Forcing Axioms and Definabilty of the Nonstationary Ideal on ω_1

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Abstract

We show that under BMM and "there exists a Woodin cardinal", the nonstationary ideal on ω_1 can not be defined by a Σ_1 formula with parameter $A \subset \omega_1$. We show that the same conclusion holds under the assumption of Woodin's $(*)$-axiom. We further show that there are universes where BPFA holds and NS_{ω_1} is $\Sigma_1(\omega_1)$-definable. Last we show that if the canonical inner model with one Woodin cardinal M_1 exists, there is a universe where NS_{ω_1} is saturated and $\Sigma_1(\omega_1)$-definable, and MA_{ω_1} holds.

1 Introduction

This article deals with the possibility of a (boldface) $\boldsymbol{\Delta_1}$-definition (over $H(\omega_2)$) of the nonstationary ideal on ω_1 in the presence of various forcing axioms. As we shall see, stronger assumptions rule out the existence of such $\boldsymbol{\Sigma_1}$-definitions, whereas weaker assumptions are consistent with such $\boldsymbol{\Sigma_1}$-definitions, even in the presence of NS_{ω_1} being saturated.

The main results are as follows.

Theorem 1.1. *Assume* BMM *and that there exists a Woodin cardinal. Then for no Σ_1-formula $\varphi(v_0, v_1)$ and no parameter $A \subset \omega_1$ does it hold that*

$$\forall S \in P(\omega_1)(S \textit{ is stationary } \Leftrightarrow \varphi(S, A)).$$

Theorem 1.2. *Assume that Woodin's axiom $(*)$ holds. Then there do not exist an $A \subset \omega_1$ and a Σ_1-formula $\varphi(-, A)$ in the language of set theory such that*

$$\forall S \in P(\omega_1)\,(S \textit{ is stationary } \Leftrightarrow \varphi(S, A)).$$

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In contrast to these two impossibility results we also obtain two theorems which show that under weaker assumptions, Σ_1 definitions of NS_{ω_1} are possible.

Theorem 1.3. *There is a universe in which* BPFA *holds and* NS_{ω_1} *is* $\Sigma_1(\omega_1)$-*definable.*

Theorem 1.4. *Assume that the canonical inner model with one Woodin cardinal* M_1 *exists. Then there is a generic extension of* M_1 *where* NS_{ω_1} *is saturated and* $\Sigma_1(\omega_1)$-*definable and* MA_{ω_1} *holds.*

The paper is organized as follows. We will prove the theorems in the order stated above, thus we start with the two impossibility results, then follow up with the two possibility results. The methods and techniques which are used in this article are quite varied and we will provide only very little preliminary definitions, instead assuming the reader knows the basics of the stationary tower forcing (see [8] for an extensive account) and $\mathbb{P}_{\max}$ (see [7] or [17]), as well as the coding technique of A. Caicedo and B. Velickovic ([1]).

2 Impossibility Results

This section collects two results which show that strong assumptions entail the impossibility of a boldface Σ_1-definition of NS_{ω_1}. We assume that the reader is familiar with Woodin's stationary tower and with $\mathbb{P}_{\max}$-forcing.

2.1 Impossibility under BMM plus the existence of a Woodin Cardinal

The goal of this section is to prove that under BMM+"there exists a Woodin cardinal", no Σ_1 formula (boldface) can define stationary subsets of ω_1 correctly. We use Bagaria's theorem ([2]) saying that BMM is equivalent to the statement that $H(\aleph_2)$ is Σ_1-elementary in the $H(\aleph_2)$ of any forcing extension in which every stationary subset of ω_1 from the ground model remains stationary.

Theorem 2.1. *Assume* BMM *and the existence of a Woodin cardinal* δ. *Then for no parameter* $A \subset \omega_1$ *and for no* Σ_1-*formula* $\varphi(-, A)$ *in the language of set theory, does* φ *define the stationary subsets of* ω_1 *i.e, we do not have that*

$$\forall T \in P(\omega_1)(T \text{ is stationary } \Leftrightarrow \varphi(T, A)).$$

Proof. Assume for a contradiction that there is a Σ_1- formula φ and a set $A \subset \omega_1$ such that $\forall T \in P(\omega_1)(T$ is stationary iff $\varphi(T, A))$. Let δ be our Woodin cardinal.

Let

$$S_0 = \{X \prec H(\omega_2) : |X| = \aleph_1 \wedge X \text{ is transitive}\}$$

and let g be $\mathbb{P}_{<\delta}$-generic over V where $\mathbb{P}_{<\delta}$ is the associated full stationary tower. Let us pick a generic filter g which contains S_0, which is possible by the stationarity of S_0. As usual we can form the generic elementary embedding in the universe $V[g]$:

$$j : V \to M \subset V[g]$$

for a transitive inner model M of $V[g]$. Membership in the generic filter g for the stationary tower forcing can be characterized using j, namely we have that

$$\forall a (a \in g \leftrightarrow j" \bigcup a \in j(a))$$

thus $S_0 \in g$ yields $j"H(\omega_2)^V \in j(S_0)$. In particular $j"H(\omega_2)^V$ is transitive, and as $H(\omega_2)^V$ is the transitive collapse of $j"H(\omega_2)^V$, we obtain that $j"H(\omega_2)^V = H(\omega_2)^V$ and that the critical point $crit(j)$ of the elementary embedding j must be $\geq \omega_2^V$. As $H(\omega_2)^V \in j(S_0)$, $|H(\omega_2)^V| = \aleph_1$ in M, so $crit(j) = \omega_2^V$.

We have $P(\omega_1) \cap V \in M$. Indeed by BMM, $2^{\aleph_1} = \aleph_2$ [16, 11], so let $f : \omega_2 \to P(\omega_1) \cap V$ be a bijection with $f \in V$. Then $j(f) \restriction \omega_2^V = f$ and $j(f) \in M$ so $P(\omega_1) \cap V = ran(f) \in M$.

It is a theorem of A. D. Taylor (see [14]) that MA_{ω_1} implies that NS_{ω_1} is not ω_1-dense. As $P(\omega_1) \cap V$ has size $|\omega_2^V| = \aleph_1$ in M, there is a stationary set $D \subset \omega_1$ in M such that $T \backslash D$ is stationary for every stationary $T \in P(\omega_1) \cap V$. By Theorem 2.5.8 of [8], $V[g] \models M^{<\delta} \cap V[g] \subset M$. Since δ remains strongly inaccessible in $V[g]$, this implies that $V[g]$ is a stationary set preserving extension of V. Further it is still true in $V[g]$ that D is stationary and $T \backslash D$ is stationary for all $T \notin \mathrm{NS}_{\omega_1}^V$.

In the next step we use the ordinary club shooting forcing $\mathbb{P}_{\omega_1 \backslash D}$ over $V[g]$ to shoot a club through the complement of D. The forcing will not destroy any stationary subsets from $V \cap P(\omega_1)$:

Fact 2.2. *If h denotes a generic filter for $\mathbb{P}_{\omega_1 \backslash D}$ over $V[g]$, then if $T \in P(\omega_1) \cap V$ is stationary in V then it will remain stationary in $V[g, h]$.*

Proof. Fix a stationary (in V) $T \in P(\omega_1) \cap V$ and let $p \in \mathbb{P}_{\omega_1 \backslash D}$ be a condition and τ be a name in $V[g]^{\mathbb{P}_{\omega_1 \backslash D}}$ such that $p \Vdash \tau$ is a club in ω_1. We shall find a $q' < p$ such that $q' \Vdash \tau \cap T \neq \emptyset$.

As $T \backslash D$ is stationary in $V[g]$, we fix a sufficiently large regular θ and may pick a countable $X \prec H(\theta)^{V[g]}$ such that $p, \mathbb{P}_{\omega_1 \backslash D}$ and τ are elments of X and which satisfies $\alpha = X \cap \omega_1 \in T \backslash D$. It is straightforward to construct a condition $q \in \mathbb{P}_{\omega_1 \backslash D}$, $q < p$ with domain α such that for every dense $D \subset \mathbb{P}_{\omega_1 \backslash D}$, $D \in X$ there is a $\xi < \alpha$ such that $q \restriction \xi \in D \cap X$. Finally $q' := q \cup \{(\alpha, \alpha)\}$ is as desired.

