CLOSED SETS WHICH CONSISTENTLY HAVE FEW TRANSLATES COVERING THE LINE

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Abstract. We characterize the compact subsets K of 2^ω for which one can force the existence of a set X of cardinality less than the continuum such that $K + X = 2^\omega$.

1. Introduction

In this this note we answer a variant of the following well-known question: For which compact subsets K of the real line can one force that the real line is covered by fewer than continuum many translations of K (as reinterpreted in the forcing extension)? This question has been considered by several authors and the following are known.

- The real line is not covered by fewer than $2^{\aleph_0}$ many translations of the ordinary Cantor set. (Gruenhage)
- If C has packing dimension less than 1 then $\mathbb{R}$ is not covered by fewer than $2^{\aleph_0}$ translations of C. (Darji-Keleti, [3]),
- There is a compact set K of measure zero such that $\mathbb{R}$ is covered by $\mathsf{cof}(\mathcal{N})$ (which is consistently $< 2^{\aleph_0}$) many translations of K (Elekes-Steprāns, [4]). The same holds in any locally compact abelian Polish group. (Elekes-Toth, [5]).

Instead of the real line, we will work in the space 2^ω, with addition as coordinatewise addition modulo 2. For all sets $X, K \subseteq 2^\omega$, and any $z \in 2^\omega$, $X \subseteq K + z$ if and only if $z \notin (2^\omega \setminus K) + X$ (this formulation uses the fact that $-z = z$ for all $z \in 2^\omega$). Replacing K with its complement, this says that 2^ω is covered by the set of translations of K by elements of X if and only if X is not covered by a single translation of $2^\omega \setminus K$. It follows that we can restrict our attention to compact sets K which are nowhere dense and have measure zero with respect to the standard product measure on 2^ω.

Lemma 1. *Let K be a closed subset of 2^ω.*

(1) If K is somewhere dense then 2^ω is covered by finitely many translations of K.

(2) If K has positive measure and $\mathsf{non}(\mathcal{N}) < 2^{\aleph_0}$ then 2^ω is covered by fewer than $2^{\aleph_0}$ many translations of K.

Proof. For the first part, if K is somewhere dense then it contains a basic open set, which implies that a finite set of translations of K covers 2^ω. For the second, if

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$\mathsf{non}(\mathcal{N}) < 2^{\aleph_0}$ there there exists a set $X \subseteq 2^\omega$ of cardinality less than $2^{\aleph_0}$ and outer measure 1. Suppose that K has positive measure. Since $X \not\subseteq (2^\omega \setminus K) + z$ for any $z \in 2^\omega$ it follows that $K + X = 2^\omega$. □

Known proofs that fewer than $2^{\aleph_0}$ many translations of a given set K do not cover 2^ω are based on the following property.

Definition 2. *Let K be a subset of 2^ω. We say that K is* small *if there exists a perfect set $P \subseteq 2^\omega$ such that for every $z \in 2^\omega$,*

$$(K + z) \cap P \text{ is countable.}$$

If K is small then we need $2^{\aleph_0}$ translations of K to cover 2^ω since we need that many translations to cover P. Furthermore, the property "K is small" is Σ^1_2 in a parameter for K, hence absolute. To see this, note that K is small if and only if there exists P such that

(1) P is closed and uncountable, and
(2) $\forall z\ (K + z) \cap P$ is countable.

The first clause is $\mathbf{\Sigma}^1_1$ and the second is $\mathbf{\Pi}^1_1$, by the well known fact that $\{W \in \mathcal{K}(2^\omega) : W \text{ is countable}\}$ is a $\mathbf{\Pi}^1_1$ set, where $\mathcal{K}(2^\omega)$ is the hyperspace of compact subsets of 2^ω (see Section 33.B of [7]).

The notion of being small can be generalized as follows:

Definition 3. *Suppose that K is a subset of 2^ω, Y is a subset of 2^ω and $\mathcal{J}$ is an ideal on Y. We say that K is $\mathcal{J}$- small if for every $z \in 2^\omega$, $(K + z) \cap Y \in \mathcal{J}$.*

In particular, K is small if it is $\mathcal{J}$-small for $\mathcal{J}$ the ideal of countable subsets of some fixed perfect set. In the cases of interest the ideal $\mathcal{J}$ is defined on $|Y|$ rather than Y so we omit mention of Y in the notation.

The following lemma connects the previous definition with the topic of this paper.

Lemma 4. *Suppose that X, Y and K are subsets of 2^ω, and that $\mathcal{J}$ is an ideal on Y such that K is $\mathcal{J}$-small. If $X + K = 2^\omega$, then $|X| \geq \mathsf{cov}(\mathcal{J})$.*

A compact subset K of 2^ω, being closed, is the set of paths through the tree $\{x \restriction n \mid x \in K, n \in \omega\}$. This tree gives rise to a natural reinterpretation of K in any forcing extension as the set of paths through T. The main result of this paper is the following.

Theorem 5. *Suppose that K is a compact set in 2^ω. Then exactly one of the following holds.*

(1) In some forcing extension, 2^ω is covered by fewer than continuum many translations of the reinterpretation of K.
(2) There exist a set $Y \subseteq 2^\omega$ of size $2^{\aleph_0}$ and an ideal $\mathcal{J}$ on Y such that
  (a) K is $\mathcal{J}$-small, and
  (b) $\mathsf{cov}(\mathcal{J}) = 2^{\aleph_0}$.

The theorem easily gives that if the second case holds, then it holds in all forcing extensions. In fact, our characterization of the dichotomy is absolute between models of set theory with the same ordinals (see Remark 33).

The paper is organized as follows. In Section 2 we give a simple criterion which implies that in a c.c.c. forcing extension fewer than $2^{\aleph_0}$ translations of K cover 2^ω. In Section 3 we give examples of sets that satisfy this criterion. Section 4 reviews basic information about Sacks forcing, and Section 5 introduces a rank function on Sacks names for reals. In Sections 6 and 7 we prove both parts of the main result and give necessary and sufficient conditions for a compact set K to cover 2^ω with fewer than $2^{\aleph_0}$ translations.

2. Special cases and simple tests

In this section we introduce a property of a set $K \subseteq 2^\omega$ which implies that in some c.c.c. forcing extension fewer than $2^{\aleph_0}$ translations of the corresponding reinterpretation of K cover 2^ω.

Definition 6. *A perfect set $K \subseteq 2^\omega$ is big if for every $n \in \omega$ there exists $j_n \in \omega$ such that for $X \subseteq 2^\omega$ and $x \in 2^\omega$, if*

(1) $|X| \leq n$,
(2) $(2^\omega \setminus K) + X \neq 2^\omega$,
(3) $x \restriction j_n \in X \restriction j_n = \{y \restriction j_n : y \in X\}$,

then

$$(2^\omega \setminus K) + (X \cup \{x\}) \neq 2^\omega.$$

We say that K is big* if $K \cap [s]$ is big for every $s \in 2^{<\omega}$ such that $K \cap [s] \neq \emptyset$.

If K is big then the collection of finite sets covered by translations of K resembles an ideal, in the following sense: if $X_0, X_1 \subseteq 2^\omega$ are sets of size n, $(2^\omega \setminus K) + X_0 \neq 2^\omega$ and $X_0 \restriction j_{2n} = X_1 \restriction j_{2n}$, then $(2^\omega \setminus K) + (X_0 \cup X_1) \neq 2^\omega$.

Lemma 7. *If K is big then K is not small.*

Proof. Suppose that $P \subseteq 2^\omega$ is a perfect set. Build recursively a sequence

$$\{x_n : n \in \omega\} \subseteq P$$

such that

(1) $Q = \mathsf{cl}(\{x_n : n \in \omega\})$ is perfect,
(2) $(2^\omega \setminus K) + \{x_0, \dots, x_n\} \neq 2^\omega$ for $n \in \omega$.

Given $\{x_0, \dots, x_n\}$ satisfying (2), choose $x_{n+1} \in P$ such that $x_{n+1} \restriction j_n = x_i \restriction j_n$ for some $i \leq n$. This will guarantee that (2) continues to hold. Condition (1) can be arranged by careful bookeeping.

By (2), $L_n = \{z \in 2^\omega : \{x_0, \dots, x_n\} \subseteq K + z\}$ is a nonempty compact set. For $z \in \bigcap_n L_n$, we have $\{x_n : n \in \omega\} \subseteq K + z$, and thus $Q \subseteq K + z$. $\square$

The following theorem is essentially proved in [5].

Theorem 8. *If K is big* then there exists $X \subseteq 2^\omega$ such that $X + K = 2^\omega$ and $|X| \leq \mathsf{cof}(\mathcal{N})$.*

The following theorem complements this result.

Theorem 9. *If K is big*, then there is a c.c.c. forcing extension in which 2^ω is covered by fewer than continuum many translations of the reinterpretation of K.*

Let $\mathbb{Q} = \{q \in 2^\omega : \forall^\infty n\ q(n) = 0\}$. Before beginning the proof, we prove the following lemma.

Lemma 10. *Suppose that $K \subseteq 2^\omega$ is big*. There exists a c.c.c. forcing notion $\mathbb{P}_K$ which adds real $z_K \in 2^\omega$ such that*

$$\Vdash_{\mathbb{P}_K} \forall x \in 2^\omega \cap \mathbf{V}\ \exists q \in \mathbb{Q}\ x \in K + z_K + q.$$

Proof of the lemma. Suppose that $K \subseteq 2^\omega$ is big*. Let $\mathbb{P}_K$ be the collection of pairs (t, X) such that

(1) $t \in 2^{<\omega}$ and X is a finite subset of 2^ω,
(2) $((2^\omega \setminus K) + X) \cap [t] \neq [t]$.

For $(t_0, X_0), (t_1, X_1) \in \mathbb{P}_K$, we put $(t_1, X_1) \geq (t_0, X_0)$ if $t_0 \subseteq t_1$ and $X_0 \subseteq X_1$. We will show that $\mathbb{P}_K$ has the required properties.

