D-spaces, irreducibility and trees

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August 14, 2008

Abstract

We show that the removal of one point from 2^{ω_1} gives a counterexample to a conjecture of Ishiu on D-spaces. We also show that Martin's Axiom implies that there are no Lindelöf non-D-spaces that can be written as union of less than continuum many compact subspaces. Finally we show that the property of being a D-space is preserved by forcing with trees of height ω.

An *open neighborhood assignment* (*ONA*) on a topological space X is a function N which assigns to each point $x \in X$ an open set $N(x)$ containing x. Given an ONA N on a space X, and subset Y of X, we let $N[Y]$ denote $\bigcup\{N(x) \mid x \in Y\}$. A space X is a *D-space* [11] if for every ONA N on X there is a closed discrete $C \subseteq X$ such that $N[C] = X$. These spaces were introduced by van Dowen in 1979[11], and while they have attracted a lot of attention in recent years[2, 3, 4, 6, 7, 8, 9, 10, 13, 14], many basic questions remain open [12]. Probably the best known is whether every regular Lindelöf space is a D-space (see [16]).

In the first section we prove that removing one point from 2^{ω_1} gives a counterexample to a conjecture of Tetsuya Ishiu, as the resulting space is irreducible but not a D-space. In the second section, we prove that, assuming Martin's Axiom, there are no "small" Lindelöf non-D-spaces, where "small" means a union of less than continuum many compact subspaces. Finally, in the third section we consider the effects of forcing with trees of height ω. For instance, we show that if T is such a tree and X is a D-space, then X remains a D-space after forcing with T.

1 Irreducibility and the Revised Range Conjecture

Tetsuya Ishiu proposed what he called the *Revised Range Conjecture*, asserting that every topological space X has a basis $\mathcal{B}$ such that for any two ONA's N_0,

*The work in this paper was supported in part by NSF Grant DMS-0401603 (third author), FAPESP Grant 2005/60183-5 (second and third authors) and CNPq (first author)

N_1 (on X) with the same range $R \subseteq \mathcal{B}$, there is a closed discrete set C_0 such that $N_0[C_0] = X$ if and only if there is a closed discrete set C_1 such that $N_1[C_1] = X$. We will see in this section that this conjecture is false.

A topological space X is said to be *irreducible* [1] if for every open cover $\mathcal{O}$ of X there is an open cover $\mathcal{O}'$ such that each element of $\mathcal{O}'$ is contained in a member of $\mathcal{O}$ and contains a point not in any other member of $\mathcal{O}'$ (such a $\mathcal{O}'$ is said to be a *minimal open refinement* of $\mathcal{O}$).

Lemma 1.1. *Let X be an irreducible space in which every open set has the same cardinality. If the Revised Range Conjecture holds for X, then X is a D-space.*

Proof. Let N be an ONA on X. We may assume that the range of N is contained in a basis $\mathcal{B}$ witnessing the Revised Range Conjecture for X. Since X is irreducible, there exists a minimal open refinement $\mathcal{O}'$ of the range of N covering X. For each $O \in \mathcal{O}'$ pick a point in O not in any other member of $\mathcal{O}'$, and let Y be the set of picked points. Then Y is a closed discrete set, and we can define a partial ONA N' on Y by letting $N'(y)$ be any member of the range of N containing the member of $\mathcal{O}'$ containing y, for each $y \in Y$. It suffices now to extend N' to an ONA on all of X with the same range as N. Since Y is closed discrete, each open set has intersection of size $|X|$ with the complement of Y. The range of N has cardinality $\kappa \leq |X|$. Let $\langle B_\alpha : \alpha < \kappa \rangle$ be a wellordering of the range of N, and choose points $\langle x_\alpha : \alpha < \kappa \rangle$ such that each $x_\alpha \in B_\alpha \setminus (Y \cup \{x_\beta : \beta < \alpha\})$, and define $N'(x_\alpha) = B_\alpha$ for each $\alpha < \kappa$. For each $x \in X \setminus (Y \cup \{x_\alpha : \alpha < \kappa\})$, let $N'(x) = N(x)$. $\square$

In [18] it was shown that the removal of one point from 2^{ω_1} gives an irreducible space. However, this space is not a D-space, as shown by the following lemma. Note that every open subset of this space has the same cardinality.

