property. Under AD, every function $F : \mathbb{R} \to \mathbb{R}$ is continuous on a comeager set.

Theorem (Chan-Jackson, AD, Almost Everywhere Continuity)

Let $\Phi : [\omega_1]^{\omega_1} \to \omega_1$. There is a club C so that $\Phi \restriction [C]^{\omega_1}$ is a continuous function: for all $f \in [C]^{\omega_1}$, there is an $\alpha < \omega_1$ so that for all $g \in [C]^{\omega_1}$, if $g \restriction \alpha = f \restriction \alpha$, then $\Phi(g) = \Phi(f)$.

The proof of this continuity result requires using a particular partition $P : [\omega_1]^{\omega_1} \to \omega_1$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club homogeneous for P, but the heart of the argument is to show that the club is homogeneous for the desired side.

To prove this, one needs to perform a construction that requires an ω_1-length dependent choice of club subsets of ω_1. This is made possible by club uniformization.

The Club Uniformization

Using the simple Kechris-Woodin generic coding function for ω_1 and category arguments,

Theorem (Chan-Jackson. Everywhere Club Uniformization)

(AD) Let $R \subseteq [\omega_1]^{<\omega_1}_* \times \text{club}_{\omega_1}$ which is $\subseteq$-downward closed, meaning for all $\sigma \in [\omega_1]^{<\omega_1}$ and $C \subseteq D$, if $R(\sigma, D)$, then $R(\sigma, C)$. Let $\tilde{R} \subseteq \mathbb{R} \times \mathbb{R}$ be the coded version of R. If $\tilde{R}$ has a uniformization, then there is a function $\Psi : [\omega_1]^{<\omega_1} \cap \mathrm{dom}(R) \to \text{club}_{\omega_1}$ so that for all $\sigma \in \mathrm{dom}(R)$, $R(\sigma, \Psi(\sigma))$. Thus under $\mathrm{AD}_{\frac{1}{2}\mathbb{R}}$, everywhere club uniformization holds.

Everywhere club uniformization fails in $L(\mathbb{R})$.

Using the Moschovakis coding lemma and the good code uniformization, one can find a club C so that associated coded relation $\tilde{R}$ is projective and hence uniformizable in AD.

The Club Uniformization

Theorem (Chan-Jackson, Almost Everywhere Club Uniformization)

(AD). For every $R \subseteq [\omega_1]^{<\omega_1} \times \text{club}_{\omega_1}$ which is $\subseteq$-downward closed. There is a club C and a function $\Psi : [C]^{<\omega_1}_* \cap \mathrm{dom}(R) \to \text{club}_{\omega_1}$ so that for all $\sigma \in [C]^{<\omega_1}_* \cap \mathrm{dom}(R)$, $R(\sigma, \Psi(\sigma))$.

The almost everywhere club uniformization is especially useful for studying the stable theory of the strong partition measure.

Theorem (Chan-Jackson-Trang; AD)

For μ_{ω_1} almost all f, $L[f]$ satisfies GCH, Σ^1_1-determinacy, the failure Δ^1_2-determinacy, and f is a generalized Prikry sequence for an inner models $L[\bar{\nu}_f]$ where $\bar{\nu}_f$ is ω_1-sequence of normal measures with discontinuous sequence of critical points.

The ω_2 Case

Example

Martin showed ${}^{\omega_1}\omega_1/\mu_1 = \omega_2$, where μ_1 is the club measure on ω_1. Let $\Phi_0 : [\omega_1]^{\omega_1} \to \omega_2$ be defined by $\Phi_0(f) = [f]_{\mu_1}$. For any club $C \subseteq \omega_1$, one has that $\Phi_0[[C]^{\omega_1}] = [C]^{\omega_1}/\mu_1$ which is cofinal through ω_2. This shows that Φ_0 cannot satisfy the continuity property for functions $\Psi : [\omega_1]^{\omega_1} \to \omega_1$. Also the set $\mathrm{GC}(\mathfrak{C}_z)$ can never be Σ^1_2 (for any choice of good coding system) since otherwise one can repeat the earlier argument with an norm $\pi : P \to \omega_2$ which satisfy ω_1-Suslin bounding to show that there is a $\delta < \omega_2$ so that $\Phi_0[[\mathfrak{C}_z]^{\omega_1}] \subseteq \delta$, which is impossible.

However for any $\omega_1 \leq \alpha < \omega_2$, $|\Phi_0^{-1}[\{\alpha\}]| = |[\omega_1]^{\omega_1}|$ since one can modify a function on some set A with $|A| = \omega_1$ and $A \notin \mu_1$ in arbitrary ways without changing the value of Φ_0. This hints at the general argument.

The ω_2 Case

Now suppose $\Phi : [\omega_1]^{\omega_1} \to \omega_2$ is the minimal counterexample to the theorem. Let $\mathfrak{L}$ the lexicographical ordering on $\omega_1 \times 2$. If $F : \mathscr{L} \to \omega_1$, let $F_i : \omega_1 \to \omega_1$ be defined by $F_i(\alpha) = F(\alpha, i)$. Define a partition $P : [\omega_1]^{\mathscr{L}}_* \to 2$ by $P(F) = 0$ if and only if $\Phi(F_0) \leq \Phi(F_1)$. By $\omega_1 \to_* (\omega_1)^{\omega_1}_2$, there is a club $C \subseteq \omega_1$ homogeneous for P. It must homogeneous taking value 0 since the other case yields a violation of the wellfoundedness of ω_2.

$\langle a_\alpha, b_\alpha, c_\alpha : \alpha < \omega_1 \rangle$ be discontinuous sequence of blocks of three consecutive elements of C. Let $h(\alpha) = c_\alpha$. Note that for any function f so that $f(\alpha) \in \{a_\alpha, b_\alpha\}$, one can find a $F \in [C]^{\mathscr{L}}_*$ so that $F_0 = f$ and $F_1 = h$. Thus by the partition, one has that $\Phi(f) \leq \Phi(h)$ for any such function f. One construct an injection $\Psi' : [\omega_1]^{\omega_1} \to [h]_{\mu_1}$. Since $[h]_{\mu_1} < \omega_2$, one essentially has a function $\Psi : [\omega_1]^{\omega_1} \to \omega_1$. Now by an application of the continuity property for ω_1, one can show that there is an $\alpha < \omega_2$ so that $|\Phi^{-1}[\{\alpha\}]| = |[\omega_1]^{\omega_1}|$. This violates the assumption that Φ was a counterexample to the theorem.

Having considered every regular cardinal, one has completed the proof of the main theorem.

Theorem (Chan)

Assume $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. Let $\kappa \in \mathrm{ON}$. Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence so that $\bigcup_{\alpha<\kappa} X_\alpha = \mathscr{P}(\omega_1)$, then there is a $\alpha < \kappa$ so that $|X_\alpha| = |\mathscr{P}(\omega_1)|$.

Thanks for listening.