$\square$

So $V[g,h]$ is a stationary set preserving extension of V. But now by our hypothesis and by elementarity of $j : V \to M$ we get that

$$M \models \varphi(D, A)$$

and hence

$$V[g] \models \varphi(D, A)$$

as φ is Σ_1, and consequentially

$$V[g,h] \models \varphi(D, A).$$

In $V[g,h]$ the set D is nonstationary, thus

$$V[g,h] \models \exists D(D \text{ is nonstationary } \wedge \varphi(D, A)).$$

This statement is Σ_1 with parameter $A \subset \omega_1$ in the language of set theory, and as BMM is assumed to hold true in V we conclude that

$$V \models \exists D(D \text{ is nonstationary } \wedge \varphi(D, A))$$

which is a contradiction. □

2.2 Impossibility under $(*)$

Our next goal is to derive the same conclusion from Woodin's $(*)$-principle. Recall that the $(*)$-principle states that

- AD holds in $L(\mathbb{R})$ and
- $L(P(\omega_1))$ is a $\mathbb{P}_{\max}$-generic extension of $L(\mathbb{R})$.

It has been shown very recently by the third author and D. Aspero that MM^{++} implies $(*)$, solving a long-standing open question. Its proof paved the way for a third impossibility result, namely that under MM, there is no $A \subset \omega_1$ and no Σ_1-formula which defines stationarity. The proof is due to the third author and Xiuyuan Sun and will appear soon (see [12]).

Theorem 2.3. *Assume that $(*)$ holds. Then there do not exist an $A \subset \omega_1$ and a Σ_1-formula $\varphi(-, A)$ in the language of set theory such that*

$$\forall T \in P(\omega_1)\, (T \textit{ is stationary } \Leftrightarrow \varphi(T, A)).$$

Proof. Let $V = L(\mathbb{R})[g]$, where g is $\mathbb{P}_{\max}$-generic over $L(\mathbb{R})$. Suppose toward a contradiction that $A \subseteq \omega_1$ and φ witness that the conclusion of the theorem fails. Since $P(\omega_1) = P(\omega_1)_G$, there exist a condition $p = \langle (M, I), b\rangle \in g$ and a set $a \in P(\omega_1)^M$ such that $A = j_{p,g}(a)$, where $j_{p,g}$ is the unique iteration of (M, I) sending b to $A_g = \bigcup\{c : \langle (N, J), c\rangle \in g\}$. By the genericity of g, it suffices to find a condition $r = \langle (P, K), c\rangle < p$ and a set $e \in K$ such that $H(\aleph_2)^P \models \varphi(e, j_{p,r}(a))$, where $j_{p,r}$ is the iteration of (M, I) sending b to c.

First, let $q = \langle (N, J), d\rangle$ any condition below p, as witnessed by the iteration $j_{p,q}\colon (M, I) \to (M^*, I^*)$. Then $|M^*| = \aleph_1$ in N, and $N \models \mathsf{MA}_{\omega_1}$. Hence by the result of Taylor cited above, there is no ω_1-dense, normal ideal in N. In particular, there must be a $T \in (J^+)^N$ such that $S\backslash T$ is an element of $(J^+)^N$ for all $S \in P(\omega_1)^{M^*} \setminus I^*$.

Again by the genericity of g, there is a condition $r_0 = \langle (P_0, K_0), c\rangle$ below q, as witnessed by the $j_{q,r_0}\colon (N, J) \to (N^*, J^*)$, such that the formula $\varphi(j_{q,r_0}(T), j_{q,r_0}(j_{p,q}(a)))$ holds in $H(\aleph_2)^{P_0}$. Since $L(\mathbb{R}) \models \mathsf{AD}$, we may also assume that there exists a Woodin cardinal in P_0, and that K_0 is $\mathrm{NS}_{\omega_1}^{P_0}$ (this follows, for instance, from Theorem 5.36 of [17] with $n = 2$, and the fact that the partial order $\mathrm{Col}(\omega_1, {<}\delta)$ forces NS_{ω_1} to be presaturated whenever δ is a Woodin cardinal).

Now, $j_{q,r_0}(j_{p,q})$ is an iteration of (M, I) to $(j_{q,r_0}(M^*), j_{q,r_0}(I^*))$ sending b to c. By the definition of the order on $\mathbb{P}_{\max}$ conditions, $J^* = N^* \cap K_0$, from which it follows that $S\backslash j_{q,r_0}(T)$ is stationary in P_0, for all S in

$$j_{q,r_0}(P(\omega_1)^{M^*} \setminus I^*).$$

Applying Fact 2.2 above, force over P_0 to make $j_{q,r_0}(T)$ nonstationary, while preserving the stationarity of each member of $j_{q,r_0}(P(\omega_1)^{M^*} \setminus I^*)$. Call this extension P_0'.

The rest of the argument consists of the standard machinery for building $\mathbb{P}_{\max}$ conditions. Let P be the result of forcing over P_0' with $\mathrm{Col}(\omega_1, {<}\delta)^{P_0'}$ followed by some c.c.c. forcing making MA_{ω_1} hold. Let P be this forcing extension, and let K be $\mathrm{NS}_{\omega_1}^P$. Then $r = \langle (P, K), c\rangle$ and $e = j_{q,r_0}(T)$ are as desired. □

3 Possibility Results

3.1 BPFA and the Π_1-definabilty of NS_{ω_1}

The goal of this section is to show that BPFA is consistent with a Σ_1-definition of $\mathrm{NS}_{\omega_1}^+$. The proof relies on a new coding technique which exploits mutually stationary sets.

3.1.1 Mutually stationary preserving forcing

Definition 3.1. *Let K be a collection of regular cardinals with bounded below κ, and suppose that we have $S_\eta \subseteq \eta$ for each $\eta \in K$. Then the collection of sets $\{S_\eta \mid \eta \in K\}$ is* mutually stationary *if and only if for all algebras $\mathcal{A}$ on*

κ, there is an $N \prec \mathcal{A}$ such that

$$\text{for all } \eta \in K \cap N,\ \sup(N \cap \eta) \in S_\eta.$$

Foreman-Magidor ([3]) show that every sequence $\vec{S}$ with $S_\eta \subset \eta \cap Cof(\omega)$ is mutually stationary. Let $\mathcal{T}_{\vec{S}}$ be the collection of all countable N such that for all $\eta_i \in N$, $\sup(N \cap \eta_i) \in S_i$.

Theorem 3.2 (Foreman-Magidor)**.** *Let $\langle \eta_i \mid i < j \rangle$ be an increasing sequence of regular cardinals. Let $\vec{S} = \langle S_i \mid i < j \rangle$ be a sequence of stationary sets such that $S_i \subseteq \eta_i \cap Cof(\omega)$. If θ is a regular cardinal greater than all η_i and $\mathcal{A}$ is a algebra on θ, then there is a $N \prec \mathcal{A}$ belongs to the class $\mathcal{T}_{\vec{S}}$. In particular, $\vec{S}$ is mutually stationary.*

From now on, we assume all stationary subsets of ordinals discussed in this section are concentrated on countable cofinality. The corresponding notion for being club in this context is that of an unbounded set which is closed under ω-sequences.