Lemma 1.2. *The space 2^{ω_1} with one point removed contains a closed copy of ω_1.*

Proof. For simplicity, let the removed point be the constant 0 function. For each $\alpha < \omega_1$, let x_α be $(\alpha \times \{0\}) \cup ((\omega_1 \setminus \alpha) \times \{1\})$. The subspace $\{x_\alpha : \alpha < \omega_1\}$ is closed. Furthermore, if for each $\beta < \omega_1$, we let $O_\beta = \{z \in 2^{\omega_1} \mid z(\beta) = 0\}$ and $I_\beta = \{z \in 2^{\omega_1} \mid z(\beta) = 1\}$, then the O_β's and I_β's generate the ω_1-topology on $\{x_\alpha : \alpha < \omega_1\}$. $\square$

2 Lindelöfness and Martin's Axiom

Our second section concerns Lindelöf non-D-spaces and Martin's Axiom (MA). Recall that Martin's Axiom is the statement that if P is a partial order without uncountable antichains, and $\mathcal{D}$ is a collection of dense subsets of P such that $|\mathcal{D}| < \mathfrak{c}$ (where $\mathfrak{c}$ denotes the cardinality of the continuum), then there is a filter $G \subseteq P$ intersecting each element of $\mathcal{D}$ (see [17], for instance). The *covering number for the meager ideal* ($cov(\mathcal{M})$) is the smallest cardinality of a family of meager sets of reals whose union is all of $\mathbb{R}$ (see [5], for instance); restated, it is the smallest cardinality of a collection $\mathcal{D}$ consisting of dense subsets of the partial order $(\omega^\omega, \subseteq)$ with the property that no filter intersects every member of $\mathcal{D}$. The Baire Category Theorem implies that $cov(\mathcal{M}) \geq \aleph_1$. Martin's Axiom (indeed, its restriction to Cohen forcing) implies that $cov(\mathcal{M}) = \mathfrak{c}$.

We first prove that there are no Lindelöf non-D-spaces of cardinality less than $cov(\mathcal{M})$. One easy consequence is that one can not prove, only assuming ZFC, that there is such space of cardinality $\aleph_1$.

We begin with the following.

Lemma 2.1. *If X is a T_1 Lindelöf space and N is an open neighborhood assignment on X, then there is a countable $Y \subseteq X$ such that for every finite $a \subseteq Y$ and every $x \in X \setminus N[a]$ there is a $y \in Y \setminus N[a]$ such that $x \in N[y]$.*

Proof. We find countable sets $Y_i \subseteq X$ $(i < \omega)$, and let $\langle a_i : i < \omega \rangle$ be a listing of all the finite subsets of $\bigcup_{i<\omega} Y_i$, such that each $a_i \subseteq \bigcup_{j \leq i} Y_j$. Let Y_0 be any countable subset of X such that $N[Y_0] = X$. Given a_i, consider the open cover of X given by the restriction of N to $a_i \cup (X \setminus N[a_i])$, and let Y_{i+1} be the set of x such that $N(x)$ is in some fixed countable subcover. Then $Y = \bigcup_{i<\omega} Y_i$ is as desired. □

Theorem 2.2. *If X is a T_1 Lindelöf space and $|X| < cov(\mathcal{M})$, then X is a D-space.*

Proof. Let $Y = \langle y_i : i < \omega \rangle$ be as in Lemma 2.1, and consider the set A of $a \in 2^\omega$ such that for each $i \in a^{-1}(1)$, $y_i \notin N[\{y_j : j \in i \cap a^{-1}(1)\}]$. Then A is a perfect subset of 2^ω, and for each $x \in X$ the set of $a \in A$ with $x \notin N[\{y_i : i \in a\}]$ is nowhere dense in A. Since $|X| < cov(\mathcal{M})$, there is an $a \in A$ such that $N[a] = X$. □

The assumptions of the Theorem 2.2 are implied by MA(Cohen forcing) when $|X| < \mathfrak{c}$ (see [5]).