To see that $\mathbb{P}_K$ is c.c.c., suppose that $\{(t_\alpha, X_\alpha) : \alpha < \omega_1\}$ is a subset of $\mathbb{P}_K$. Without loss of generality we can assume that there exist $t \in 2^{<\omega}$ and $n \in \omega$ such that $t_\alpha = t$ and $|X_\alpha| = n$ for all $\alpha < \omega_1$. Furthermore, we can assume that $X_\alpha \restriction j_{2n} = X_\beta \restriction j_{2n}$ for $\alpha, \beta < \omega_1$, where j_{2n} is as in the definition of big*. It follows then from the definition of big* that $(t, X_\alpha \cup X_\beta) \in \mathbb{P}_K$ is a condition extending both (t_α, X_α) and (t_β, X_β).

Let $z_K = \bigcup\{t : (t, X) \in G\}$, where G is the generic filter.

Now suppose that $(t, X) \in \mathbb{P}_K$, $|X| = n$ and $x \in 2^\omega$. Find $q \in \mathbb{Q}$ such that $q + x \restriction j_n \in X \restriction j_n$. Since K is big*, it follows that $(t, X \cup \{x + q\}) \in \mathbb{P}_K$. Furthermore,

$$(t, X \cup \{x + q\}) \Vdash_{\mathbb{P}_K} x \in K + z_K + q.$$

In particular,

$$\mathbf{V}^{\mathbb{P}_K} \models 2^\omega \cap \mathbf{V} \subseteq K + z_K + \mathbb{Q},$$

which finishes the proof. $\square$

Proof of Theorem 9. Let $\mathbf{V}[g]$ be a c.c.c. extension of the universe satisfying $\neg\,\mathsf{CH}$ and let $\mathbb{P}_{\omega_1}$ be the finite support iteration of $\mathbb{P}_K$ of length $\aleph_1$ defined in $\mathbf{V}[g]$. Let H be $\mathbf{V}[g]$-generic for $\mathbb{P}_{\omega_1}$. For each $\alpha < \omega_1$, let H_α denote the restriction of H to the first α many stages of $\mathbb{P}_{\omega_1}$, and let z_α be the generic real added at the αth stage. Let $X = \{z_\alpha + q : \alpha < \omega_1, q \in \mathbb{Q}\}$. For each $x \in 2^\omega \cap \mathbf{V}[g, H]$ there is an $\alpha < \omega_1$ such that $x \in \mathbf{V}[g, H_\alpha]$, and it follows that for some $q \in \mathbb{Q}$, $x \in K + z_\alpha + q$. Thus in $\mathbf{V}[g, H]$, $2^\omega \subseteq X + K$ and $|X| < 2^{\aleph_0}$. $\square$

3. Examples of big sets and small sets

In this section we will provide some examples of small sets and big* sets. Let $\{I_n : n \in \omega\}$ be a partition of ω into finite sets of increasing size and let K_n be a subset of 2^{I_n}, for each $n \in \omega$. Consider sets of form $K = \prod_n K_n$. This is a typical compact set in 2^ω whose combinatorial properties are hereditary with respect to all full subtrees, i.e. subtrees of form $K \cap [s]$, where $K \cap [s] \neq \emptyset$ and $s \in 2^{<\omega}$. In particular if such set is big it is also big*.

Theorem 11. *If* $\lim_{n\to\infty} \dfrac{|K_n|}{|2^{I_n}|} = 1$ *then* K *is big**.

We use the following lemma.

Lemma 12 ([5])**.** *Suppose that* $I \subseteq \omega$ *is finite,* $n \in \omega$ *and* $C \subseteq 2^I$ *is such that* $\dfrac{|C|}{2^{|I|}} \geq 1 - \dfrac{1}{n+1}$. *For any* $X \subseteq 2^I$ *of size* $\leq n$ *there exists* $t \in 2^I$ *such that* $t + X \subseteq C$.

Proof. For any $s \in X$,

$$\frac{|\{t \in 2^I : t + s \notin C\}|}{|2^I|} \leq \frac{1}{n+1}.$$

Thus

$$\frac{|\{t \in 2^I : \exists s \in X \ t+s \notin C\}|}{|2^I|} \le \frac{n}{n+1} < 1.$$

□

Proof of Theorem 11. For each $n \in \omega$, let $j_n = \sum_{m \le k} |I_i|$, where k is such that $|K_j|/2^{|I_j|} \ge 1 - \dfrac{1}{n+2}$ for all $j \ge k$. Then for any $n \in \omega$, and $X \subseteq 2^\omega$ of size n and any $x \in 2^\omega$, repeated application of Lemma 12 will produce a translation as desired (the initial segment of the translation up to j_n being given by the assumption that some translation already covers X). □

If the sets I_n are large enough then we can chose sets K_n $(n \in \omega)$ so that

$$1 - \frac{1}{n+1} \le \frac{|K_n|}{|2^{I_n}|} \le 1 - \frac{1}{2n+1}$$

holds for all $n \in \omega$. Then $K = \prod_{n \in \omega} K_n$ has measure zero since $\prod_{n \in \omega} \dfrac{1}{2n+1} = 0$.

The next two lemmas show that if $\lim_{n \to \infty} \dfrac{|K_n|}{|2^{I_n}|} < 1$ then K may be small or big*, depending on the choice of K_n's. In the following lemma, the sets K_n can be chosen so that the ratios $\dfrac{|K_n|}{|2^{I_n}|}$ are eventually any given dyadic rational value in the interval $[0, \dfrac{1}{2}]$.

Lemma 13. *For each $n \in \omega$, let J_n be a nonempty proper subset of I_n, and let K_n be the set of $s \in 2^{I_n}$ such that $s(i) = 0$ for all $i \in J_n$. Then $K = \prod_{n \in \omega} K_n$ is small.*

Proof. Put $J = \bigcup_n J_n$ and let $P = \{x \in 2^\omega : \forall n \notin J \ x(n) = 0\}$. For each $z \in 2^\omega$, $(K+z) \cap P$ has at most one element. □

Lemma 14. *Fix a sequence of positive reals $\{\varepsilon_n : n \in \omega\}$. There exists a sequence $K_n \subseteq 2^{I_n}$ such that for each n, $|K_n|/2^{|I_n|} \le \varepsilon_n$ and $K = \prod_{n \in \omega} K_n$ is big*.*

Lemma 14 can be proved in the same way as Lemma 12, with the following theorem (which is Theorem 3.3 of [1], with $1 - \varepsilon$ in place of ε) used instead of Lemma 11.

Theorem 15 ([1])**.** *Suppose that $m \in \omega$ and $0 < \delta < 1 - \epsilon < 1$. There exists $n \in \omega$ such that for every finite set $I \subseteq \omega$ of size at least n, there exists a set $C \subseteq 2^I$ such that $\varepsilon + \delta \ge |C| \cdot 2^{-|I|} \ge \varepsilon - \delta$ and for every set $X \subseteq 2^I$, $|X| \le m$*

$$\left| \frac{|\bigcap_{s \in X}(C+s)|}{2^{|I|}} - \varepsilon^{|X|} \right| < \delta.$$

Theorem 15 says that we can choose C is such a way that for all sequences $s_1, \ldots, s_m \in 2^I$ the sets $s_1 + C, \ldots, s_m + C$ are probabilistically independent with error δ.

Proof of Lemma 14. Thus, if we choose δ to be much smaller than ε^m, then if $|X| < m$ it follows that $\bigcap_{s \in X}(C+s) \neq \emptyset$. In particular, if $t \in \bigcap_{s \in X}(C+s)$ then $t + X \subseteq C$.

The rest of the argument is just like in Theorem 11. □

4. The Sacks model

In the following section we will describe necessary and sufficient conditions for a compact set K to (consistently) cover 2^ω by fewer than $2^{\aleph_0}$ translations. This characterization is intrinsically connected to the Sacks model.

The Sacks model, obtained by a length ω_2 countable support iteration of perfect set forcing, is a natural candidate to witness that $\aleph_1$ translations of a compact set K covers 2^ω. This follows from Zapletal's work on tame cardinal invariants in [13]. More specifically, we have the following:

Definition 16. *A tame cardinal invariant is defined as*

$$\min\{|A| : A \subseteq \mathbb{R} \ \& \ \phi(A) \& \ \psi(A)\}$$

where $\phi(A)$ is a statement of the model $\langle TC(A), \in, A\rangle$ and $\psi(A)$ is a statement of form "$\forall x \in \mathbb{R}\ \exists y \in A\ \theta(x, y)$", where θ is a formula whose quantifiers range over reals and ω only.

If $K \subseteq 2^\omega$ is a compact set than

$$\min\{|A| : A \subseteq 2^\omega \ \ \forall x \in 2^\omega\ \exists y \in A\ x + y \in K\}$$

is a tame cardinal invariant.

Theorem 17 (Zapletal [13]). *Assuming the existence of a proper class of inaccessible cardinals δ which are limits of Woodin cardinals and of $<\delta$-strong cardinals, if $\mathfrak{x}$ is a tame cardinal invariant, and $\mathfrak{x} < 2^{\aleph_0}$ holds in a set forcing extension, then $\mathfrak{x} < 2^{\aleph_0}$ holds in the iterated Sacks extension.*

A natural attempt would be to show that if K is not small then in the Sacks model $\mathbf{V}^{\mathbb{S}_{\omega_2}}$,

$$\forall x \in 2^\omega\ \exists z \in \mathbf{V} \cap 2^\omega\ x \in K + z.$$

Translating to the Sacks model it would suffice that the following statement holds:

Proposition 18 (false). *Suppose that $p \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in 2^\omega$. Then there exists $p' \geq p$ and a perfect set $P \subseteq 2^\omega$ such that for every perfect set $Q \subseteq P$ there exists $q \geq p'$ such that $q \Vdash \dot{x} \in Q$.*

Indeed, suppose that K is not small and let $p \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in 2^\omega$. If there is $p' \geq p$ and $x \in \mathbf{V} \cap 2^\omega$ such that $p' \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} = x$ then any $z \in (K + x) \cap \mathbf{V}$ will be as required. Otherwise, find $p' \geq p$ and P as in Proposition 18. Since K is not small there is $z \in 2^\omega$ such that $P \cap (K + z)$ is uncountable. Let $Q \subseteq P \cap (K + z)$ be a perfect set. It follows that there is $q \geq p'$ such that $q \Vdash_{\mathbb{S}_{\omega_2}} \dot{x} \in Q \subseteq K + z$. Since $\dot{x}$ was arbitrary, this finishes the proof.

