THE EXTENDER ALGEBRA AND PRESERVING STATIONARY SETS

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Abstract. We give an instance when the extender algebra can preserve stationary subsets of ω_1. In particular, we show that for any model operator satisfying certain conditions (satisfied by the currently known minimal inner models for large cardinal statements), any Ω-consistent statement about a rank initial segment of the universe can be forced over canonical model containing $H(\omega_2)$ while preserving stationary subsets of ω_1. This is a variation of Theorem 10.13 of [8].

We will use the following phrasing of Woodin's extender algebra theorem (see [7, 4, 6, 2]). The notion of iterability here and in the statement of our main theorem refers to the existence of iteration strategies for iteration trees of arbitrary length.

Theorem 0.1. *Let M be an iterable model, and let δ be a Woodin cardinal in M. Then for any set x and any $\lambda < \delta$ there is an elementary embedding $j\colon M \to M^*$ with critical point greater than λ such that x is M^*-generic for a partial order in M^* of cardinality $j(\delta)$.*

The following well-known fact is used to produce $\mathbb{P}_{max}$ conditions from large cardinals, and will be used in our argument in almost the same way. Proofs appears in [5, 3].

Lemma 0.2. *Let θ be a regular cardinal, suppose that T is a weakly homogeneous tree on $\omega \times Z$ in $H(\theta)$, for some set Z. Let $\gamma \geq 2^\omega$ be an ordinal such that there exists a countable collection Σ of γ^+-complete measures witnessing the weak homogeneity of T. Assume that there is a measurable cardinal in the interval (γ, θ).*

Then for every elementary submodel X of $H(\theta)$ of cardinality less than γ with T, Σ, $\gamma \in X$, there is an elementary submodel Y of $H(\theta)$ containing X such that $Y \cap \theta$ is uncountable, $Y \cap \gamma = X \cap \gamma$ and $p[T \cap Y] = p[T]$.

Fixing a recursive bijection $\pi\colon \omega\times\omega \to \omega$, we use the following coding of elements of $H(\omega_1)$ by subsets of ω: $x \subseteq \omega$ codes $a \in H(\omega_1)$ if

$$\langle \omega, \{(n,m) \mid \pi(n,m) \in x\}\rangle \cong \langle \{a\} \cup tc(a), \in\rangle,$$

where $tc(a)$ is the transitive closure of a. Under this coding, the relations " $\in$ " and "=" are both Σ^1_1, since permutations of ω can give rise to different codes for the same set. We say that a function $f\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is *invariant in the codes* if whenever x and y code the some element of $H(\omega_1)$, $f(x)$ and $f(y)$ do as well. Note that if a function $f\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is universally Baire and invariant in the codes, it induces a class function from V to V: for any set Z in any $H(\kappa)$, letting f^* denote the extension of f in the $Coll(\omega,\kappa)$-extension, the set coded by $f^*(x)$ exists already in V, for x any subset of ω in this extension coding Z.

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The statement of our main theorem uses the notion of A-closure from Woodin's Ω-logic. For our purposes, we need to know only that if A is a universally Baire function which is invariant in the codes and N is an A-closed model, then N is closed under the induced class function from the previous paragraph (see [8, 1]).

The proof of our main theorem uses another notion of iterability, that of producing wellfounded models under generic iterated embeddings using the stationary tower. The following fact is proved in [8].

Theorem 0.3. *Suppose that Q is a transitive model containing ω_1 in which countable ordinals $\delta < \lambda$ are a Woodin cardinal and a strongly inaccessible cardinal, respectively. Then Q and V^Q_λ are both iterable with respect to $\mathbb{Q}^Q_{<\delta}$.*

Theorem 0.4. *Suppose that*

- *$\delta_0 < \delta_1$ are a Woodin cardinals below a measurable cardinal.*
- *A is a set of reals coding a function that takes each real x to a model $M(x)$ of ZFC + T containing x and an iteration strategy for $M(x)$, in a way that is invariant for some coding of elements of $H(\omega_1)$ by reals, such that A and $\omega^\omega \setminus A$ are δ_1^+-weakly homogeneously Suslin;*
- *ϕ is a statement of the form "some rank initial segment of the universe satisfies ψ", for some statement ψ;*
- *for every real r, ϕ holds in an A-closed model of ZFC containing r.*

Then for every set $Z \in V_{\delta_1}$ such that $M(Z)$ is NS_{ω_1}-correct, ϕ can be forced over $M(Z)$ by a forcing preserving stationary subsets of ω_1.