Corollary 2.3 (MA(Cohen forcing))**.** *If X is a Lindelöf space such that $|X| < 2^\omega$, then X is a D-space.*

Corollary 2.4 (MA(Cohen forcing))**.** *If X is a hereditary Lindelöf space such that it is not a D-space, then $|X| = 2^\omega$.*

Proof. This is immediate, since if Y is a hereditary Lindelöf space then $|Y| \leq 2^\omega$, by a result of de Groot (see [15]). □

Modifying the proof of Theorem 2.2, we can obtain a stronger result. First we note the following consequence of Lemma 2.1.

Lemma 2.5. *Let X be a Lindelöf space and N be an open neighborhood assignment on X. Then there is a countable $Y \subseteq X$ such that for every finite $a \subseteq Y$, there is $b \subseteq Y \setminus N[a]$ such that $X = N[a] \cup N[b]$.*

Let (X, τ) be a topological space. Let $f : \omega^{<\omega} \longrightarrow X \times \tau$ be a function. If $s \in \omega^{<\omega}$ and $f(s) = (x, V)$, then we denote by $f_X(s) = x$ and by $f_\tau(s) = V$.

The idea for the next lemma is the following: we will construct an ω-tree using the Y given by the previous lemma. The successors of every element of the tree will be all the points of Y that are "not yet covered" by our construction. At the same time we will assure that every finite subset of Y that is not yet covered can be added to the tree in finitely many steps.

Lemma 2.6. *Let (X, τ) be a Lindelöf space and N be an open neighborhood assignment on X. Then there is $f : \omega^{<\omega} \smallsetminus \{\emptyset\} \longrightarrow X \times \tau$ such that:*

(i) if $s \in \omega^{<\omega} \smallsetminus \{\emptyset\}$ then $f_\tau(s) \subseteq N(f_X(s))$;

(ii) if r is a branch of $\omega^{<\omega}$, then $\{f_X(s) : s \in r\}$ is closed discrete in $\bigcup\{f_\tau(s) : s \in r\}$;

(iii) if $C \subseteq X$ is compact, then $D_C = \{s \in \omega^{<\omega} : C \subseteq \bigcup_{t \leq s} f_\tau(t)\}$ is dense in $\omega^{<\omega}$.

Proof. Le Y be as given by Lemma 2.5. We will define $f : \omega^{<\omega} \smallsetminus \{\emptyset\} \longrightarrow Y \times \tau$ by recursion on the length of s in such a way that:

(a) if $s \in \omega^{<\omega}$ then for every $n \in \omega$ and every nonzero $k \leq |s|$, $f_X(s^\frown n) \notin f_\tau(s \restriction k)$;

(b) if $s \in \omega^{<\omega}$ then $f_\tau(s) = N(f_X(s)) \smallsetminus F$ where F is a finite subset of $Y \smallsetminus \{f_X(s)\}$;

(c) for every $s \in \omega^{<\omega}$, if $y \in Y \smallsetminus \bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\}$, then there is an $n \in \omega$ such that $y = f_X(s^\frown n)$;

(d) if $y = f_X(s^\frown n)$ for some $s \in \omega^{<\omega}$ and $n \in \omega$, then for each finite $F \subseteq (Y \cap N(y)) \smallsetminus \{y\}$ there is a $k \in \omega$ such that $f(s^\frown k) = (y, N(y) \smallsetminus F)$;

Note that we can make this construction since Y is countable and so is $[Y]^{<\omega}$.

First we will show that if r is a branch of $\omega^{<\omega} \smallsetminus \{\emptyset\}$, then $\{f_X(s) : s \in r\}$ has no accumulation points in $\bigcup\{f_\tau(s) : s \in r\}$. Let $x \in \bigcup\{f_\tau(s) : s \in r\}$. We will show that it is not an accumulation point of $\{f_X(s) : s \in r\}$. Let $s \in r$ such that $x \in f_\tau(s)$. Note that $f_X(t) \notin f_\tau(s)$ for every $t \in r$, $t > s$. Then x is separated from these points and, since there are only finitely many points more in r, we have that x is not an accumulation point.