Definition 3.3. *Suppose $\vec{S} = \{S_\eta \mid \eta \in K\}$ is mutually stationary and that for every $\eta \in K$, S_η is stationary, co-stationary in $\eta \cap Cof(\omega)$. We say a forcing poset $\mathbb{P}$ is $\vec{S}$-preserving if the following holds: Suppose $\theta > 2^{|\mathbb{P}^+|}$ is regular. Suppose M is a countable elementary submodel of $H(\theta)$ with $\{\mathbb{P}, \vec{S}\} \subset M$ and $M \in \mathcal{T}_{\vec{S}}$. Suppose $p \in \mathbb{P} \cap M$. Then there exists a $(M, \mathbb{P})$-generic condition q extending p.*

Remark

1. Any proper forcing is $\vec{S}$-preserving.
2. When $K = \{\omega_1\}$ and $\vec{S} = S \subset \omega_1$, the definition of $\vec{S}$-preserving is identical to the usual definition of S-proper forcing.
3. Let $\vec{S}$ be such that each $S_\eta \in \vec{S}$ is stationary, co-stationary in $\eta \cap Cof(\omega)$. Then an example of a non-proper, $\vec{S}$ preserving forcing is the forcing poset $Club(S_\eta)$, i.e, the forcing which adds an unbounded subset to S_η which is closed under ω-sequences, via countable approximations.

The iteration theorems for countable support iterated proper forcing can be generalized to $\vec{S}$-preserving forcing.

Lemma 3.4. *If $\langle P_i, \dot{Q}_i \mid i < \alpha \rangle$ is a countable support iterated forcing system and for each $i < \alpha$, $\Vdash_{P_i}$ "$\dot{Q}_i$ is $\vec{S}$-preserving" then P_α is $\vec{S}$-preserving.*

Proof. (Sketch, following the proof of [13], Theorem 3.2) We will only need to show by induction on $j \leq \alpha$ that for any $N \in \mathcal{T}_{\vec{S}}$, if $j \in N$, then:

$(*)_N$ For all $i < j$, $i \in N$ and for all $p \in N \cap P_j$ and $q \in P_i$ is (N, O_i) generic condition extending $p \restriction i$, there is an $r \in P_j$ such that r is (N, P_j)-generic condition extending p and $r \restriction i = q$.

The statement $(*)_N$ is identical to the statement $(*)$ in the proof of [13], Theorem 3.2. It can be checked that the original proof also works here. □

In our proof of the main theorem of this section, we will use forcings which have a specific form, so they get a name.

Definition 3.5. *Let κ be an inaccessible cardinal. Let $\vec{S} = \langle S_i \mid i < \kappa \rangle$ be mutually stationary. We say a forcing poset $\mathbb{P}$ is a $\vec{S}$-coding if $\delta \leq \kappa$ and $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ satisfies the following:*

- *$\mathbb{P}$ is a countable support iteration*
- *For any $\alpha < \delta$, one of the followings holds:*
  1. *Assume that α is inaccessible and $\mathbb{P}_\alpha$ is forcing equivalent to a forcing of size less or equal than α.*[1] *Assume that in $V^{\mathbb{P}_\alpha}$, $\langle B_\beta \mid \beta < 2^\alpha \rangle$ is an enumeration of V_α. Then $\dot{Q}_\alpha$ is allowed to be*

     $$\prod_{j \in B_\beta} Club(S_{\alpha\cdot(\beta+1)+2j}) \times \prod_{j \notin B_\beta} Club(S_{\alpha\cdot(\beta+1)+2j+1})$$

     using countable support.
  2. *In all other cases, we have that $\Vdash_{\mathbb{P}_\alpha} \dot{Q}_\alpha$ is proper.*

Let η be an regular cardinal, we say $\mathbb{P}$ is an η-$\vec{S}$ coding if (1) is replaced by

(1') $\alpha \geq \eta$ is inaccessible and $\mathbb{P}_\alpha$ is forcing equivalent to a forcing of size less or equal than α. In $V^{\mathbb{P}_\alpha}$, $\langle B_\beta \mid \beta < 2^\alpha \rangle$ is an enumeration of V_α. Then $\dot{Q}_\alpha$ is allowed to be

$$\prod_{j \in B_\beta} Club(S_{\alpha\cdot(\beta+1)+2j}) \times \prod_{j \notin B_\beta} Club(S_{\alpha\cdot(\beta+1)+2j+1})$$

By Lemma 3.4, if $\mathbb{P}$ is a $\vec{S}$-coding forcing, then $\mathbb{P}$ is $\vec{S}$-preserving. It is also clear that an iteration is a $\vec{S}$-coding forcing if and only all its initial segments are $\vec{S}$-coding forcing.

Lemma 3.6. *Suppose that $\vec{S}$ is stationary, co-stationary. Suppose $\mathbb{P}$ is an $\vec{S}$ coding forcing. Then for any $i \in \kappa$, the followings are equivalent:*

(a) $\Vdash_{\mathbb{P}} S_i$ contains an ω-club.

[1] We say two forcing P and Q are equivalent if their Boolean completions $B(P)$ and $B(Q)$ are isomorphism.

(b) there are β, α, j and a set $B_\beta \subset V_\alpha$ such that $j \in B_\beta$ if $\beta \cdot (\alpha+1) + 2j = i$ and i is even and $j \notin B_\beta$ if $\beta \cdot (\alpha+1) + 2j + 1 = i$ and i is odd.

Proof. ((b) → (a)) Follow from the definition of the forcing.

((a) → (b)) Fix an i and assume without loss of generality that i is even. Write $i = \alpha \cdot (\beta + 1) + 2j$ and suppose for a contradiction that j is not an element of B_β. By the definition of an $\vec{S}$ coding forcing, we must have added a club through $S_{\alpha(\beta+1)+2j+1}$ instead. Let $\vec{T}$ be the sequence $\langle T_k \mid k < \kappa \rangle$, where $T_k = S_k$ if $k \neq i$ and $T_i = \eta_i \setminus S_i$. It follows from Theorem 3.2 again that $\vec{T}$ is mutually stationary. We will prove that $\mathbb{P}$ is $\vec{T}$-preserving to derive a contradiction. Indeed we shall see that $\vec{T}$-perservation implies that $\eta_i \setminus S_i$ must remain stationary after forcing with $\mathbb{P}$, yet $\mathbb{P} \Vdash$ "S_i contains an ω-club" which is impossible.

To see that $\vec{T}$ preserving forcings preserve the stationarity of every $S_{\eta_i} \in \vec{T}$, we only need to note that for any name $\dot{C}$ of a subset of η_i which is unbounded and ω-closed, and any countable elementary substructure N which contains $\dot{C}$ and for which $\sup(N \cap \eta_i) \in S_{\eta_i}$, any $(N, \mathbb{P})$-generic condition q, q forces $\dot{C} \cap (S_{\eta_i}) \neq \emptyset$.

Next we show by induction that each Q_β is forced to be $\vec{T}$-preserving. Work in $V[G_\beta]$. If $\dot{Q}_\beta/G_\beta$ is proper, then it is also $\vec{T}$-preserving. Otherwise, (1) holds. Now $\dot{Q}_\beta/G_\beta$ is a countable support product of shooting club forcing. Fix any $N \in \mathcal{T}_T$ which is a countable substructure of $H(\theta)^{V[G_\beta]}$. For any $p \in N \cap \dot{Q}_\beta$, we can construct a countable decreasing sequence of conditions $\langle p_i \mid i < \omega \rangle$ meeting all dense set in N. Define q by setting $q(i)$ to be the closure of $\bigcup_{n<\omega} p_n(i)$ if $i \in N$ and trivial otherwise. Note any nontrivial $q(i)$ is equal to $\bigcup_{n<\omega} p_n(i) \cup \{\sup(N \cap \eta_i)\}$, where $\eta_i = \sup(S_i)$ is a regular cardinal. As $N \in \mathcal{T}_T$ and no β, α, j witness that $\beta \cdot (\alpha+1) + 2j = i$ or $\beta \cdot (\alpha + 1) + 2j + 1 = i$, we have $\sup(N \cap \eta_i) \in S_i$, whenever $q(i)$ is non-trivial. Hence q is a condition. □

The proof also show that $\vec{S}$-coding preserves stationary subset of ω_1 if $\sup(S_0) > \omega_1$. As a Corollary of Lemma 3.6 and the definition of $\vec{S}$-coding, in any generic extension by $\vec{S}$-coding and any even i, at most one of S_i and S_{i+1} contains a club.