Proposition 18 is true for a single Sacks forcing but fails for an iteration of two or more Sacks reals. To see this note that if $(p, \dot{q}) \Vdash_{\mathbb{S}\star\mathbb{S}} \dot{x} \in 2^\omega$, then $(p, \dot{q})$ can be represented as a closed subset $\bar{p} \subseteq 2^\omega \times 2^\omega$, where $p = \{x : (\bar{p})_x \neq \emptyset\}$, and $(\bar{p})_x \in \mathbb{S}$ whenever $(\bar{p})_x \neq \emptyset$. Furthermore, we can find a one-to-one continuous function $f : \bar{p} \longrightarrow 2^\omega$ such that $\bar{p} \Vdash_{\mathbb{S}\star\mathbb{S}} \dot{x} = f(s_0, s_1)$, where s_0, s_1 are first and second Sacks reals. Let $x_0 \in p$ be a real that is not Sacks-generic (for example a real that is in $\mathbf{V}$) , and put $Q = \{z : \exists y \in (\bar{p})_{x_0}\ z = f(x_0, y)\}$. Clearly Q is a perfect set (since $(\bar{p})_{x_0}$ is and f is one-to-one) and $\bar{p} \Vdash \dot{x} \notin Q$ (since x_0 is not Sacks-generic).

In spite of this counterexample, the basic idea in the Proposition 18 is sound and in the sequel we will look for a largeness condition on Q such that Proposition 18 is true for the iteration as well. Then we will require that K is such that for some $z \in 2^\omega$, $P \cap (K + z)$ satisfies this condition.

We begin with a review of well known properties of Sacks forcing and its iterations.

Sacks forcing $\mathbb{S}$ is defined as the collection of perfect subtrees of $2^{<\omega}$ ordered by inclusion (we write $T \geq T'$ to indicate that $T \subseteq T'$). We will often identify a tree T with the corresponding (perfect) set $[T]$ consisting of its branches, and use letters p, q, etc. to refer to these perfect sets. Given a closed set $p \subseteq 2^\omega$, we let $\mathsf{split}(p)$ be the set of $s \in 2^{<\omega}$ such that $s^\frown\langle 0\rangle$ and $s^\frown\langle 1\rangle$ are both initial segments of members of p. For each $n \in \omega$, we let $\mathsf{split}_n(p)$ be the set of $s \in \mathsf{split}(p)$ having exactly n proper initial segments in $\mathsf{split}(p)$.

For $T, T' \in \mathbb{S}$ and $n \in \omega$ define

$$T \geq_n T' \iff T \geq T' \ \& \ T{\restriction}n = T'{\restriction}n.$$

Lemmas 19-23 are taken from [2] (which in turn is modeled after [8]). Lemmas 19 and 20 are well known (see, for instance, pages 244-245 of [6]).

Lemma 19. *Suppose that $p \in \mathbb{S}$ and $p \Vdash_{\mathbb{S}} \dot{x} \in 2^\omega$. For every $n \in \omega$ there exist $q \geq_n p$ and a continuous function $F : [q] \longrightarrow 2^\omega$ such that $q \Vdash_{\mathbb{S}} \dot{x} = F(\dot{g})$, where $\dot{g}$ is the canonical name for the generic real.*

Moreover, we can require that for every $v \in \mathsf{split}_n(q)$ and any $x_1, x_2 \in [q_v]$, $F(x_1){\restriction}n = F(x_2){\restriction}n$.

Lemma 20. *Suppose that $p \in \mathbb{S}$, $n \in \omega$ and $p \Vdash_{\mathbb{S}} \dot{x} \in 2^\omega$. Let $F : [p] \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{S}} \dot{x} = F(\dot{g})$.*

There exists $q \geq p$ such that $F{\restriction}[q]$ is constant, or there exists $q \geq_n p$ such that $F{\restriction}[q]$ is one-to-one. In particular, the generic real is minimal.

For each ordinal $\gamma \leq \omega_2$, we let $\mathbb{S}_\gamma$ denote the countable support iteration of $\mathbb{S}$ of length γ. So $\mathbb{S}_\gamma$ is the set of functions p such that

1. $\mathsf{dom}(p) = \gamma$,
2. $\mathsf{supp}(p) = \{\beta : p(\beta) \neq \emptyset\}$ is countable,
3. $\forall \beta < \gamma \ p{\restriction}\beta \Vdash_{\mathbb{S}_\beta} p(\beta) \in \mathbb{S}$.

For $F \in [\gamma]^{<\omega}$, $n \in \omega$, and $p, q \in \mathbb{S}_\gamma$ define

$$q \geq_{F,n} p \iff q \geq p \ \& \ \forall \beta \in F \ q{\restriction}\beta \Vdash_{\mathbb{S}_\beta} q(\beta) \geq_n p(\beta).$$

For $p \in \mathbb{S}_\gamma$ let $\mathsf{cl}(p)$ be the smallest set $w \subseteq \gamma$ such that p can be evaluated using the generic reals $\langle \dot{g}_\beta : \beta \in w\rangle$. In other words, $\mathsf{cl}(p)$ consists of those $\beta < \gamma$ such that the transitive closure of p (as a set) contains a $\mathbb{S}_\beta$-name for an element of $\mathbb{S}$. It is well-known [12] that $\{p \in \mathbb{S}_\gamma : \mathsf{cl}(p) \in [\gamma]^{\leq\omega}\}$ is dense in $\mathbb{S}_\gamma$.

Suppose that $p \in \mathbb{S}_\gamma$, $w = \mathsf{cl}(p)$ is countable and $\gamma_p = \mathsf{ot}\big(\mathsf{cl}(p)\big)$. Let $\mathbb{S}_w$ be the countable support iteration of $\mathbb{S}$ with domain w. In other words, consider the countable support iteration $\langle \mathcal{P}_\beta, \dot{\mathcal{Q}}_\beta : \beta < \sup(w)\rangle$ such that

$$\forall \beta < \sup(w) \ \Vdash_{\mathcal{P}_\beta} \dot{\mathcal{Q}}_\beta \simeq \begin{cases} \mathbb{S} & \text{if } \beta \in w \\ \emptyset & \text{if } \beta \notin w \end{cases}.$$

It is clear that $\mathbb{S}_w$ is forcing-equivalent to $\mathbb{S}_{\gamma_p}$. Moreover, we can view the condition p as a member of $\mathbb{S}_w$.

Let γ be a countable ordinal and $p \in \mathbb{S}_\gamma$. Define $\overline{p} \subseteq (2^\omega)^\gamma$ as follows:

$\langle x_\beta : \beta < \gamma \rangle \in \overline{p}$ if for every $\beta < \gamma$,

$$x_\beta \in \Big[p(\beta)\big[\langle x_\gamma : \gamma < \beta \rangle\big]\Big].$$

Note that $p(\beta)[\langle x_\gamma : \gamma < \beta \rangle]$ is the interpretation of $p(\beta)$ using reals $\langle x_\gamma : \gamma < \beta \rangle$ so it may be undefined if these reals are not sufficiently generic.

For a set $G \subseteq (2^\omega)^\gamma$, $u \subseteq \gamma$, and $x \in (2^\omega)^u$ let

$$(G)_x = \{y \in (2^\omega)^{\gamma \setminus u} : \exists z \in G \; z{\restriction}u = x \;\&\; z{\restriction}(\gamma \setminus u) = y\},$$

and for $\beta \in \gamma$ let $(G)_\beta = \{x(\beta) : x \in G\}$.

We say that $p \in \mathbb{S}_\gamma$ is good if

1. $\overline{p}$ is compact,
2. for every $\beta < \gamma$ and $x \in \overline{p{\restriction}\beta}$, $\overline{p[x]} = (\overline{p})_x$ and $\overline{p(\beta)[x]} = ((\overline{p})_x)_\beta$.
3. $\overline{p}$ is homeomorphic to $(2^\omega)^\gamma$ via a homeomorphism h such that for every $\beta < \gamma$ and $x \in \overline{p{\restriction}\beta}$, $h{\restriction}((\overline{p})_x)_\beta$ is a homeomorphism between $((\overline{p})_x)_\beta$ and 2^ω.

Lemma 21. *$\{p \in \mathbb{S}_\gamma : \overline{p}$ is good$\}$ is dense in $\mathbb{S}_\gamma$.*

From now on we will always work with conditions p such that $\overline{p}$ is good.

As in the lemma 19 we show that:

Lemma 22. *Suppose that $p \in \mathbb{S}_\gamma$ and $p \Vdash_{\mathbb{S}_\gamma} \dot{x} \in 2^\omega$. Then there exists $q \geq p$ and a continuous function $F : \overline{p} \longrightarrow 2^\omega$ such that $q \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \gamma \rangle$ is the sequence of generic reals.*

The following lemma is an analogue of Lemma 20.

Lemma 23. *Suppose that $p \in \mathbb{S}_\gamma$, $n \in \omega$ and $p \Vdash_{\mathbb{S}_\gamma} \dot{x} \in 2^\omega$. Let $F : \overline{p} \longrightarrow 2^\omega$ be a continuous function such that $p \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, where $\dot{\mathbf{g}} = \langle \dot{g}_\beta : \beta < \gamma \rangle$ is the sequence of generic reals. There exists $q \geq p$ such that exactly one of the following conditions hold:*

(1) $F{\restriction}\overline{q}$ is constant,
(2) $F{\restriction}\overline{q}$ is one-to-one.

5. A rank function

In this section we will work towards formulating a correct version of Proposition 18. Let K be a perfect subset of 2^ω and fix a tree $\widetilde{T}$ such that $K = [\widetilde{T}]$.