Note that, by Lemma 2.5, we have that, for every $s \in \omega^{<\omega} \smallsetminus \{\emptyset\}$,

$$\bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\} \cup \bigcup_{n \in \omega} f_\tau(s^\frown n) = X.$$

For each $C \subseteq X$, let D_C denote the set of $s \in \omega^\omega \setminus \{\emptyset\}$ such that C is contained in $\bigcup\{f_\tau(s \restriction k) : 0 < k \leq |s|\}$. We will show that when C is compact, D_C is dense in $\omega^\omega \setminus \{\emptyset\}$. Fix $s \in \omega^{<\omega}$ and suppose $s \notin D_C$. Then $C \smallsetminus \bigcup\{f_\tau(s \restriction k) : 0 < k \leq$

$|s|\}$ is a compact subset covered by the family $(f_\tau(s^\frown n))_{n\in\omega}$. Then, there are $k_1, ..., k_n \in \omega$ such that

$$C \smallsetminus \bigcup\{f_\tau(s\restriction k) : 0 < k \leq |s|\} \subseteq f_\tau(s^\frown k_1) \cup \cdots \cup f_\tau(s^\frown k_n).$$

We can suppose that $f_X(s^\frown k_i) \notin f_\tau(s^\frown k_j)$ for all $1 \leq i < j \leq n$ by property (d). Thus, we can choose $p_1, ..., p_n \in \omega$ such that $f(s^\frown p_1) = f(s^\frown k_1)$, $f(s^\frown p_1^\frown p_2) = f(s^\frown k_2)$, ..., $f(s^\frown p_1^\frown p_2^\frown \cdots ^\frown p_n) = f(s^\frown k_n)$, by property (c). Note that $C \subseteq \bigcup_{t \leq s^\frown p_1^\frown p_2^\frown \cdots ^\frown p_n} f_\tau(t)$. □

It was already known that every σ-compact space is a D-space (see [6], for instance). Lemma 2.6 allows us to improve this result.

Theorem 2.7. *Every Lindelöf space which is a union of fewer than $cov(\mathcal{M})$ many compact spaces is a D-space.*

Proof. Suppose that κ is a cardinal less than $cov(\mathcal{M})$, and that $X = \bigcup_{\xi<\kappa} C_\xi$, where each C_ξ is compact. Let $f : \omega^{<\omega} \longrightarrow X \times \tau$ be the function given by Lemma 2.6. Note that for every $\xi < \kappa$ we have that D_{C_ξ} (as defined in the proof of Lemma 2.6) is dense in $\omega^{<\omega}$. Then there is a branch r of $\omega^{<\omega}$ such that $r \cap D_{C_\xi} \neq \emptyset$ for each $\xi < \kappa$. Thus $\bigcup_{s\in r} f_\tau(s) \supset \bigcup_{\xi<\kappa} C_\xi = X$. Since $\{f_X(s) : s \in r\}$ is closed discrete in $\bigcup_{s\in r} f_\tau(s)$, we have that $\{f_X(s) : s \in r\}$ is closed discrete in X. □

3 Forcing with trees of height ω

The results of the previous section suggest that some basic facts about D-spaces may be independent of ZFC. While we do not have such a result, we present in this section two facts about D-spaces and forcing which may be of some use. These facts concern forcing with trees of height ω, and apply the approach of the previous section.

Theorem 3.1. *If X is a D-space and T is a tree of height ω, then X remains a D-space after forcing with T.*

Proof. Let $\dot{N}$ be a T-name for an ONA on X, and let $\mathcal{B}$ be the set of open subsets of X which are forced by some condition to be in the range of the realization of $\dot{N}$. Let $\langle p_\alpha : \alpha < \kappa\rangle$ be a wellordering of the elements of T such that shorter elements are listed before longer ones. We define recursively on α closed discrete sets D_α $(\alpha < \kappa)$ and functions $f_\alpha \colon D_\alpha \to T$ and $h_\alpha \colon D_\alpha \to \mathcal{B}$ such that, letting Y_α be the set of x in any D_β $(\beta < \alpha)$ such that $f_\beta(x) \geq p_\alpha$:

- for all x in any D_α, $f_\alpha(x) \leq p_\alpha$ and $f_\alpha(x) \Vdash \dot{N}(\check{x}) = \check{h_\alpha(x)}$;
- for all $\alpha < \kappa$ and for all $y \in X$, either there exist $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$ such that $y \in h_\beta(x)$ or there exists an $x \in D_\alpha$ such that $y \in h_\alpha(x)$;
- if $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$, then $h_\beta(x) \cap D_\alpha = \emptyset$.

(Note that since $f_\beta(x) \leq p_\beta$ for each $\beta < \kappa$ and each $x \in X$, $p_\beta \geq p_\alpha$ whenever $D_\beta \cap Y_\alpha$ is nonempty; in particular there are only finitely many such β, so Y_α is closed discrete.)

Supposing that we have constructed D_α, f_α and h_α for all $\beta < \alpha$, let

$$E_\alpha = \bigcup\{h_\beta(x) \mid \beta < \alpha \wedge x \in D_\beta \cap Y_\alpha\}.$$

We define a new ONA N_α as follows. For each $x \in E_\alpha$, let $N_\alpha(x) = E_\alpha$. For each $x \in X \setminus E_\alpha$, pick a condition $p(x) \leq p_\alpha$ and an element $B(x) \in \mathcal{B}$ such that $p(x) \Vdash \dot{N}(\check{x}) = B(x)\check{}$, and let $N_\alpha(x) = B(x)$. Then there is a closed discrete set D^*_α such that $N_\alpha[D^*_\alpha] = X$. Let $D_\alpha = D^*_\alpha \setminus E_\alpha$. For each $x \in D_\alpha$, let $f_\alpha(x) = p(x)$ and let $h_\alpha(x) = B(x)$. This completes the construction.

Let g be a V-generic path through T. For each $\alpha \in \kappa$, let C_α be the set of $x \in D_\alpha$ such that $f_\alpha(x) \in g$. Let

$$C = \bigcup\{C_\alpha \mid \alpha \in \kappa\}.$$

By genericity $\dot{N}_g[C] = X$. We will we done once we show that C is closed discrete.

Pick a point y in X. There is a $x \in C_\beta$ for some $\beta \in \kappa$ such that $y \in \dot{N}_g(x)$. Since $f_\beta(x) \in g$, $\dot{N}_g(x) = h_\beta(x)$. Fix $\gamma < \kappa$ such that $p_\gamma \in g$, $\gamma > \beta$ and $p_\gamma \leq f_\beta(x)$. Since $D_\alpha \cap h_\beta(x) = \emptyset$ for all $\alpha > \beta$ with $p_\alpha \leq f_\beta(x)$, y is not in the closure of

$$\bigcup\{D_\alpha \mid \gamma \leq \alpha < \kappa, p_\alpha \in g\},$$

which contains $\bigcup\{C_\alpha : \gamma \leq \alpha < \kappa\}$. On the other hand,

$$C \subseteq \bigcup\{D_\alpha : p_\alpha > p_\gamma\} \cup \bigcup\{C_\alpha \mid \gamma \leq \alpha < \kappa\}.$$

Since $\bigcup\{D_\alpha : p_\alpha > p_\gamma\}$ is a finite union of closed discrete sets, y is not in the closure of $\bigcup\{D_\alpha : p_\alpha > p_\gamma\} \setminus \{y\}$, either, which shows that C is closed discrete. ☐

A similar argument shows the following result, where we start with a Lindelöf space in the ground model. If T is a tree and S is a subset of T, we say that S can be refined to an antichain if there is a function $a \colon S \to T$ such that $a(s) \leq s$ for all $s \in S$, and such that the range of S is an antichain.