Lemma 3.7. *Suppose $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \delta \rangle$ is a countable support iteration. Suppose for any $\alpha > 0$, $\dot{Q}_\alpha$ is forced to be η_α-$\vec{S}$ coding of length $l(\alpha)$, where $\eta_\alpha = \max\{|\mathbb{P}^+_\alpha|, \Sigma_{\beta<\alpha} l(\beta)\}$. Also let η_0 be regular. Then $\mathbb{P}$ is forcing equivalent to a η_0-$\vec{S}$ coding.*

Proof. Let $\epsilon_\alpha = \Sigma_{\beta<\alpha} l(\beta)$ for any $\alpha \leq \delta$. We define a countable support iteration $\mathbb{P}' = \langle P'_\alpha, \dot{Q}'_\alpha \mid \alpha < \epsilon_\delta \rangle$. By induction on $\alpha < \delta$, we give the definition of P'_{ϵ_α}. We will also verify that P'_{ϵ_α} is forcing equivalent to P_α and P'_{ϵ_α} is a η_0-$\vec{S}$ coding.

If α is limit. Then P'_{ϵ_α} is the countable support limit of P'_γ for $\gamma < \epsilon_\alpha$. Hence P'_{ϵ_α} is the countable support limit of P'_{ϵ_γ} for $\gamma < \alpha$. As P'_{ϵ_γ} is forcing equivalent to P_γ. P'_{ϵ_α} is forcing equivalent to the countable support limit of P_γ for $\gamma < \alpha$. Hence P'_{ϵ_γ} is forcing equivalent to P_α.

If $\alpha = \beta+1$. Let $\epsilon_\alpha = \epsilon_\beta + l(\alpha)$. Now $\dot{Q}_\beta$ is a name for a forcing iteration $\langle P^\alpha_\gamma, \dot{Q}^\alpha_\gamma \mid \gamma < l(\alpha)\rangle$. Now as P_β is forcing equivalent to P'_{ϵ_β}. We can view $\dot{Q}_\alpha$ as a P'_{ϵ_β} name. By induction on $\gamma < l(\alpha)$, we can set $\dot{Q}'_{\epsilon_\beta+\gamma}$ be $P'_{\epsilon_\beta+\gamma}$ name of $\dot{Q}^\alpha_\gamma$ and verify that $P'_{\epsilon_\beta+\gamma}$ is forcing equivalent to $P'_{\epsilon_\beta} * P^\alpha_\gamma$.

It remains to check that P'_{ϵ_α} is a η_0-$\vec{S}$ coding. Fix a $\beta < \epsilon_\alpha$, with $\dot{Q}_\beta$ is not forced to be proper. There is γ such that $\beta \in [\epsilon_\gamma, \epsilon_{\gamma+1})$. β' is inaccessible cardinal greater then η_γ. Let $\beta = \epsilon_\gamma + \beta'$. As $\epsilon_\gamma < \eta_\gamma$ and β is inaccessible. $\beta = \beta'$ is inaccessible. Now P'_β is forcing equivalent to $P_\gamma * P^\gamma_\beta$. P_γ is of size less than β and P^γ_β is forced to be forcing equivalent to a forcing of size less of equal to β. Hence P'_β is also forcing equivalent to a forcing of size less of equal to β. The rest of (2) is routine to check. □

We are mainly interested in executing $\vec{S}$-coding forcing over L. Now we can define the coding machinery to be used in the latter section.

Definition 3.8. *Suppose α is in inaccessible in L and $X \subseteq P(\alpha)$. We say $\vec{S}$ codes $X \subset \alpha$ if*

1. *For any even $i \in [\alpha, (\alpha^+)^L)$, one of S_i and S_{i+1} contains a club.*
2. *For any $x \in X$, there is a $\beta < \alpha$ such that*

$$j \in x \text{ if and only if } S_{\alpha\cdot(\beta+1)+2i} \text{ contains a club}$$

and

$$j \notin x \text{ if and only if } S_{\alpha\cdot(\beta+1)+2i+1} \text{ contains a club}$$

Let $\vec{C} = \langle C_i \mid i \in [\alpha, (\alpha^+)^L)\rangle$ be a club sequence which witnesses (2), then we say that $\vec{C}$ is a $\vec{S}$ code for X.

A useful fact is the upward absoluteness of the coding between certain pair of models.

Lemma 3.9. *Suppose $N \subset M$ are two ZFC$^-$ transitive models. $N \models \vec{C}$ is a $\vec{S}$ code for $X \subset \alpha$, α is inaccessible in L^M, $((\alpha^+)^L)^M = ((\alpha^+)^L)^N$. Suppose $M \models$ for any even $i \in [\alpha, (\alpha^+)^L)$, at most one of S_i and S_{i+1} contains a club. Then $M \models \vec{C}$ is a $\vec{S}$ code for X.*

Proof. This follows from the definition. Note that being a club is an absolute property between transitive models. □

Lemma 3.10. *Suppose $\mathbb{P}$ is a $\vec{S}$-coding over L, α is inaccessible, $X \subseteq P(\alpha)$ and S codes X, then $X = P(\alpha)^{L[G_\alpha]}$.*

Proof. By Lemma 3.6, for any i, S_i contains a club if there is some stage α such that $\dot{Q}_\alpha$ satisfies (2) in the definition. But $\dot{Q}_\alpha$ forces S codes $P(\alpha)^{L[G_\alpha]}$ in $L[G_{\alpha+1}]$. Then by Lemma 3.9, S codes $P(\alpha)^{L[G_\alpha]}$ in $L[G]$. □

3.1.2 A model of BPFA and Δ_1-definalbility of NS_{ω_1}

Theorem 3.11. *Suppose that $V = L$ and δ is a reflecting cardinal. Then there is a forcing poset P such that in L^P, the following statements hold:*

1. *BPFA*
2. *$\omega_1 = \omega_1^L$ and $\omega_2 = \delta$.*
3. *The nonstationary ideal on ω_1 is $\Sigma_1(\{\omega_1\})$-definable over $\langle H(\omega_2), \in\rangle$.*

Proof. We first choose a sequence $\langle S_\alpha \mid \alpha \in \mathrm{Lim}(\delta)\rangle$ uniformly in $\alpha < \delta$ satisfying

- $S_\alpha \subset \alpha$
- If α is a regular cardinal, then S_α is stationary co-stationary in $\alpha \cap \mathrm{Cof}(\omega)$.

The existence of such a sequence $\langle S_\alpha \mid \alpha \in \mathrm{Lim}(\delta)\rangle$ follows from the fact that $\diamondsuit_\lambda$ holds in L for any L-cardinal λ and is a routine construction.

Now we define the forcing poset P. The forcing $P = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha < \delta\rangle$ will be a countable support iteration of length δ. We require the size of each iterand to be smaller than δ. As a consequence P satisfies δ-cc. We demand that $\dot{Q}_\alpha$ is trivial unless α is an inaccessible cardinal and $P_\alpha \subset L_\alpha$. We split into two cases if α is inaccessible.

- (α is Mahlo) We follow an idea from Goldstern-Shelah [4]. Choose a $A \subset \omega_1$ and a Σ_1 formula $\psi(x, y)$. A and ψ is chosen in a bookkeeping way so that during the whole iteration, each pair (A, ψ) will be dealt with unboundedly many times. Since δ is reflecting, whenever there is a α-$\vec{S}_{>\alpha}$ coding forcing poset Q which adds a witness to $\psi(x, A)$, there is such a Q of size less than δ. In this case, let $\dot{Q}_\alpha$ be the name of Q and use this forcing at stage α. When there is no such Q, we set $\dot{Q}_\alpha$ to be trivial forcing.
- (α is inaccessible but not Mahlo) $\dot{Q}_\alpha$ is trivial unless $|P_\alpha| \leq \alpha$. Work in $L[G_\alpha]$, where G_α is a L-P_α generic filter. $|P_\alpha| = \alpha$ is inaccessible. We choose $\langle B_\beta \mid \beta < 2^\alpha\rangle$ to be an enumeration of $P(\alpha)^{V[G_\alpha]}$. Let Q be the forcing

$$\prod_{i \in B_\beta} Club(S_{\alpha\cdot(\beta+1)+2i}) \times \prod_{i \notin B_\beta} Club(S_{\alpha\cdot(\beta+1)+2i+1})$$

with countable support. Then we force with Q at stage α.