Our main objective is to find property of K which will lead to the following dichotomy:

Suppose that $\mathbf{V} \models \mathsf{GCH}$ is a model containing K. Either

$$\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega$$

or, in all outer models of ZFC,

$$\forall X \subseteq 2^\omega \; (|X| < 2^{\aleph_0} \rightarrow K + X \neq 2^\omega).$$

We need only look at iterations of Sacks forcing of countable length.

Lemma 24. *The following are equivalent for a model $\mathbf{V} \models \mathsf{GCH}$:*

(1) $\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega$,
(2) for every $\gamma < \omega_1$, $\mathbf{V}^{\mathbb{S}_\gamma} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega$.

Proof. Implication (1) → (2) is obvious. To show that (2) → (1) observe that every real in $\mathbf{V}^{\mathbb{S}_{\omega_2}}$ depends only on countably many Sacks reals. If $\mathbf{G} \subseteq \mathbb{S}_{\omega_2}$ is a generic filter over $\mathbf{V}$ and $x \in \mathbf{V}[\mathbf{G}] \cap 2^\omega$ then there exists a countable ordinal γ and a $\mathbf{H} \subseteq \mathbb{S}_\gamma$ generic filter over $\mathbf{V}$ which belongs to $\mathbf{V}[\mathbf{G}]$ such that $x \in \mathbf{V}[\mathbf{H}]$. It follows that $\mathbf{V}[\mathbf{H}] \models \exists z \in 2^\omega \cap \mathbf{V}\ x \in K + z$ and, by absoluteness, the same holds in $\mathbf{V}[\mathbf{G}]$. □

Definition 25. *For $\gamma < \omega_1$ let $\mathbf{Q}_\gamma$ be the collection of triples $\vec{p} = (p, F, T)$ where $p \in \mathbb{S}_\gamma$ is good and $F\colon \bar{p} \to [T]$ is a homeomorphism.*

Elements of $\mathbf{Q}_\gamma$ represent $\mathbb{S}_\gamma$-names for real numbers. By Lemma 22, when $p \Vdash_{\mathbb{S}_\gamma} \dot{x} \in 2^\omega$ we can find a homeomorphism $F : \bar{p} \longrightarrow P$ such that $p \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, possibly after passing to a stronger condition.

By combining F with a homeomorphism between $\bar{p}$ and $(2^\omega)^\gamma$, we can assume that all elements of $\mathbf{Q}_\gamma$ are of form $((2^\omega)^\gamma, F, T)$. This is equivalent to the homegeneity of $\mathbb{S}_\gamma$.

Since F is a homeomorphism, every branch of T reconstructs the entire generic sequence of γ Sacks reals.

Definition 26. *Suppose that $(p, F, T) \in \mathbf{Q}_\gamma$. For $u \in \mathsf{split}(T)$ let $\mathsf{proj}_\alpha(u)$ be the portion of α-th Sacks real computed by u.*

The notation $\mathsf{proj}_\alpha(u)$ suppresses the parameter (p, F, T), which will be clear in context. Since $[u]$ is a clopen set, $\mathsf{proj}_\alpha(u) \neq \emptyset$ only for finitely many $\alpha < \gamma$. More precisely, we have the following:

Lemma 27. *For every $u \in \mathsf{split}(T)$ there is $A_u \in [\gamma]^{<\omega}$ such that*
$$F^{-1}([u]) = \{x \in \bar{p} : \forall \alpha \in A_u\ \ \mathsf{proj}_\alpha(u) \subseteq x(\alpha)\}.$$

Let $R \subseteq (2^{<\omega})^\gamma$ be the tree generated by the family
$$\{\langle \mathsf{proj}_\alpha(u) : \alpha \in A_u \rangle : u \in \mathsf{split}(T)\}.$$

It is easy to see that

Lemma 28. $\langle x_\alpha : \alpha < \gamma \rangle \in \bar{p} \iff \forall \alpha < \gamma\ \forall n\ x_\alpha{\restriction}n \in R(\alpha)$.

Lemma 29. *Suppose that $(p, F, T) \in \mathbf{Q}_\gamma$. For every $v \in \mathsf{split}(T)$, and any $\delta \in \gamma$ there are nodes $t_0, t_1 \in \mathsf{split}(T)$ such that*

(1) $v \subseteq t_0, t_1$,
(2) $\mathsf{proj}_\delta(t_0)$, $\mathsf{proj}_\delta(t_1)$ are incompatible,
(3) $\mathsf{proj}_\alpha(t_0) = \mathsf{proj}_\alpha(t_1)$ for $\alpha < \delta$.

Proof. Let $A_v \in [\gamma]^{<\omega}$ be such that $F^{-1}[v] = \{x \in \bar{p} : \forall \alpha \in A_v\ \ \mathsf{proj}_\alpha(u) \subseteq x(\alpha)\}$. Choose two branches $x_0, x_1 \in F^{-1}[v]$ such that $x_0(\alpha) = x_1(\alpha)$ for all $\alpha < \delta$ and $x_0(\delta) \neq x_1(\delta)$. Recall that we assumed that $\dot{x}$ depends on all Sacks reals so this is always possible. Now $F(x_0)$ and $F(x_1)$ are two branches extending v. Let $n \in \omega$ be so large that $\mathsf{proj}_\delta(F(x_0){\restriction}n)$, $\mathsf{proj}_\delta(F(x_1){\restriction}n)$ are incompatible.

Now let $t_0 = F(x_0){\restriction}n$ and $t_1 = F(x_1){\restriction}n$. Since $x_0(\alpha) = x_1(\alpha)$ for $\alpha < \delta$, it follows that $\mathsf{proj}_\alpha(t_0) = \mathsf{proj}_\alpha(t_1)$ for all $\alpha < \delta$. □

In the proof above, n may have to be quite large to determine that $x_0(\delta){\restriction}n \neq x_1(\delta){\restriction}n$, and its value depends on F and T. To illustrate this point suppose that we are dealing with just two Sacks reals and $\dot{x}$ is a name for the sum of them. Even if we know that the first digit of $\dot{x}$ is 0 we only know that the first digits of both Sacks reals are the same. It depends on the tree T how far we have to extend v to determine the value of the first digit of either Sacks real.

Definition 30. *Given a tree $T \subseteq 2^{<\omega}$ we let $\mathsf{obj}(T)$ be the collection of triples $x = (n_x, t_x, s_x)$ such that*

(1) $n_x \in \omega$,
(2) $t_x \subseteq T$ is a finite tree whose all maximal nodes have length n_x,
(3) $s_x \in 2^{n_x}$.

For $x = (n_x, t_x, s_x)$ and $y = (n_y, t_y, s_y)$ we say that $x \geq y$ if

(1) $n_x \geq n_y$,
(2) $t_x \cap 2^{n_y} = t_y$,
(3) $s_y \subseteq s_x$.

Let $\mathbf{0}$ be $(0, \emptyset, \emptyset)$, the smallest element in $\mathsf{obj}(T)$.

The following definition is modeled after Lemma 29.

Definition 31. *Suppose that*

- *$\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$;*
- *$x = (n_x, t_x, s_x)$ is in $\mathsf{obj}(T)$;*
- *v is a maximal node of t_x;*
- *$\xi < \gamma$.*

We say that y is a response to $(\vec{p}, x)$-challenge (v, ξ) if

(1) $y \geq x$,
(2) there are maximal nodes $t_0, t_1 \in t_y$ such that
  (a) $v \subseteq t_0, t_1$,
  (b) $\mathsf{proj}_{\xi+1}(t_0)$, $\mathsf{proj}_{\xi+1}(t_1)$ are incompatible and
  (c) $\forall \zeta \leq \xi$ $\mathsf{proj}_\zeta(t_0) = \mathsf{proj}_\zeta(t_1)$.

Definition 32. *Suppose that $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$ and $K = [\widetilde{T}]$ is a fixed compact set. The rank function $\mathsf{rk}_{\vec{p}} : \mathsf{obj}(T) \longrightarrow \omega_1 \cup \{\infty\}$ is defined as follows.*

(1) $\mathsf{rk}_{\vec{p}}(x) = 0$ if $t_x + s_x \not\subseteq \widetilde{T} \cap 2^{n_x}$,
(2) $\mathsf{rk}_{\vec{p}}(x) \geq \alpha > 0$ if for every $\beta < \alpha$, and every $(\vec{p}, x)$-challenge (v, ξ) there exists a response $y \in \mathsf{obj}(T)$ with $\mathsf{rk}_{\vec{p}}(y) \geq \beta$.

In other words,

$$\mathsf{rk}_{\vec{p}}(x) = \min_{\xi<\gamma} \min_{v \in t_x \cap 2^{n_x}} \sup\{\mathsf{rk}_{\vec{p}}(y) + 1 : y \geq x,\ y \text{ responds to } (\vec{p}, x)\text{-challenge } (v, \xi)\}.$$

Let $\mathsf{rk}_{\vec{p}}(x) = \infty$ if $\mathsf{rk}_{\vec{p}}(x) \geq \alpha$ for all α.

Remark 33. *For $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$, the members of $\mathsf{obj}(T)$ are hereditarily finite, and the function $\mathsf{rk}_{\vec{p}}$ depends only on $\mathsf{obj}(T)$ and $\vec{p}$. It follows that $\mathsf{rk}_{\vec{p}}$ takes the same values in every wellfounded model of ZFC containing $\vec{p}$. Similarly, the existence of a countable ordinal γ and $\vec{p} \in \mathbf{Q}_\gamma$ such that the corresponding rank function $\mathsf{rk}_{\vec{p}}$ takes value γ at $\mathbf{0}$ is a Σ^1_2 statement, so absolute to models of ZFC containing ω_1.*

Lemma 34. *If $x \leq y$ then $\mathsf{rk}_{\vec{p}}(x) \geq \mathsf{rk}_{\vec{p}}(y)$.*

Proof. If (v, ξ) is a $(\vec{p}, y)$-challenge then $(v \restriction n_x, \xi)$ is a $(\vec{p}, x)$-challenge. □

Lemma 35. *Suppose that $x \in \mathsf{obj}(T)$ and $y \geq x$ is a response to $(\vec{p}, x)$-challenge (v, ξ). Then there exists a minimal $x \leq y' \leq y$ which responds to (v, ξ).*

Proof. Suppose that $x = (n_x, t_x, s_x)$ and $y = (n_y, t_y, s_y)$. First find $n_x \leq n_{y'} \leq n_y$ such that $t_0\restriction n_{y'}, t_1\restriction n_{y'}$ are still responses to (v, ξ). Let $t_{y'}$ consist of these two nodes plus one extension of length $n_{y'}$ for each maximal node of t_x. □

Observe that in the definition of rank we can limit ourselves to extensions that are minimal in the above sense.