Proof. Let $\kappa > \delta_1$ be measurable, and let θ be a regular cardinal greater than 2^κ. Let λ be a strongly inaccessible cardinal between δ_0 and δ_1 with $Z \in V_\lambda$. Applying Lemma 0.2, let Y be an elementary submodel of $H(\theta)$ with κ, δ_0, δ_1, Z and A as members such that $Y \cap \delta_1$ is uncountable, $Y \cap \lambda$ is countable, and such that there exist trees S and T on $\omega \times \gamma$ (for some ordinal γ) in Y such that $p[S \cap Y] = A$, $p[T \cap Y] = \omega^\omega \setminus A$.

Let Q be the transitive collapse of Y and let S_Q, T_Q, κ_Q, δ_{0Q}, δ_{1Q}, λ_Q and Z_Q be the images of S, T, κ, δ_0, δ_1, λ and Z under this collapse. Let P denote $V^Q_{\kappa_Q}$. Since $\omega_1 \subset Q$, Q is iterable with respect to $\mathbb{Q}^Q_{<\delta_{0Q}}$, and therefore P is as well. Let N be a countable A-closed model of ZFC + ϕ with P as an element. Let γ be an ordinal such that $V^N_\gamma \models \psi$. Let $j \colon P \to P'$ be an iterated generic elementary embedding of length ω_1^N in N via $\mathbb{Q}^P_{<\delta_{0Q}}$ such that P' is correct about stationary subsets of ω_1 in N ([8]). Thus induces an iteration of Q with the same generic filters, such that P' is a rank initial segment of the corresponding final model Q'. We let j denote the entire embedding from Q to Q'.

Since Q' is wellfounded and the projections of $j(S_Q)$ and $j(T_Q)$ are disjoint in Q', they are disjoint in V as well. Since $p[S] = p[S_Q] \subset p[j(S_Q)]$ and $p[T] = p[T_Q] \subset p[j(T_Q)]$, it follows that $p[j(S_Q)] = p[S]$ and $p[j(T_Q)] = p[T]$. Therefore, $M(j(Z_Q))$ exists in Q' and is coded by a real in the projection of $p[jS_Q]$ in any $Coll(\omega, j(Z_Q))$ extension of Q' (this is just to say that $M(j(Z_Q))$ is definable in Q' from $j(S_Q)$ and $j(Z_Q)$, which since $M(Z)$ is definable in the same way from S and Z means that facts about $M(j(Z_Q))$ true in Q' will be true about $M(Z)$ in $H(\theta)$).

Since N is A-closed, $M(j(Z_Q))$ is in N as well. Furthermore, $M(j(Z_Q))$ is NS_{ω_1}-correct in P' and thus in N. Let $k \colon M(j(Z_Q)) \to M^*$ be an elementary embedding in N with critical point greater than 2^{ω_1} such that V^N_γ is generic over M^*. Then

M^* and $M(j(Z_P))$ have the same $\mathcal{P}(\omega_1)$, so M^* is NS_{ω_1}-correct in N, so M^* must be NS_{ω_1}-correct in $M^*[V_\gamma^N]$. Furthermore,

$$V_\gamma^{M^*[V_\gamma^N]} = V_\gamma^N.$$

Therefore, $M^*[V_\gamma^N]$ is an NS_{ω_1}-preserving forcing extension of M^* satisfying ϕ, so $M(j(Z_Q))$, $M(Z_Q)$ and $M(Z)$ all also have NS_{ω_1}-preserving forcing extensions satisfying ϕ. $\square$

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