It follows from the definition of P that for any $\alpha < \delta$, the forcing $\dot{Q}_\alpha$ is forced to be a $\alpha - \vec{S}$ coding forcing. Applying Lemma 3.7, we know that P is a $\vec{S}$-coding forcing. Moreover, for any α Mahlo, the tail P/P_α is a α-$\vec{S}_{>\alpha}$ coding.

Now as P is $\vec{S}$-preserving, P preserves ω_1. On the other hand, P is δ-c.c and P preserves cardinals above δ. By the definition of the forcing in the inaccessible but not Mahlo case, all cardinals below δ are collapsed to ω_1. In summary, $\omega_1^{L^P} = \omega_1^L$ and $\omega_2^{L^P} = \delta$.

Lemma 3.12. $P \Vdash \mathsf{BPFA}$.

Proof. Work in $L[G]$. Let $A \subset \omega_1$ and $Q \in L[G]$ be a proper forcing which adds a witness to the Σ_1-formula $\psi(x, A)$. Now let α be a stage such that $A \in L[G_\alpha]$, α is Mahlo and (A, ψ) is to be dealt with at stage α. In $L[G]$, the forcing $P/P_\alpha * Q$ is a $\alpha - \vec{S}_{>\alpha}$ coding forcing adding a witness to $\psi(x, A)$. Hence, we must be in the first case of the definition of our iteration at stage α and Q_α is a $\alpha - \vec{S}_{>\alpha}$ coding forcing which adds a witness to $\psi(x, A)$. Thus $H(\omega_2)^{L[G_{\alpha+1}]} \models \exists x \psi(x, A)$. By upward absoluteness, $H(\omega_2)^{L[G]} \models \exists x \psi(x, A)$. □

Lemma 3.13. *Work in $L[G]$. Let S be a subset of ω_1, then the following are equivalent:*

(a) S is stationary.

(b) there is a α inaccessible in L with $P_\alpha \subseteq L_\alpha$ and S is stationary in $L[G_\alpha]$.

(c) there is a α inaccessible in L, there is $\vec{C} \in L[G_{\alpha+1}]$ which is a $\vec{S}$ code for $P(\alpha)^{L[G_\alpha]}$, and there is a transitive model M of ZF^- such that $\vec{C} \in M$ and M thinks that S is stationary.

Proof. ((a) → (c)) Let α be inaccessible but not Mahlo and consider stage α of the iteration which we can assume to be such that $S \in L[G_\alpha]$ and $|P_\alpha| \leq \alpha$. Now Q_α forces the existence of a $\vec{C}$-sequence which is an $\vec{S}$ code for $P(\alpha)^{L[G_\alpha]}$. Any transitive M which contains $\vec{C}$ is as desired in (c), moreover M will automatically think that S is stationary if S is stationary in $L[G]$.

((b) → (a)) The tail of forcing P/P_α is α-$S_{>\alpha}$ coding. Now the proof of Lemma 3.6 shows P/P_α preserves stationarity of S.

((c) → (b)) If M is as in the assumption, $P(\alpha)^{L[G_\alpha]}$ is a subset of M, and if M thinks that S is stationary, so must $V_{\alpha+1}^{L[G_\alpha]} = P(\alpha)^{L[G_\alpha]}$, and hence $L[G_\alpha]$. □

We now present the Σ_1 definition of stationary subsets of ω_1 over $(H(\omega_2), \in, \{\omega_1\})$. Let $\psi(x)$ describe the following statement: there are objects A and M such that

(i) $A = \langle C_i \mid i \in [\alpha, \beta) \rangle$ is a sequence with C_i is an ω-club in $\sup(C_i \cap Ord) = \eta_i$.

(ii) All η_i are regular cardinals in L. $\eta_\alpha = \alpha$ is inaccessible in L.

(iii) M is a transitive ZFC$^-$ model with $\omega_1 = \omega_1^M$, $((\alpha^+)^L)^M = (\alpha^+)^L$, and $x \in M$.

(iv) $M \models x \subset \omega_1$ is stationary.

(v) $M \models A$ is a $\vec{S}$ codes for $P(\alpha)$.

It is routine to check (i), (iii), (iv) and (v) are all Σ_1 over $\langle H(\omega_2), \in, \{\omega_1\}\rangle$. For (ii), as $L[G] \models BPFA$, we can apply a trick of Todorcevic (cf. Proof of Lemma 4, [15]) to get Σ_1 formulas $\psi_0(x)$ and $\psi_1(x)$ such that $H(\omega_2) \models \psi_0(\beta)$ if and only if β is a regular cardinal in L and $H(\omega_2) \models \psi_1(\beta)$ if and only if β is an inaccessible cardinal in L.

Here $\psi_0(x)$ is the formula describing the existence of a specialization function of the tree T_x, where T_x is derived from the canonical global square sequence in L. It is a consequence of BPFA that x is uncountable regular in L if and only if such a specialization function exists. Now $\psi_1(x)$ is a formula says α is regular and a limit of regular cardinals in L.

It is now clear that $\psi(x)$ is Σ_1 over $\langle H(\omega_2), \in, \{\omega_1\}\rangle$. What is left is to show that $\psi(x)$ indeed characterizes stationarity.

Lemma 3.14. *In $L[G]$, for any $x \subseteq \omega_1$, $H(\omega_2) \models \psi(x)$ if and only if x is stationary.*

Proof. If $x \subset \omega_1$ is stationary, then, by Lemma 3.13 (c), we do immediately get a witness $M \in H(\omega_2)$ and an A so that $\psi(x)$ holds.

Conversely, let x, A, M be as given by the definition of $\psi(x)$, we check that x is stationary. But if $\psi(x)$ holds, then M thinks that A is a $\vec{S}$-code for $P(\alpha)$. By item *(ii)* of ψ, this also means that A is a $\vec{S}$-code for $P(\alpha)$ and by (c) of Lemma 3.13 it is true that x is indeed stationary. □

□

3.2 A model for MA_{ω_1}, NS_{ω_1} being saturated and $\Delta_1(\{\omega_1\})$-definable

In this section we improve an earlier result of [6] namely we show that given a Woodin cardinal, there is a model such that NS_{ω_1} is saturated, Δ_1-definable with parameters from $H(\omega_2)$ while Martin's Axiom also holds true. If one forces over the canonical inner model with one Woodin cardinal M_1, then the construction yields a model where additionally NS_{ω_1} is definable with ω_1 as the only parameter.

3.2.1 Short summary of the main features of the model W_1

The proof relies heavily on the coding machinery introduced in [6], where it is shown that, given a Woodin cardinal δ, then there is a universe such that NS_{ω_1} is saturated and $\Delta_1(\vec{C}, \vec{T}^0)$-definable, where $\vec{C}$ is an arbitrary ladder system on ω_1 and $\vec{T}^0$ is an independent sequence (independence will be defined in a moment below) of Suslin trees of length ω. As this coding is rather convoluted, we will not define it here in detail but instead only highlight the most important notions and features of it. Our notation will be exactly as there.