EXAMPLES $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = 1$ if there exists $\xi < \gamma$ such that for every $n \in \omega$, if $t_0, t_1 \in 2^n$ and

(1) $\mathsf{proj}_{\xi+1}(t_0) \neq \mathsf{proj}_{\xi+1}(t_1)$ and
(2) $\forall \zeta \leq \xi \;\; \mathsf{proj}_\zeta(t_0) = \mathsf{proj}_\zeta(t_1)$,

then

$$\neg\exists s \in 2^n \; t_0 + s, t_1 + s \in \widetilde{T} \cap 2^n.$$

In other words, for every response y to $(\vec{p}, \mathbf{0})$-challenge $(\emptyset, \xi)$, $\mathsf{rk}_{\vec{p}}(y) = 0$.

For arbitrary $x = (n_x, t_x, s_x)$ the same holds except that the $(\vec{p}, x)$-challenge would be of form (v, ξ) for some $v \in t_x$ and then we also require that $v \subseteq t_0, t_1$, $s_x \subseteq s$ and $n \geq n_x$.

Similarly $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = 2$ if for every $(\vec{p}, \mathbf{0})$-challenge $(\emptyset, \xi)$ there is a response $y \geq \mathbf{0}$ such that $\mathsf{rk}_{\vec{p}}(y) = 1$.

Lemma 36. *Suppose that $\mathsf{rk}_{\vec{p}}(x) = \infty$ and $\xi < \gamma$. Then there exists $y \geq x$ such that*

(1) $\mathsf{rk}_{\vec{p}}(y) = \infty$,
(2) for every maximal node $v \in t_x$, y responds to the $(\vec{p}, x)$-challenge (v, ξ).

Proof. Let $v_1, \ldots, v_k$ be a list of maximal nodes of t_x. Let $x_0 = x$ and define by recursion a sequence $x_1, \ldots, x_k = y$ such that for every $i < k$,

(1) $x_{i+1} \geq x_i$,
(2) $\mathsf{rk}_{\vec{p}}(x_i) = \infty$,
(3) for every $j > i$, v_j has a unique maximal extension $v_j^\star$ in t_{x_i},
(4) x_{i+1} is a response to the $(\vec{p}, x_i)$-challenge $(v_{i+1}^\star, \xi)$.

If x_i is already constructed then by the induction hypothesis v_i has a unique extension $v_i^\star$ in x_i. Let x_{i+1} be any maximal extension of x_i responding to $(v_i^\star, \xi)$ with $\mathsf{rk}_{\vec{p}}(x_{i+1}) = \infty$. It is easy to see that $y = x_k$ has required properties. □

The definition of rank depends on the set K. The following examples relate it to the concepts from previous sections. The two lemmas below follow from the general theorem which we are aiming to prove but here we will provide a direct argument.

Lemma 37. *Suppose that $\vec{p} = (p, F, T)$ and $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$. Then there exists $z \in 2^\omega$ such that $K \cap (z + [T])$ is uncountable. In particular, if $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ then K is not small.*

Proof. Suppose that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$. Recursively construct a sequence $\langle x_k : k \in \omega\rangle$ such that for every k,

(1) $x_k = \langle n_{x_k}, t_{x_k}, s_{x_k}\rangle$,
(2) $x_{k+1} \geq x_k$,
(3) $\mathsf{rk}_{\vec{p}}(x_k) = \infty$,
(4) x_{k+1} responds to all $(\vec{p}, x_k)$-challenges $(v, 1)$ for each maximal node $v \in t_{x_k}$.

For the step (4) we use Lemma 36 with $\xi = 1$.

Let $\bar{T} = \bigcup_k t_{x_k}$ and $z = \bigcup_k s_{x_k}$. It follows that $\bar{T}$ is a perfect tree and $[\bar{T}] \subseteq [\widetilde{T}] + z$. $\square$

Lemma 38. *Suppose that K is big*. Then $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ for all $\gamma < \omega_1$ and all $\vec{p} \in \mathbf{Q}_\gamma$.*

Proof. Fix $\gamma < \omega_1$ and let $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$. It suffices to to find a tree $T' \subseteq T$ and a real $z \in 2^\omega$ such that

(1) $[T'] \subseteq [T] + z$,
(2) for all $v \in \mathsf{split}(T')$ and all $\delta < \gamma$ there are nodes t_0, t_1 such that
 (a) $v \subseteq t_0, t_1$,
 (b) $\mathsf{proj}_{\delta+1}(t_0)$, $\mathsf{proj}_{\delta+1}(t_1)$ are incompatible,
 (c) $\forall \eta \leq \delta\ \ \mathsf{proj}_\eta(t_0) = \mathsf{proj}_\eta(t_1)$.

If we succeed in finding such T' and z then for every $x = (n_x, t_x, s_x) \in \mathsf{obj}(T)$ satisfying

(1) $t_x \subseteq T' \cap 2^{n_x}$
(2) $s_x \subseteq z$

we have $\mathsf{rk}_{\vec{p}}(x) > 0$. Consequently, $\mathsf{rk}_{\vec{p}}(x) = \infty$.

We will refine the argument in Lemma 7. Let $\{\eta_n : n \in \omega\}$ be the enumeration of γ. We build inductively a function $f \in \omega^{<\omega}$ and find reals $\{y_s : s \in \omega^{<\omega}\ \&\ s(i) < f(i) \text{ for } i < |s|\}$ and a sequence of integers $\{k_n : n \in \omega\}$ such that

(1) $\mathsf{cl}(\{y_s : s < f\})$ is a perfect set,
(2) $(2^\omega \setminus K) + \{y_s : s < f,\ |s| < n\} \neq 2^\omega$ for $n \in \omega$,
(3) for every $t \in \{y_s{\restriction}k_n : s < f,\ |s| < n\}$ and every η_m, $m \leq n$ there are reals $y_{s^\frown i}, y_{s^\frown j}$ with $i, j < f(|s|)$ such that
 (a) $t \subseteq y_{s^\frown i}, y_{s^\frown j}$,
 (b) $\mathsf{proj}_{\eta_m+1}(y_{s^\frown i}{\restriction}k_{n+1})$, $\mathsf{proj}_{\eta_m+1}(y_{s^\frown j}{\restriction}k_{n+1})$ are incompatible,
 (c) $\forall \eta \leq \eta_m\ \ \mathsf{proj}_\eta(y_{s^\frown i}{\restriction}k_{n+1}) = \mathsf{proj}_\eta(y_{s^\frown j}{\restriction}k_{n+1})$.

Suppose that $\{y_s : s < f, |s| < n\}$ are given. For each already constructed real and each γ_m, $m \leq n$ we need to construct two reals satisfying (3). This requirement determines the value of $f(|s|)$. Condition (1) is guaranteed by (3) and condition (2) will be satisfied as long as for every $i < f(|s|)$, $y_{s^\frown i}{\restriction}j_{f(|s|)} \in \{y_s{\restriction}j_{f(|s|)} : s < f, |s| < n\}$ where j is the function from Definition 6. Finally, reals $y_{s^\frown i}$ for $i < f(|s|)$ can be found using lemma 29, and lastly k_{n+1} can be chosen large enough so that (3) holds.

Arguing as in Lemma 7 we find $z \in 2^\omega$ such that $\{y_s : s < f\} \subseteq K + z$. Let T' be a tree such that $[T']$ is the closure of $\{y_s : s < f\}$. Observe that T' has the required properties. $\square$

The following theorem is a refinement of Theorem 5. It characterizes sets K that require than continuum translations to cover 2^ω.

Theorem 39. *Suppose that K is a compact subset of 2^ω. If for some $\gamma < \omega_1$ there is $\vec{p} \in \mathbf{Q}_\gamma$ such that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) < \omega_1$ then 2^ω is not covered by less than $\aleph_0$ translations of K.*

If for every $\gamma < \omega_1$ and every $p \in \mathbf{Q}_\gamma$, $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ then for a model $\mathbf{V} \models \mathsf{GCH}$,

$$\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega.$$

6. The consistency result

In this section we will show the second part of Theorem 39.

Theorem 40. *If for every $\gamma < \omega_1$ and every $p \in \mathbf{Q}_\gamma$, $\mathsf{rk}_{\vec{p}}(\mathbf{0}) = \infty$ then for a model $\mathbf{V} \models \mathsf{GCH}$,*

$$\mathbf{V}^{\mathbb{S}_{\omega_2}} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega.$$

The proof of this theorem will occupy the rest of this section. As we already remarked earlier it suffices to show that for every $\gamma < \omega_1$,

$$\mathbf{V}^{\mathbb{S}_\gamma} \models K + (\mathbf{V} \cap 2^\omega) = 2^\omega.$$

Fix $\gamma < \omega_1$. We have to show that for every real $x \in \mathbf{V}^{\mathbb{S}_\gamma} \cap 2^\omega$ there exists $z \in \mathbf{V} \cap 2^\omega$ such that $x \in K + z$.