Theorem 3.2. *If X is a Lindelöf space and T is a tree of height ω such that every countable subset of T can be refined to an antichain, then X is a D-space after forcing with T.*

Proof. Let $\dot{N}$ be a T-name for an ONA on X, and let $\mathcal{B}$ be the set of open subsets of X which are forced by some condition to be in the range of the realization of $\dot{N}$. Let $\langle p_\alpha : \alpha < \kappa \rangle$ be a wellordering of the elements of T such that shorter elements are listed before longer ones. We define recursively on α countable sets $D_\alpha \subseteq X$ $(\alpha < \kappa)$ and functions $f_\alpha \colon D_\alpha \to T$ and $h_\alpha \colon D_\alpha \to \mathcal{B}$ such that, letting Y_α be the set of x in any D_β $(\beta < \alpha)$ such that $f_\beta(x) \geq p_\alpha$:

- for all x in any D_α, $f_\alpha(x) \le p_\alpha$ and $f_\alpha(x) \Vdash \dot{N}(\check{x}) = h_\alpha(x)\check{}$;
- for all $\alpha < \kappa$ and for all $y \in X$, either there exist $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$ such that $y \in h_\beta(x)$ or there exists an $x \in D_\alpha$ such that $y \in h_\alpha(x)$;
- if $\beta < \alpha$ and $x \in D_\beta \cap Y_\alpha$, then $h_\beta(x) \cap D_\alpha = \emptyset$;
- the range of each f_α is an antichain.

(Note that since $f_\beta(x) \le p_\beta$ for each $\beta < \kappa$ and each $x \in X$, $p_\beta \ge p_\alpha$ whenever $D_\beta \cap Y_\alpha$ is nonempty; in particular there are only finitely many such β, so Y_α is finite.)

Supposing that we have constructed D_α, f_α and h_α for all $\beta < \alpha$, let

$$E_\alpha = \bigcup \{h_\beta(x) \mid \beta < \alpha \wedge x \in D_\beta \cap Y_\alpha\}.$$

We define a new ONA N_α as follows. For each $x \in E_\alpha$, let $N_\alpha(x) = E_\alpha$. For each $x \in X \setminus E_\alpha$, pick a condition $p(x) \le p_\alpha$ and an element $B(x) \in \mathcal{B}$ such that $p(x) \Vdash \dot{N}(\check{x}) = B(x)\check{}$, and let $N_\alpha(x) = B(x)$. Then there is a countable set D^*_α such that $N_\alpha[D^*_\alpha] = X$. Let $D_\alpha = D^*_\alpha \setminus E_\alpha$. For each $x \in D_\alpha$, let $f_\alpha(x)$ be a condition below $p(x)$ in such a way that the range of f_α is an antichain, and let $h_\alpha(x) = B(x)$. This completes the construction.

Let g be a V-generic path through T. For each $\alpha \in \kappa$, let C_α be the set of $x \in D_\alpha$ such that $f_\alpha(x) \in g$. Let

$$C = \bigcup \{C_\alpha \mid \alpha \in \kappa\}.$$

By genericity $\dot{N}_g[C] = X$. We will we done once we show that C is closed discrete.

Pick a point y in X. There is a $x \in C_\beta$ for some $\beta \in \kappa$ such that $y \in \dot{N}_g(x)$. Since $f_\beta(x) \in g$, $\dot{N}_g(x) = h_\beta(x)$. Fix $\gamma < \kappa$ such that $p_\gamma \in g$, $\gamma > \beta$ and $p_\gamma \le f_\beta(x)$. Since $D_\alpha \cap h_\beta(x) = \emptyset$ for all $\alpha > \beta$ with $p_\alpha \le f_\beta(x)$, y is not in the closure of

$$\bigcup \{D_\alpha \mid \gamma \le \alpha < \kappa, p_\alpha \in g\},$$

which contains $\bigcup \{C_\alpha : \gamma \le \alpha < \kappa\}$. On the other hand, letting

$$Z = \bigcup \{\{z \in D_\alpha \mid f_\alpha(z) \in g\} : \alpha < \gamma\},$$

$$C \subseteq Z \cup \bigcup \{C_\alpha \mid \gamma \le \alpha < \kappa\}.$$

Since Z is a finite set, y is not in the closure of $Z \setminus \{y\}$, either, which shows that C is closed discrete. □

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