We shall use Suslin trees on ω_1 for creating a $\mathbf{\Sigma_1}$-definition of stationarity. To facilitate things, the trees should satisfy a certain property:

Definition 3.15. *Let $\vec{T} = (T_\alpha : \alpha < \kappa)$ be a sequence of Suslin trees. We say that the sequence is an independent family of Suslin trees if for every finite set $e = \{e_0, e_1, ..., e_n\} \subset \kappa$ the product $T_{e_0} \times T_{e_1} \times \cdots \times T_{e_n}$ is a Suslin tree again.*

Independent sequences of Suslin trees can be used to code arbitrary information using the two well-known and mutually exclusive ways to destroy a Suslin tree, namely either shooting a branch through the tree or specializing it. More precisely, given a set $X \subset \omega_1$ and an independent sequence of Suslin trees $\vec{T} = (T_i : i < \omega_1)$, we can code the characteristic function of X into $\vec{T}$ via forcing with the finitely supported product of

$$\mathbb{P}_i = \begin{cases} T_i & \text{if } i \in X \\ Sp(T_i) & \text{if } i \notin X \end{cases}$$

where T_i just denotes the forcing one obtains when forcing with the Suslin tree T_i (which adds a cofinal branch to T_i) and $Sp(T_i)$ denotes the forcing which spezializes the tree T_i. Note that the independence of $\vec{T}$ buys us that the finitely supported product has the ccc as well. We will eventually use this mechanism to create a generic extension of V with a $\Sigma_1(\vec{C}, \vec{T}^0)$-definition of being stationary on ω_1.

The construction of such a universe shall be sketched now. We start with a universe V with one Woodin cardinal δ with $\diamondsuit$. We fix a ladder system $\vec{C}$ and an ω-sequence of independent Suslin trees $\vec{T}^0$ and start a first, nicely supported iteration (using Miyamoto's nice iterations, see [10]) of length δ over V which combines Shelah's proof of the saturation of NS_{ω_1} from a Woodin cardinal with the forcings invented by A. Caicedo and B. Velickovic (see [1]) and other forcings whose purpose is to create a model W_0 which will have several features listed below which will turn our to be useful.

In a next step we force over W_0 with a variant of almost disjoint coding which is due to L. Harrington (see [5]), which is used to ensure that over the resulting generic extension of W_0, denoted by W_1, there is a $\Sigma_1(\vec{C}, \vec{T}^0)$-definable ω_2-sequence of $\aleph_1$-sized, transitive models which are sufficiently smart to determine whether a member is a stationary subset of ω_1 or a Suslin tree in W_0. These sufficiently smart models are called suitable, and the purpose of suitable models is that they can correctly compute a fixed, independent sequence $\vec{T} = (T_\alpha : \alpha < \omega_2)$ of Suslin trees. Due to this correctness, we can identify whether an ω_1-block of trees is in $\vec{T}$ in a $\Sigma_1(\vec{C}, \vec{T}^0)$-way. The sequence $\vec{T}$ will be used later to code up being stationary in a $\Sigma_1(\vec{T}^0, \vec{C})$-way.

To summarize the above, starting from an arbitrary V which contains a Woodin cardinal δ with $\diamondsuit$, and fixing a ladder system on ω_1 $\vec{C}$ and an ω-sequence of independent Suslin trees $\vec{T}^0$, we create first a generic extension W_0 and then a further generic extension W_1 such that in W_1 the following holds:

1. $\delta = \aleph_2$.

2. In W_1, the nonstationary ideal is saturated and the saturation is ccc-indestructible.

3. Every real in W_0 is coded by a triple of limit ordinals (α, β, γ) below ω_2 relative to the ladder system $\vec{C}$ (in the sense of Caicedo-Velickovic, see [6], pp. Theorem 18 (‡)).

4. Every subset $X \subset \omega_1$, $X \in W_0$ is coded by a real $r_X \in W_0$ relative to the fixed almost disjoint family of reals F we recursively obtain from our ladder system $\vec{C}$.

5. There is an ω_2-sequence of independent Suslin trees $\vec{T} = (T_i : \omega < i < \omega_2) \in W_1$ whose initial segments are uniformly and correctly definable in suitable models. The set of suitable models is itself $\Sigma_1(\vec{C}, \vec{T}^0)$-definable in W_1 using as parameters the ladder system $\vec{C}$, and one ω-block of independent Suslin tree $\vec{T}^0 = (T_n : n \in \omega)$. As a consequence, $\vec{T}$ is $\Sigma_1(\vec{C}, \vec{T}^0)$-definable over W_1.

6. The definition of $\vec{T}$ remains valid in all generic extensions of W_1 by forcings with the countable chain condition. So W_1 is a reasonable candidate for a ground model using coding forcings which themselves have the ccc.

In a second iteration using W_1 as the ground model, we force with coding forcings using Suslin trees that are applied to make NS_{ω_1} $\Sigma_1(\vec{C}, \vec{T}^0)$-definable. We force with a finitely supported iteration of ccc forcings over W_1. As a consequence we preserve the saturation of the nonstationay ideal and the sequence $\vec{T}$ is still $\Sigma_1(\vec{C}, \vec{T}^0)$-definable. The only forcings which are used in this second iteration are the Suslin trees from our independent sequence $\vec{T}^{>0} = (T_i : i > \omega)$, which we either specialize or destroy via the addition of an ω_1-branch. We will use a bookkeeping function and start to write characteristic functions of every stationary subset of ω_1 into ω_1-blocks of $\vec{T}^{>0}$ using either the spezialization forcing or shooting a branch through elements of $\vec{T}^{>0}$.

This will eventually yield a universe W_{ω_2} where the nonstationary ideal remains saturated and where stationary subsets of ω_1 can be characterized as follows:

Fact 3.16. *There are $\Sigma_1(\vec{C}, \vec{T}^0)$-formulas $\Phi(r)$ and $\Psi(S)$ where the formula $\Phi(r)$ defines a set of reals such that every member is an almost disjoint code for an $\aleph_1$-sized, transitive models which can be used to compute the sequence $\vec{T}^{>0}$ of Suslin trees correctly (these models are the suitable models mentioned earlier). The formula $\Psi(S)$ then defines stationary subsets of W_{ω_2} in the following way:*

$\Psi(S)$ *if and only if there is an $\aleph_1$-sized, transitive model N which contains $\vec{C}$ and $\vec{T}^0$ such that N models that*

- *There exists a real x such that $\Phi(x)$ holds, i.e. x is a code for a suitable model M.*
- *There exists an ordinal α in the suitable model M such that $\vec{T}'$ is the α-th ω_1 block of the definable sequence of independent Suslin trees as computed in M and N sees a full pattern on $\vec{T}'$.*
- *$\forall \beta < \omega_1$ ($\beta \in S$ if and only if $\vec{T}'(\beta)$ has a branch).*
- *$\forall \beta < \omega_1$ ($\beta \notin S$ if and only if $\vec{T}'(\beta)$ is special).*

Note that $\Psi(S)$ is of the form $\exists N(N \models ...)$, thus Ψ is a Σ_1-formula.

3.2.2 Forcing over W_1

We shall show how to modify the just sketched construction to obtain a model where additionally MA_{ω_1} holds. As before we will construct the model W_1. We will proceed however not coding up stationary subsets of ω_1 as we do in [6] but instead code up a second ω_2-block of generically added Suslin trees first.

In an ω_2-length iteration we first use Tennenbaum's forcing over W_1 to add an independent sequence of Suslin trees of length ω_2 and use our definable independent sequence $\vec{T}$ to code up the added Suslin trees. First let us briefly recall the definition of Tennenbaum's forcing.

Definition 3.17. *Tennenbaum's forcing $\mathbb{P}_T$ consists of conditions which are finite trees $(T, <_T)$, $T \subset \omega_1$ such that $\alpha < \beta$ if $\alpha <_T \beta$, and $(T_1, <_{T_1}) < (T_2, <_{T_2})$ holds if $T_2 \subset T_1$ and $<_{T_2} = <_{T_1} \cap (T_2 \times T_2)$.*

It is well-known that $\mathbb{P}_T$ is Knaster and adds generically a Suslin tree to the ground model.