Suppose that $x \in \mathbf{V}^{\mathbb{S}_\gamma} \cap 2^\omega$. Without loss of generality, x depends on all Sacks reals, that is γ is minimal. We can find $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$ such that $p \Vdash_{\mathbb{S}_\gamma} \dot{x} = F(\dot{\mathbf{g}})$, where $\mathbf{g} = \langle g_\beta : \beta < \gamma \rangle$ is the sequence of Sacks reals. As before we can assume that $\bar{p} = (2^\omega)^\gamma$. We need to find $q \in \mathbb{S}_\gamma$ and $z \in \mathbf{V} \cap 2^\omega$ such that $q \Vdash_{\mathbb{S}_\gamma} \dot{x} \in K + z$. We will construct sequences $\langle x_k = (n_{x_k}, t_{x_k}, s_{x_k}) : k \in \omega \rangle$ and $\langle \xi_k : k \in \omega \rangle$ such that

1. $\forall \xi < \gamma\ \exists^\infty k\ \xi_k = \xi$,
2. $x_{k+1} \geq x_k$,
3. $\mathsf{rk}_{\vec{p}}(x_k) = \infty$,
4. x_{k+1} responds to every $(\vec{p}, x_k)$-challenge (v, ξ_k).

Suppose that x_k is already constructed. To get x_{k+1} apply Lemma 36 with $\xi = x\xi_k$.

Let $\bar{T} = \bigcup_k t_{x_k}$ and $z = \bigcup_k s_{x_k}$. It follows that $[\bar{T}] + z \subseteq [\widetilde{T}] = K$, that is $[\bar{T}] \subseteq K + z$.

Lemma 41. *There exists $q \in \mathbb{S}_\gamma$ such that $\bar{q} = F^{-1}([\bar{T}])$.*

Proof. This lemma finishes the proof, as $q \Vdash_{\mathbb{S}_\gamma} \dot{x} \in [\bar{T}] \subseteq K + z$.

Let $Q = F^{-1}([\bar{T}])$, we want to show that there is $q \in \mathbb{S}_\gamma$ such that $\bar{q} \subseteq Q$. It suffices to show that for every $\beta < \gamma$ and every $x \in (2^\omega)^\beta$, $((Q)_x)_\beta$ is a perfect set provided that $((Q)_x)_\beta \neq \emptyset$. In other words, whenever x simulates the first β Sacks reals, $((Q)_x)_\beta$ is supposed to be a Sacks condition determined by x. Note that $((Q)_x)_\beta$ is a closed set, so it a set of branches of some tree. Choose a $v \in 2^{<\omega}$ such that $[v] \cap ((Q)_x)_\beta \neq \emptyset$. It remains to check that v has two incompatible extensions t_0, t_1 such $[t_0] \cap ((Q)_x)_\beta \neq \emptyset$ and $[t_1] \cap ((Q)_x)_\beta \neq \emptyset$. Let $x^\star \in (2^\omega)^\gamma$ be such that $x^\star{\restriction}\beta = x$ and $v \subseteq x^\star(\beta)$ and let $y^\star = F(x^\star)$. By Lemma 27 for each $n \in \omega$ there is A_n such that $F^{-1}([y^\star{\restriction}n]) = \{x : \forall \alpha \in A_n\ \ \mathsf{proj}_\alpha(y^\star{\restriction}n) \subseteq x(\alpha)\}$. Let n and k be chosen so large that

1. $\beta = \xi_k$,
2. $v \subseteq \mathsf{proj}_\beta(y^\star{\restriction}n)$,
3. $y^\star{\restriction}n$ is a maximal node in t_{x_k}.

In other words, at this step we will produce nodes t_0, t_1 such that

1. $y^\star{\restriction}n \subseteq t_0, t_1$,
2. $\mathsf{proj}_\beta(t_0)$, $\mathsf{proj}_\beta(t_1)$ are incompatible,
3. $\forall \zeta < \beta\ \ \mathsf{proj}_\zeta(t_0) = \mathsf{proj}_\zeta(t_1)$.

It follows that t_0 and t_1 are two incompatible extensions of v in $((Q)_x)_\beta$. ☐

7. Coherent club-guessing principles

The argument in Section 8 uses the coherent club-guessing principle given by Theorem 46 below. First we prove Theorem 42, a stronger version of the restriction of Theorem 46 to the case of successors of regular cardinals. The material in this section is entirely due to the third author, but the proof of Theorem 42 was provided to us by Assaf Rinot. The sets C_α in Theorem 42 are not closed, but they are cofinal. In Theorem 46 the condition of cofinality is dropped as well.

Theorem 42. *Let λ be a regular uncountable cardinal, let $\theta < \lambda$ be a limit ordinal, and let S be a stationary subset of λ^+ consisting of ordinals of cofinality $\mathsf{cof}(\theta)$. Then there exists a sequence $\langle C_\alpha \mid \alpha < \lambda^+\rangle$ such that for each club $E \subseteq \lambda^+$ there exists an $\alpha \in S$ such that*

(1) $\sup(C_\alpha) = \alpha$,
(2) $\mathsf{ot}(C_\alpha) = \theta$,
(3) $C_\beta = C_\alpha \cap \beta$ *for all* $\beta \in C_\alpha$,
(4) $C_\alpha \subseteq S \cap E$.

Proof. For each ordinal $\alpha < \lambda^+$, fix an injection $d_\alpha : \alpha \to \lambda$, and for each $\beta < \lambda$, let a^β_α denote $d^{-1}_\alpha[\beta]$. Then for each $\alpha < \lambda^+$, $\langle a^\beta_\alpha \mid \beta < \lambda\rangle$ is a continuous, $\subseteq$-increasing chain in $[\alpha]^{<\lambda}$ with union α. For each $\alpha < \lambda^+$, let F_α be the set of $\gamma < \lambda$ such that, for all $\beta \in a^\gamma_\alpha$, $a^\gamma_\beta = a^\gamma_\alpha \cap \beta$. Then each F_α is club subset of λ.

Since $\theta < \lambda$, club many ordinals below λ^+ of cofinality $\mathsf{cof}(\theta)$ contain a cofinal set of ordertype θ.

Given a set $E \subseteq \lambda^+$, and $\beta < \lambda$, let $E(\beta)$ be the set of all $\alpha \in S$ for which the following hold:

(1) $\beta \in F_\alpha$;
(2) $\mathsf{ot}(E \cap S \cap \alpha) = \alpha$;
(3) $\sup(E \cap S \cap a^\beta_\alpha) = \alpha$;
(4) $\mathsf{ot}(E \cap S \cap a^\beta_\alpha) = \mathsf{ot}(a^\beta_\alpha)$ contains a cofinal subset of ordertype θ.

Note that if $E \subseteq E'$ are subsets of λ and $\beta < \lambda$, then $E(\beta) \subseteq E'(\beta)$.

Lemma 43. *There exists a $\beta^* < \lambda$ for which $E(\beta^*)$ is nonempty whenever E is a club in λ^+.*

Proof. To prove the claim, suppose otherwise. Then for each $\beta < \lambda$ we may pick a club $E_\beta \subseteq \lambda^+$ for which $E_\beta(\beta) = \emptyset$. Let $E = \bigcap_{\beta<\lambda} E_\beta \setminus \lambda$. Since E is club in λ^+, we may fix an $\alpha \in E \cap S$ such that $\mathsf{ot}(E \cap S \cap \alpha) = \alpha$.

As $\mathrm{cf}(\alpha) < \mathrm{cf}(\lambda)$, the set $D = \{\beta < \lambda : \sup(E \cap S \cap a^\beta_\alpha) = \alpha\}$ is co-bounded in λ. Furthermore, continuity entails that the set

$$D' = \{\beta \in D : \mathsf{ot}(E \cap S \cap a^\beta_\alpha) = \mathsf{ot}(a^\beta_\alpha)\}$$

is club in λ. Pick $\beta \in D' \cap F_\alpha$ such that $\mathsf{ot}(a^\beta_\alpha)$ contains a cofinal subset of ordertype θ. Then since $E \subseteq E_\beta$, we get that $\mathsf{ot}(E_\beta \cap S \cap a^\beta_\alpha) = \mathsf{ot}(a^\beta_\alpha)$. So $\alpha \in E_\beta(\beta)$, contradicting the choice of E_β. This completes the proof. □

Let $\beta^* < \lambda$ be as given by Lemma 43.

Lemma 44. *There exists a club $E^* \subseteq \lambda^+$ such that for every club $D \subseteq \lambda^+$, the set $\{\alpha \in E^*(\beta^*) : a^{\beta^*}_\alpha \cap E^* \subseteq D\}$ is nonempty.*

Proof. To prove this, suppose otherwise. Then there exists a $\subseteq$-decreasing sequence $\langle G_\beta : \beta < \lambda\rangle$ of club subsets of λ^+ such that

(1) $G_0 = \lambda^+$,
(2) for every $\beta < \lambda$, the set $\{\alpha \in G_\beta(\beta^*) : a_\alpha^{\beta^*} \cap G_\beta \subseteq G_{\beta+1}\}$ is empty,
(3) for every limit ordinal $\gamma < \lambda$, $G_\gamma = \bigcap_{\beta<\gamma} G_\beta$.

Let $G = \bigcap_{\beta<\lambda} G_\beta$, and pick $\alpha \in G(\beta^*)$. Then $\alpha \in G_\beta(\beta^*)$ for all $\beta < \lambda$, hence $\langle a_\alpha^{\beta^*} \cap G_\beta : \beta < \lambda\rangle$ must be a strictly-decreasing sequence of subsets of $a_\alpha^{\beta^*}$, contradicting the fact that $|a_\alpha^{\beta^*}| < \lambda$. This completes the proof. □

Let $E^* \subseteq \lambda^+$ be as given by Lemma 44.

