An extendible structure with a rigid elementary extension

Paul B. Larson* — Miami University, Oxford, Ohio USA

Saharon Shelah† — Hebrew University of Jerusalem, Rutgers University

December 6, 2017

A countable structure is said to be *extendible* if it is $\mathcal{L}_{\infty,\aleph_0}$-elementarily equivalent to an uncountable structure. A theorem of Su Gao says that a countable structure is extendible if and only if its automorphism group is not closed in the space of injections from the domain of the structure to itself. In particular, an extendible structure has infinitely many automorphisms. In this note we give an example of an extendible countable structure (M_1 below) with a rigid elementary extension (M_2).

A set $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is said to be *independent* if for all disjoint finite $\mathcal{F}_0, \mathcal{F}_1 \subseteq \mathcal{A}$,

$$\bigcap \mathcal{F}_0 \cap \bigcap \{\omega \setminus A : A \in \mathcal{F}_1\}$$

is infinite. A natural recursive argument builds a tree $T \subseteq 2^{<\omega}$ whose infinite paths correspond to an independent family of cardinality $2^{\aleph_0}$. Splitting this family into countably many pieces of cardinality $2^{\aleph_0}$, and then modifying the members of each piece accordingly, one can produce an independent family $\mathcal{A}$ with the additional property that for all finite disjoint $u, v \subseteq \omega$ there are $2^{\aleph_0}$ many $A \in \mathcal{A}$ with $A \cap (u \cup v) = u$. Let call such a family an *improved independent* set.

0.1 Definition. We define a *good independent sequence* to be a set

$$\bar{A} = \langle A_{\alpha,n} : \alpha \leq 2^{\aleph_0}, n < \omega \rangle$$

such that

1. $\{A_{\alpha,n} : \alpha \leq 2^{\aleph_0}, n < \omega\}$ is an independent family;
2. the function $(\alpha, n) \mapsto A_{\alpha,n}$ is injective;

*Supported in part by NSF Grant DMS-1201494.

†Research partially supported by NSF grant no: 1101597, and by the European Research Council grant 338821. Paper No. 1130 on Shelah's list.

3. for all $k, m, n < \omega$ and all $u_\ell \subseteq m$ $(\ell < n)$, there exist $2^{\aleph_0}$ many ordinals α such that

   (a) $\alpha = k \mod \omega$,

   (b) $\forall \ell < n \ A_{\alpha,\ell} \cap m = u_\ell$,

4. for all $i < j < \omega$ there exist an $n \in \omega$ such that $|A_{2^{\aleph_0},n} \cap \{i, j\}| = 1$.

Given an improved independent set $\mathcal{A}$, one can easily find a good independent sequence whose members are in $\mathcal{A}$. We let $\bar{A}$ be one such family.

We let τ be the vocabulary consisting of unary predicates P and Q, and binary predicates R_n $(n \in \omega)$. We let N_2 be the following τ-structure.

- The set of elements is of N is $\{b_i : i < \omega\} \cup \{c_\alpha : \alpha \leq 2^{\aleph_0}\}$, with no repetition.
- $Q^{N_2} = \{b_i : i < \omega\}$
- $P^{N_2} = \{c_\alpha : \alpha \leq 2^{\aleph_0}\}$
- $R_n^{N_2} \subseteq Q^{N_2} \times P^{N_2}$ is defined as follows:
  - $(b_i, c_{2^{\aleph_0}}) \in R_n^{N_2}$ if and only if $i \in A_{2^{\aleph_0},n}$;
  - if $\alpha < 2^{\aleph_0}$ and $\alpha = m \mod \omega$, then $(b_i, c_\alpha) \in R_n^{N_2}$ if and only if $n \leq m$ and $i \in A_{\alpha,n}$.

Let N_1 be the restriction of N_2 to

$$\{b_i : i < \omega\} \cup \{c_\alpha : \alpha < 2^{\aleph_0}\}.$$

Let $\mathcal{X} \prec (H((2^{\aleph_0})^+), \in)$ be countable, with $\bar{A} \in \mathcal{B}$. We let

- M_1 be the restriction of N_2 to $\{b_i : i < \omega\} \cup \{c_\alpha : \alpha \in 2^{\aleph_0} \cap \mathcal{X}\}$;
- M_2 be the restriction of N_2 to $\{b_i : i < \omega\} \cup \{c_\alpha : \alpha \in (2^{\aleph_0} + 1) \cap \mathcal{X}\}$.

For each $m \in \omega$, let $\tau_m = \{P, Q, R_n : n \leq m\}$. Proposition 0.2 shows that M_1 is extendible, and that M_1 is elementary in M_2. Proposition 0.3 shows that M_2 is rigid.