So we start with the model W_1 as our ground model. Let $\vec{C}$ be our fixed ladder system on ω_1 and let $\vec{T}^0$ be a fixed independent sequence of Suslin trees of length ω. In W_1 there is a $\Sigma_1(\vec{C}, \vec{T}^0)$-definable ω_2-sequence of ω_1-blocks of independent Suslin trees $\vec{T} = (\vec{T}^\alpha : \alpha < \omega_2)$, where for every $0 \neq \alpha < \omega_2$, $\vec{T}^\alpha = (T^\alpha_\eta : \eta < \omega_1)$, and $\vec{T}$ forms an independent sequence of Suslin trees.

Over W_1 we start a finitely supported iteration $\mathbb{Q} = ((\mathbb{Q}_\alpha, \dot{\mathbb{R}}_\alpha) : \alpha < \omega_2)$ and let H_α denote the generic filter for $\mathbb{Q}_\alpha$. For every $\alpha < \omega_2$, using $W_1[G_\alpha]$ as the ground model, $\dot{\mathbb{R}}^{G_\alpha}_\alpha$ is defined to be $\mathbb{Q}^1_\alpha * \mathbb{Q}^2_\alpha$, and $\mathbb{Q}^1_\alpha$ is Tennenbaum's $\mathbb{P}_T$ and $\mathbb{Q}^2_\alpha$ codes up the generically by $\mathbb{Q}^1_\alpha$ added tree, called $h_\alpha \subset \omega_1$ in the following way:

$$\mathbb{Q}^2_\alpha = \begin{cases} T^\alpha_\eta & \text{if } \eta \in h_\alpha \\ Sp(T^\alpha_\eta) & \text{if } \eta \notin h_\alpha \end{cases}$$

It is immediate that the resulting universe $W_2 = W_1[H_{\omega_2}]$ is a ccc extension of W_1, thus NS_{ω_1} remains saturated and as in [?] one can show that the generically added sequence of Suslin trees $(h_\alpha : \alpha < \omega_2)$ is an independent, $\Sigma_1(\vec{C}, \vec{T}^0)$-definable sequence of Suslin trees via the formula:

Fact 3.18. *There is a $\Sigma_1(\vec{C}, \vec{T}^0)$-formula $\Psi(h)$ which defines the generically added Suslin trees $(h_\alpha : \alpha < \omega_2)$ of W_2:*

$\Psi(h)$ *if and only if there is an $\aleph_1$-sized, transitive model N which contains $\vec{C}$ and $\vec{T}^0$ such that N models that*

- *There exists a real x such that $\Phi(x)$ holds, i.e. x is a code for a suitable model M.*
- *There exists an ordinal α in the suitable model M such that $\vec{T}'$ is the α-th ω_1 block of the definable sequence of independent Suslin trees as computed in M and N sees a full pattern on $\vec{T}'$.*
- $\forall \beta < \omega_1$ *($\beta \in h$ if and only if $\vec{T}'(\beta)$ has a branch).*
- $\forall \beta < \omega_1$ *($\beta \notin h$ if and only if $\vec{T}'(\beta)$ is special).*

Note that $\Psi(h)$ is of the form $\exists N (N \models ...)$, thus Ψ is a Σ_1-formula.

In a second step, we use W_2 as our ground model and force in an ω_2-length, finitely supported iteration MA_{ω_1} while simultaneously coding up stationary subsets of ω_1 using the boldface Σ_1-definable sequence of trees $(h_\alpha : \alpha < \omega_2)$. We write $\vec{h}^\alpha$ for the α-th ω_1-block of elements of the sequence $(h_\beta : \beta < \omega_2)$ and let h^α_β denote the β-th element of the α-th ω_1-block.

We do the usual forcing to code characteristic functions of stationary subsets of ω_1 into $\vec{h}^\alpha$, but additionally we feed in forcings of size $\aleph_1$ with the countable chain condition to produce a model of MA_{ω_1}. Note that as all the forcings we use have the ccc and thus we will preserve the saturation of NS_{ω_1}.

In order to prevent the forcings we use to get MA_{ω_1} from damaging our codes, we will force MA_{ω_1} in a "diagonal way". We define a finitely supported iteration $((\mathbb{R}_\alpha, \dot{\mathbb{S}}_\alpha) : \alpha < \omega_2)$ of ccc forcing over W_2 inductively using a bookkeeping function F. We let $F \in W_2$, $F : \omega_2 \to \omega_2 \times \omega_2 \times 2$ such that for every $(\alpha, \beta, i) \in \omega_2 \times \omega_2 \times 2$, $F^{-1}(\alpha, \beta, i)$ is an unbounded subset in $\omega_2 \times \omega_2 \times 2$. Assume we are at stage $\alpha < \omega_2$ and we have already defined the iteration $\mathbb{R}_\alpha$ up to $\alpha < \omega_2$. We let I_α denote the generic filter for $\mathbb{R}_\alpha$. We also assume by induction that every of the first α-many ω_1-blocks of trees $\vec{h}^\beta$, $\beta < \alpha$ has already been used for codings, but the elements of the sequences $(\vec{h}^\eta : \eta \geq \alpha)$ still form an independent sequence of Suslin trees in $W_2[I_\alpha]$. The forcing we have to use next is determined by the value of $F(\alpha)$.

1. If $F(\alpha) = (\beta, \gamma, 0)$, then we look at the β-th stationary subset S of ω_1 in $W_2[I_\gamma]$ and use the α-th ω_1-block of $\vec{h}$, $\vec{h}^\alpha = (h^\alpha_\eta : \eta < \omega_1)$ to code up S i.e. we will force with $\dot{\mathbb{S}}^{I_\alpha}_\alpha := \prod_{i<\omega_1} \mathbb{P}_i$ with finite support, where
$$\mathbb{P}_i = \begin{cases} h^\alpha_\eta & \text{if } \eta \in S \\ Sp(h^\alpha_\eta) & \text{if } \eta \notin S \end{cases}$$
where h^α_η here is considered as a forcing notion when forcing with the tree and $Sp(h^\alpha_\eta)$ denotes the specialization forcing for the tree h^α_η.

2. If $F(\alpha) = (\beta, \gamma, 1)$, then we look at the β-th forcing $\mathbb{B}$ of size $\aleph_1$ in $W_2[I_\gamma]$ which has the countable chain condition as seen in the universe $W_2[I_\alpha]$. We can consider the iteration $((\mathbb{R}_\zeta : \zeta < \alpha) * \mathbb{B})$ which has a dense subforcing of size $\aleph_1$ (the dense set is just the set of conditions in $\mathbb{R}_\alpha * \mathbb{B}$ which are fully decided), ccc forcing in W_2 and can thus be seen as a subset of ω_1 in W_2. As $W_2 = W_1[H_{\omega_2}]$, there is a stage $\nu < \omega_2$ such that (a forcing equialent to) $((\mathbb{R}_\zeta : \zeta < \alpha) * \mathbb{B}) \in W_1[H_\nu]$. Now if $\nu \leq \alpha$ we let $\dot{\mathbb{S}}^{I_\alpha}_\alpha$ be $\mathbb{B}$. Otherwise we force with the trivial forcing.

In the first case of the definiton we will write codes in the sequence $(\vec{h}^\alpha : \alpha < \omega_2)$ of blocks of Suslin trees. The definition of the second case ensures that we will not accidentally write an unwanted pattern when forcing for MA_{ω_1}.