Lemma 45. *There exists an ordinal $\tau^* < \lambda$ which contains a cofinal subset of ordertype θ such that for every club $D \subseteq \lambda^+$, the set $\{\alpha \in E^*(\beta^*) : a_\alpha^{\beta^*} \cap E^* \subseteq D \wedge \mathsf{ot}(a_\alpha^{\beta^*}) = \tau\}$ is nonempty.*

Proof. Again, to prove this, suppose otherwise. Then for every ordinal $\tau < \lambda$ which contains a cofinal subset of ordertype θ, there exists a club $D_\tau \subseteq \lambda^+$ for which

$$\{\alpha \in E^*(\beta^*) : a_\alpha^{\beta^*} \cap E^* \subseteq D_\tau \ \wedge \mathsf{ot}(a_\alpha^{\beta^*}) = \tau\}$$

is empty. Let D be the intersection of these sets D_τ. By the choice of E^* we may pick an $\alpha \in E^*(\beta^*)$ such that $a_\alpha^{\beta^*} \cap E^* \subseteq D$. Let $\tau = \mathsf{ot}(a_\alpha^{\beta^*})$. Since $\alpha \in E^*(\beta^*)$, τ contains a cofinal subset of ordertype θ, contradicting the fact $a_\alpha^{\beta^*} \cap E^* \subseteq D_\tau$. This completes the proof □

Let τ^* be as given by the previous Lemma. As τ^* contains a cofinal subset of ordertype θ, we may fix a cofinal subset $u \subseteq \tau^*$ of order-type θ. For each $\alpha < \lambda^+$, let

$$C_\alpha = \{\beta \in E^* \cap S \cap a_\alpha^{\beta^*} : \mathsf{ot}(a_\beta^{\beta^*}) \in u\}.$$

Let us see that $\langle C_\alpha : \alpha < \lambda^+\rangle$ works. Suppose that we are given a club $E \subseteq \lambda^+$. Applying the choice of τ^*, pick $\alpha \in E^*(\beta^*)$ such that $a_\alpha^{\beta^*} \cap E^* \subseteq E$ and $\mathsf{ot}(a_\alpha^{\beta^*}) = \tau^*$. Then:

(1) $\alpha \in S$;
(2) $\sup(E^* \cap S \cap a_\alpha^{\beta^*}) = \alpha$;
(3) $C_\alpha \subseteq E^* \cap S \cap a_\alpha^{\beta^*} \subseteq S \cap E$;
(4) $\beta^* \in F_\alpha$, so for all $\gamma \in a_\alpha^{\beta^*}$, we have $a_\gamma^{\beta^*} = a_\alpha^{\beta^*} \cap \gamma$, and $C_\gamma = C_\alpha \cap \gamma$;
(5) $\mathsf{ot}(E^* \cap S \cap a_\alpha^{\beta^*}) = \mathsf{ot}(a_\alpha^{\beta^*}) = \tau^*$;
(6) $\mathsf{ot}(C_\alpha) = \mathsf{ot}(u) = \theta$.

This completes the proof. □

Given a set C of ordinals, and an ordinal $\beta < \sup(C)$, we let $\mathsf{next}_C(\beta)$ denote $\min(C \setminus (\beta + 1))$.

Theorem 46. *Suppose that λ is an uncountable cardinal, and let γ be a countable ordinal. There exists a sequence $\bar{C} = \{C_\alpha : \alpha < \lambda^+\}$ such that*

(1) $\forall\alpha < \lambda^+ \ C_\alpha \subseteq \alpha$,
(2) if $\beta \in C_\alpha$ then $C_\beta = C_\alpha \cap \beta$,
(3) $S = \{\alpha < \lambda^+ : \mathsf{ot}(C_\alpha) = \gamma\}$ is stationary,
(4) if $E \subseteq \lambda^+$ is a club then the set

$$\mathsf{gd}(E) = \{\alpha \in S \cap E : \forall\beta \in C_\alpha \ [\beta, \mathsf{next}_{C_\alpha}(\beta)) \cap E \neq \emptyset\}$$

is stationary.

Proof. In the case where λ is regular, this follows from Theorem 42, by replacing each C_α given there (with S as the set of all ordinals below λ^+ of countable cofinality) with $C_\alpha \cap \beta$ for β minimal violating condition (2) of the statement of this theorem (and leaving C_α as is if there is no such β).

We now prove the theorem assuming only $\lambda \geq \omega_2$, following the argument on pages 93-94 of [11]. By Conclusion 1.7 and Claim 1.3 of [9], there exist a stationary $S_0 \subseteq \lambda^+ \cap cof(\aleph_1) \setminus (\omega_1 + 1)$, a club $E_0 \subseteq \lambda^+$ and a sequence $\langle C^0_\alpha : \alpha < \lambda^+\rangle$ such that

(1) each C^0_α is a closed subset of the corresponding α,
(2) each nonaccumulation point of each C^0_α is a successor ordinal,
(3) whenever $\beta \in C^0_\alpha$ is a nonaccumulation point of C^0_α, $C^0_\beta = C^0_\alpha \cap \beta$,
(4) for every $\alpha \in S_0 \cap E_0$, $\mathsf{ot}(C^0_\alpha) = \omega_1$ and $\alpha = \sup(C^0_\alpha)$.

We may assume that $S_0 \subseteq E_0$. Let $\langle C^1_\alpha : \alpha < \lambda^+\rangle$ be the sequence formed by removing from each C^0_α all of its accumulation points. Then $\langle C^1_\alpha : \alpha < \lambda^+\rangle$ retains properties (1) - (4), except that the sets C^1_α need not be closed.

Given sets C, F, let $gl(C, F)$ denote the set $\{\sup(\beta \cap F) : \beta \in C \wedge \beta > \min(F)\}$. By [10] (Sh365, Claim 2.3 (2), for id^b), there is a club $E_1 \subseteq \lambda^+$ such that for each club $E \subseteq E_1$, the set of $\alpha \in S_0$ for which $gl(C^1_\alpha, E_1) \subseteq E$ is stationary (otherwise we can derive a descending ω-sequence of ordinals from a $\subseteq$-decreasing ω_2-sequence of club sets $F_\gamma \subseteq \lambda^+$, where each $F_{\gamma+1}$ witnesses that F_γ is not as desired). For each $\alpha \in \lambda^+$, let

$$C^2_\alpha = \{\beta \in C^1_\alpha : \beta = \min(C^1_\alpha \setminus \sup(\beta \cap E_1)) \wedge \beta > \min(E_1)\}.$$

Let us check that $\langle C^2_\alpha : \alpha < \lambda^+\rangle$ satisfies item (4) of the conclusion of the theorem (using S_0, which will be a subset of the desired S). Fix $E \subseteq \lambda^+$ club. It suffices to consider the case where E consists of limit points of E_1. Fix $\alpha \in S_0 \cap E$ for which $gl(C^1_\alpha, E_1) \subseteq E$, and fix $\beta \in C^2_\alpha$. Let $\beta' = \mathsf{next}_{C^2_\alpha}(\beta)$. Then $\beta < \sup(\beta' \cap E_1) \in E \cap \beta'$ (since β' is a successor ordinal).

Finally, for each $\alpha < \lambda^+$, let $C_\alpha = \{\beta \in C^2_\alpha : \mathsf{ot}(C^2_\alpha \cap \beta) < \gamma\}$. Then the sequence $\langle C_\alpha : \alpha < \lambda^+\rangle$ is as desired. $\square$

Condition (4) implies that for stationary many α is S there is an element of E between any two consecutive elements of C_α. By removing the least element of C_α we can also assume that $\min(C_\alpha) \cap E \neq \emptyset$ whenever $\alpha \in \mathsf{gd}(E)$ and E is a club. Observe that coherence condition (2) implies that for any $\alpha, \beta \in S$, if $\delta = \sup(C_\alpha \cap C_\beta)$ then $C_\alpha \cap \delta = C_\beta \cap \delta$.

Remark 47. *Suppose $\bar{C}$ and γ are as in Theorem 46, and that γ is a limit ordinal. Define C'_α, for $\alpha < \lambda^+$ by letting each C'_α be the set of $\beta \in C_\alpha$ for which the ordertype of $C_{alpha} \cap \beta$ has the form $\delta + n$, for δ either 0 or a limit ordinal, and $n \in \omega$ even. Then $\{C'_\alpha : \alpha < \lambda^+\}$ also satisfies the conclusion of the theorem, with part 4 strengthened so that*

$$\mathsf{gd}'(E) = \{\alpha \in S \cap E : \forall \beta \in C_\alpha\ (\beta, \mathsf{next}_{C_\alpha}(\beta)) \cap E \neq \emptyset\}$$

is stationary. The corresponding strengthened version of Theorem 46 for nonlimit γ can be obtained similarly, starting from a sequence $\bar{C}$ corresponding to some $\gamma' \geq \gamma$.

8. The ZFC result

In this section we will prove Theorem 39, and the second half of Theorem 5 in the corresponding case. Since these follow easily from CH (since K is nowhere dense), we assume otherwise.

To each sequence $\bar{C}$ as in Theorem 46 (for notational purposes, we may assume that γ is the maximal ordertype of elements of $\bar{C}$, and thus that γ is determined by $\bar{C}$) we associate an ideal $\mathcal{J}_{\bar{C}}$ on S (also determined by $\bar{C}$) generated by $\{S \setminus \mathsf{gd}(E) : E \subseteq \lambda^+ \text{ club}\}$.

Lemma 48. *The additivity of $\mathcal{J}_{\bar{C}}$ is λ^+. In particular, $\mathsf{cov}(\mathcal{J}_{\bar{C}}) = \lambda^+$.*

Proof. Suppose that $\{I_\alpha : \alpha < \lambda\} \subseteq \mathcal{J}_{\bar{C}}$. Find clubs $E_\alpha \subseteq \lambda^+$ such that $I_\alpha \subseteq S \setminus \mathsf{gd}(E_\alpha)$ for $\alpha < \lambda$. Note that $\bigcup_{\alpha<\lambda} I_\alpha \subseteq S \setminus \mathsf{gd}(\bigcap_{\alpha<\lambda} E_\alpha) \in \mathcal{J}_{\bar{C}}$. □

For each countable ordinal γ we define an ideal $\mathcal{J}_\gamma$ on $2^{\aleph_0}$ as follows. If $2^{\aleph_0}$ is a successor of an uncountable cardinal, that is if $2^{\aleph_0} = \lambda^+$ for some uncountable λ then $\mathcal{J}_\gamma = \mathcal{J}_{\bar{C}}$, where $\bar{C}$ is any sequence as in Remark 47 (so with the strengthened version of part 4) with respect to γ.