Proposition 0.2. *The following elementarity relations hold.*

1. $N_1 \prec N_2$.
2. $M_1 \prec M_2$.
3. $M_1 \prec_{\infty,\aleph_0} N_1$

Proof. For part (1), it suffices to prove that for every $m \in \omega$, $N_1 \upharpoonright \tau_m \prec N_2 \upharpoonright \tau_m$. In fact we will show that $N_1 \upharpoonright \tau_m \prec_{\infty,\aleph_0} N_2 \upharpoonright \tau_m$ by showing that player II has a winning strategy in the back-forth-game of length ω between these two structures, below the play pairing c_m with $c_{2^{\aleph_0}}$. By Karp's Theorem (Corollary 3.5.3 of [2], for instance), this suffices. To show that II has such a strategy, let F be the set of relation-preserving maps f from

$$\{b_i : i < \omega\} \cup \{c_\alpha : \alpha < 2^{\aleph_0}\}$$

to

$$\{b_i : i < \omega\} \cup \{c_\alpha : \alpha \leq 2^{\aleph_0}\}$$

with $f(c_m) = c_{2^{\aleph_0}}$ and $\alpha = \beta \mod \omega$ whenever $\alpha \neq k$ and $f(c_\alpha) = c_\beta$. It suffices then to show that for each f, each $i \in \omega$ and each $\alpha < 2^{\aleph_0}$, there exists an $f' \in F$ containing f with b_i and c_α in both the domain and the range of f'. The independence of $\mathcal{A}$ implies that elements of the form b_i can be added. Item (3b) of Definition 0.1 implies that elements of the form c_α can be dealt with.

Part (2) follows from part (1) and the elementarity of $\mathcal{X}$ in $H((2^{\aleph_0})^+)$.

Part (3) follows from the fact that player II has a winning strategy in the back-and-forth game of length ω between M_1 and N_1 (again using Karp's Theorem). For this, let F be the set of relation-preserving maps f from

$$\{b_i : i < \omega\} \cup \{c_\alpha : \alpha \in 2^{\aleph_0} \cap \mathcal{X}\}$$

to

$$\{b_i : i < \omega\} \cup \{c_\alpha : \alpha < 2^{\aleph_0}\}$$

with $\alpha = \beta \mod \omega$ whenever $f(c_\alpha) = c_\beta$. It suffices then to show that II has a response to any move by player I meeting this condition. This follows just as in the proof of part (1). □

Proposition 0.3. *Every substructure of N_2 containing M_2 is rigid.*

Proof. Let M a substructure of N_2 containing M_2 and let π be an automorphism of M. Since $c_{2^{\aleph_0}}$ is the only $c \in P^M$ such that for each $n \in \omega$ there is a $b \in Q^M$ wth $(b, c) \in R_n^M$, $\pi(c_{2^{\aleph_0}}) = c_{2^{\aleph_0}}$. Now $\pi[Q^{M_2}] = Q^{M_2}$, and if $i < j < \omega$ then, by condition (4) of Definition 0.1, for some $n \in \omega$,

$$(b_i, c_{2^{\aleph_0}}) \in R_n^M \leftrightarrow (b_j, c_{2^{\aleph_0}}) \notin R_n^M,$$

so π is the identity function on Q^{M_2}.

Finally, for each c_α in P^M, and for each $i \in \omega$, $(b_i, c_\alpha) \in R_0^M$ if and only if $i \in A_{\alpha,0}$. Since π fixes $\{b_i : i \in \omega\}$ pointwise, and the sets $A_{\alpha,0}$ ($\alpha < 2^{\aleph_0}$) are distinct, π must be the identity function on P^M. □

Say that a set $Z \subseteq (2^{\aleph_0} + 1)$ is *robust* if for each $k \in \omega$ there are infinitely many $\alpha \in Z$ with $\alpha = k \mod \omega$. For each $Z \subseteq (2^{\aleph_0} + 1)$, let M_Z be the restriction of M_1 to $\{b_i : i \in \omega\} \cup \{c_\alpha : \alpha \in Z\}$. The arguments above show the following facts, which imply that there are, up to isomorphism, continuum many countable models elementarily equivalent to M_1.

- If $Z \subseteq (2^{\alpha_0} + 1)$ is robust, then M_Z is elementarily equivalent to M_1.
- If Z_1, Z_2 are robust subsets of $(2^{\aleph_0} + 1)$ with $2^{\aleph_0} \in Z_1 \cap Z_2$, then either $Z_1 = Z_2$ or M_{Z_1} and M_{Z_2} are nonisomorphic.

One can also modify the example above to make N_1 and N_2 arbitrarily large, simply by taking as many disjoint copies of N_1 or N_2 as desired.

0.4 Question. Can an extendible countable model have at least one but only countably many rigid elementary extensions?

References

[1] S. Gao, *On automorphism groups of countable structures*, J. Symbolic Logic 63 (3), 1998, 891–896

[2] W. Hodges, **Model theory**, Encyclopedia of Mathematics and its Applications, 42. Cambridge University Press, Cambridge, 1993