Lemma 3.19. *Assume we are at stage α of our iteration and we are in the nontrivial part of case 2 of the definition, thus we force with an $\aleph_1$-sized $\dot{\mathbb{S}}^{I_\alpha}_\alpha = \mathbb{B}$. Then, if $I_{\alpha+1}$ is $\mathbb{R}_{\alpha+1} = \mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha$-generic over $W[H_{\omega_2}]$, all the Suslin trees in $\vec{h}_\zeta$, $\zeta \geq \alpha$ remain Suslin trees in $W[H_{\omega_2}][I_{\alpha+1}]$.*

Proof. Assume that we are at stage α, thus the model we have produced so far is $W_1[H_{\omega_2}][I_\alpha]$ and we force with $\dot{\mathbb{S}}_\alpha$ which is a ccc forcing of size $\aleph_1$

in $W_1[H_\nu][I_\alpha]$ for $\nu \le \alpha$. Consider some block $\vec{h}^\zeta$, $\zeta \ge \alpha$ in the universe $W_1[H_{\omega_2}][\mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha]$. The latter universe is obtained via the iteration

$$(\mathbb{Q}_\zeta \, : \, \zeta < \omega_2) * \mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha = (\mathbb{Q}_\zeta \, : \, \zeta \le \nu) * (\mathbb{Q}_\zeta/\mathbb{Q}_\nu \, : \zeta > \nu) * \mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha$$

and the right hand side can be rewritten as

$$(\mathbb{Q}_\zeta \, : \, \zeta \le \nu) * ((\mathbb{Q}_\zeta/\mathbb{Q}_\nu \, : \zeta > \nu) \times (\mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha)).$$

As we can switch the order in products, the latter can be written as

$$(\mathbb{Q}_\zeta \, : \, \zeta \le \nu) * ((\mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha) \times (\mathbb{Q}_\zeta/\mathbb{Q}_\nu \, : \zeta > \nu)),$$

and consequentially the $W_1[H_\nu]$-generic filter H_{ν,ω_2} for the tail $(\mathbb{Q}_\zeta/\mathbb{Q}_\nu \, : \zeta > \nu, \zeta < \omega_2)$ remains generic over the model $W_1[(\mathbb{Q}_\zeta \, : \, \zeta \le \nu) * (\mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha)]$. This means in particular that the generically added Suslin trees $\vec{h}^\zeta, \zeta > \nu$ for Tennenbaum's forcing for adding a Suslin tree which are elements in H_{ν,ω_2} remain generic even over the ground model $W_1[H_\nu * I_{\alpha+1}]$. Now Tennebaum's forcing is computed in an absolute way in every universe with the same ω_1, and trivially, every generic filter for it is a Suslin tree. Hence we obtain that every h^ζ_η, $\zeta > \nu, \eta < \omega_1$ is a Suslin tree in $W_1[H_\nu * I_{\alpha+1}]$, thus every h^ζ_η, $\zeta \ge \alpha \ge \nu$ is a Suslin tree in $W_1[H_{\omega_2} * I_{\alpha+1}]$ as claimed. □

After ω_2-many steps we arrive at $W_2[I_{\omega_2}]$ which has the desired properties. The first thing to note is that we are in full control of the codes which are written into the sequences of blocks of Suslin trees $(\vec{h}^\alpha \, : \, \alpha < \omega_2)$.

Lemma 3.20. *In $W_2[I_{\omega_2}]$, a set $S \subset \omega_1$ is stationary if and only if there is an $\alpha < \omega_2$ such that*

$$\forall \beta < \omega_1((\beta \in S \leftrightarrow h^\alpha_\beta \textit{ has a branch }) \textit{ and } (\beta \notin S \leftrightarrow h^\alpha_\beta \textit{ is special })).$$

Proof. If S is stationary then the rules of the iteration guarantee that there is such an $\alpha < \omega_2$ with the desired properties. On the other hand if there is an $\alpha < \omega_2$ such that the α-th block $\vec{h}^\alpha$ sees a certain 0,1-pattern then by the last Lemma, this pattern must come from the first case in the definition of our iteration. Hence S has to be stationary. □

Theorem 3.21. *In $W_2[I_{\omega_2}]$ MA_{ω_1} holds, NS_{ω_1} is saturated and $\Delta_1(\vec{C}, \vec{T}^0)$-definable.*

Proof. That NS_{ω_1} is saturated is clear as $W_2[I_{\omega_2}]$ is a ccc extension of W_0 and NS_{ω_1} is saturated in W_0 and ccc indestructible. The proof that MA_{ω_1} holds in $W_2[I_{\omega_2}]$ is also clear as a standard computation yields that the continuum is $\aleph_2$ in $W_2[I_{\omega_2}]$. Hence, it is sufficient to show that MA_{ω_1} holds for ccc posets of size $\aleph_1$. Let $\mathbb{P} \in W_2[I_{\omega_2}]$ be such, then there is a stage $\nu < \omega_2$ such that $\mathbb{P} \in W_2[I_\nu]$. The rules of the iteration yield that we will consider $\mathbb{P}$

unboundedly often after stage ν. Thus there will be a stage $\alpha < \omega_2$ such that $\mathbb{P}$ is considered by the bookkeeping F and $\mathbb{R}_\alpha * \mathbb{P}$ is an element of $W_1[H_\nu]$ for $\nu \leq \alpha$, and hence we used $\mathbb{P}$ in the iteration $(\mathbb{R}_\eta : \eta < \omega_2)$, so MA_{ω_1} holds.

In order to see that every $S \in W_2[I_{\omega_2}]$ has a $\Sigma_1(\vec{C}, \vec{T}^0)$-definition we exploit the fact that the trees $(h_\alpha : \alpha < \omega_2)$ are $\Sigma_1(\vec{C}, \vec{T}^0)$-definable in $W_1[I_{\omega_2}]$. We claim that the following $\Sigma_1(\vec{C}, \vec{T}^0)$-formula $\varphi(S)$ defines being stationary in $W_2[I_{\omega_2}]$:

> $\varphi(S)$ if and only if there exists a triple of (M_1, M_2, M_3) of transitive models of size $\aleph_1$ such that $M_1 \subset M_2 \subset M_3$, M_1 is a suitable model and M_2 sees a full pattern on the trees in some ω_1-block $\vec{T}^\alpha$. This pattern itself yields an ω_1-block of trees $\vec{h}^\beta$ and M_3 sees a full pattern on $\vec{h}^\beta$ and this pattern is the characteristic function for S.

Note that the formula $\varphi(S)$ is of the form $\exists M_1, M_2, M_3\, \sigma(M_1, M_2, M_3, S)$ and σ is Δ_1 as all the statements in σ are of the form $M_i \models ...$ which is a Δ_1-formula.

By absoluteness and the way we defined our iteration, it is clear that if $S \subset \omega_1$ is stationary in $W_2[I_{\omega_2}]$, then $\varphi(S)$ holds.

On the other hand, if $\varphi(S)$ is true and M_1, M_2 and M_3 are witnesses to the truth of $\varphi(S)$, then, as they see full patterns, their local patterns must coincide with the patterns in the real world $W_2[I_{\omega_2}]$. But the last Lemma ensures that the patterns of $W_2[I_{\omega_2}]$ characterize stationarity, so the proof is finished. $\square$

As in [6], instead of working in an arbitrary V with a Woodin cardinal, we can work over the canonical inner model with one Woodin cardinal M_1. This has the advantage, that we can replace the two parameters $\vec{C}$ and $\vec{T}^0$ by just $\{\omega_1\}$. We will not go into any details and just claim that the above proof can be applied over M_1, with all the modifications exactly as in [6]. We therefore obtain the last theorem of this article.

Theorem 3.22. *Let M_1 be the canonical inner model with one Woodin cardinal. Then there is a generic extension of M_1, in which NS_{ω_1} is saturated, $\Sigma_1(\omega_1)$-definable, and MA_{ω_1} holds.*

3.2.3 Open questions

We end with a couple of natural problems which remain open.

Question 1. *Assume* PFA. *Is there a Σ_1-formula and a set $A \subset \omega_1$ such that*

$$\forall S \in P(\omega_1)(S \text{ is stationary } \Leftrightarrow \varphi(S, A))?$$

Question 2. *Assume the existence of a Woodin cardinal. Is there a universe in which* BPFA *holds, NS_{ω_1} is $\boldsymbol{\Delta_1}$-definable over $H(\omega_2)$ and NS_{ω_1} is saturated?*

Question 3. *Assume the existence of a reflecting cardinal. Is* BPFA *consistent with the non-existence of a $\Sigma_1(\omega_1)$-definition of NS_{ω_1}?*

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