If $2^{\aleph_0}$ is a limit cardinal then we fix an increasing sequence of uncountable cardinals $\{\lambda_\eta : \eta < \mathsf{cf}(2^{\aleph_0})\}$ converging to $2^{\aleph_0}$ together with guessing sequences $\{\bar{C}_{\lambda_\eta^+} : \eta < \mathsf{cf}(2^{\aleph_0})\}$ as in Remark 47, with respect to γ. The ideal $\mathcal{J}_\gamma$ is then defined on $\prod_{\eta<\mathsf{cf}(2^{\aleph_0})} \lambda_\eta^+$ as follows:

$$X \in \mathcal{J}_\gamma \iff \forall \eta\ (X)_\eta \in \mathcal{J}_{\bar{C}_{\lambda_\eta^+}},$$

where $(X)_\eta = \{\alpha : (\eta, \alpha) \in X\}$. Observe that in either case $\mathsf{cov}(\mathcal{J}) = 2^{\aleph_0}$.

We say that a set $A \subseteq 2^\omega$ is *$\mathcal{J}_\gamma$-small* if there exists a set $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$ such that for every $z \in 2^\omega$ $\{\alpha < 2^{\aleph_0} : y_\alpha \in z + A\} \in \mathcal{J}_\gamma$. If A is $\mathcal{J}_\gamma$-small, then 2^ω is not covered by fewer than $2^{\aleph_0}$ many translations of A.

We can now rephrase Theorem 39 as follows.

Theorem 49. *Let K be a compact subset of 2^ω. If for some $\gamma < \omega_1$ there is $\vec{p} \in \mathbf{Q}_\gamma$ such that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) < \omega_1$ then K is $\mathcal{J}_\gamma$-small.*

Let $\gamma < \omega_1$, $\vec{p} \in \mathbf{Q}_\gamma$ be such that $\mathsf{rk}_{\vec{p}}(\mathbf{0}) < \omega_1$. Without loss of generality we can assume that $\vec{p} = ((2^\omega)^\gamma, F, T)$ for some tree T and homeomorphism F.

Fix a cardinal λ such that $2^{\aleph_0} > \lambda \geq \aleph_1$, and a set of distinct reals

$$X = \{x_\xi : \xi < \lambda^+\} \subseteq 2^\omega.$$

Let $\bar{C} = \langle C_\alpha : \alpha \in \lambda^+\rangle$ and S be as in Remark 47, with respect to γ. For each $\alpha \in S$ let

$$y_\alpha = F(\langle x_\beta : \beta \in C_\alpha\rangle),$$

and let $Y = \{y_\alpha : \alpha < \lambda^+\}$. Since each C_α for $\alpha \in S$ has order type γ,

$$\langle x_\beta : \beta \in C_\alpha\rangle) \in (2^\omega)^\gamma,$$

so the reals y_α are well defined.

Now the reals y_α are defined in such a way that

$$\bigcup\{\mathsf{proj}_\delta(y_\alpha \restriction n) : n \in \omega\} = x_\xi,$$

where ξ is the δ-th element of C_α. In other words, reals from X pretend to be Sacks reals.

For each $z \in 2^\omega$ let $S_z = \{\alpha \in S : y_\alpha \in K + z\}$.

Theorem 50. *For every $z \in 2^\omega$, $S_z \in \mathcal{J}_{\bar{C}}$.*

Theorem 49 follows immediately. If $2^{\aleph_0} = \lambda^+$ for some uncountable λ then $\mathcal{J}_\gamma = \mathcal{J}_{\bar{C}}$. If $2^{\aleph_0}$ is a limit cardinal then $\mathcal{J}_\gamma = \prod_{\eta < \mathsf{cf}(2^{\aleph_0})} \mathcal{J}_{\bar{C}_{\lambda_{\eta^+}}}$ and the component $\mathcal{J}_{\bar{C}_{\lambda_{\eta^+}}}$ witnesses that at least λ_η^+ translations of K are needed. Since $\sup\{\lambda_\eta^+ : \eta < \mathsf{cf}(2^{\aleph_0})\} = 2^{\aleph_0}$ we are done.

The rest of this section is devoted to the proof of Theorem 50. Supposing that the theorem is false, we let $z^\star \in 2^\omega$ be such that $S_{z^\star} \notin \mathcal{J}_{\bar{C}}$.

Lemma 51. *The set $\mathsf{gd}'(E) \cap S_{z^\star}$ is stationary for every club $E \subseteq \lambda^+$.*

Proof. If $\mathsf{gd}'(E) \cap S_{z^\star} \cap E' = \emptyset$ for some club E' then $\mathsf{gd}'(E \cap E') \cap S_{z^\star} = \emptyset$. In particular $S_{z^\star} \in \mathcal{J}_{\bar{C}}$. □

Lemma 52. *There exists a perfect tree $Q \subseteq 2^{<\omega}$ such that for every node $t \in Q$*

$$\{\alpha \in S_{z^\star} : t \subseteq y_\alpha\} \notin \mathcal{J}_{\bar{C}}.$$

Proof. Let $Z_0 = \{y_\alpha : \alpha \in S_{z^\star}\} = \{y_\alpha : y_\alpha \in z^\star + K\}$. By the Cantor-Bendixon theorem there exists a perfect tree $Q_0 \subseteq 2^{<\omega}$ and a countable set C_0 such that $\mathsf{cl}(Z_0) = [Q_0] \cup C_0$. For $t \in Q_0$ let $S_t = \{\alpha \in S_{z^\star} : t \subseteq y_\alpha\}$ and let

$$Q_1 = \{t \in Q_0 : S_t \notin \mathcal{J}_{\bar{C}}\}.$$

Note that Q_1 is a tree without terminal nodes.

Let $Z_1 = [Q_1]$. If Z_1 is uncountable then by applying the Cantor-Bendixon theorem again we get a perfect tree Q such that $Z_1 = [Q] \cup C_1$. The tree Q has the required property.

Suppose otherwise and let $E_0 = \lambda^+ \setminus \{\alpha : y_\alpha \in Z_1 \cup C_0\}$ and for $t \in Q_0 \setminus Q_1$ let E_t be a club of λ^+ such that $E_t \cap S_t = \emptyset$. Put $E = E_0 \cap \bigcap_{t \in Q_0 \setminus Q_1} E_t$. It follows that $S_{z^\star} \cap E = \emptyset$, a contradiction. □

Lemma 53. *There exists a sequence $\langle E_\xi, \bar{N}_\xi : \xi < \omega_1\rangle$ such that for each $\xi < \omega_1$,*

(1) E_ξ is a club subset of λ^+,
(2) $E_\xi \subseteq \bigcap_{\zeta<\xi} E_\zeta$,
(3) $\bar{N}_\xi$ is a sequence $\langle N_{\xi,\alpha} : \alpha \in E_\xi\rangle$ such that for each $\alpha \in E_\xi$,
  (a) $N_{\xi,\alpha} \prec \mathbf{H}(\lambda^{++})$,
  (b) $\lambda + 1 \subseteq N_{\xi,\alpha}$, $z^\star, \bar{C}, \widetilde{T}, Y \in N_{\xi,\alpha}$,
  (c) $|N_{\xi,\alpha}| = \lambda$,
  (d) for all $\beta \in \alpha \cap E_\xi$, $N_{\xi,\beta} \subseteq N_{\xi,\alpha}$, and if α is a limit point of E_ξ then $N_{\xi,\alpha} = \bigcup_{\beta \in \alpha \cap E_\xi} N_{\xi,\beta}$,
  (e) for all $\beta < \alpha$, $\langle N_{\xi,\delta} : \delta \in \beta \cap E_\xi\rangle \in N_{\xi,\alpha}$,
  (f) $\{E_\zeta : \zeta < \xi\} \in N_{\xi,\alpha}$,
  (g) $N_{\xi,\alpha} \cap \lambda^+ = \alpha$.

Observe that (3)(f) is the only condition imposing dependence between different sequences $\bar{N}_\xi$.

Proof of Lemma 53. Suppose that $\langle \bar{N}_\zeta, E_\zeta\rangle$ for $\zeta < \xi$ are already given. Let $\{N_\alpha : \alpha < \lambda^+\}$ be a continuous sequence of models satisfying condition (3)(a)-(f). Let C be

$$\{\alpha : N_\alpha \cap \lambda^+ = \alpha\}.$$

Since $\langle N_\alpha : \alpha < \lambda^+\rangle$ is continuous, C is a club. Put $E_\xi = C \cap \bigcap_{\zeta<\xi} E_\zeta$ and let $\bar{N}_\xi = \{N_\alpha : \alpha \in E_\xi\}$. Observe that E_ξ and $\bar{N}_\xi$ are as required. □

Definition 54. *Suppose that $\vec{p} = (p, F, T) \in \mathbf{Q}_\gamma$. A triple $(x, \xi, \bar{\alpha})$ is* suitable *if*

(1) $x = (n_x, t_x, s_x) \in \mathsf{obj}(T)$,
(2) $s_x = z^\star \restriction n_x$,
(3) $\mathsf{rk}_{\vec{p}}(x) < \xi$,
(4) $\bar{\alpha} = \langle \alpha_v : v \in t_x \cap 2^{n_x}\rangle$ *is such that*
    (a) *for each* $v \in t_x \cap 2^{n_x}$, $\alpha_v \in \mathsf{gd}'(E_\xi) \cap S_{z^\star}$,
    (b) $v \subseteq y_{\alpha_v}$.

Recall that $\alpha_v \in S_{z^\star}$ means that $y_{\alpha_v} \in z^\star + K$. Observe that if $\mathsf{rk}_{\vec{p}}(\mathbf{0}) < \xi$ then $(\mathbf{0}, \xi, \alpha_{\langle\rangle})$ is suitable whenever $\alpha_{\langle\rangle} \in \mathsf{gd}'(E_\xi) \cap S_{z^\star}$. The following lemma gives the desired contradiction.

Lemma 55. *If $(x, \xi, \bar{\alpha}_x)$ is suitable then there exists a suitable $(y, \zeta, \bar{\alpha}_y)$ such that $y \geq x$ and $\zeta < \xi$.*

Proof of Lemma 55. Suppose that $(x, \xi, \bar{\alpha}_x)$ is suitable. Since $\mathsf{rk}_{\vec{p}}(x) < \xi$ there exist $\zeta < \xi$ and a $(\vec{p}, x)$-challenge (v, δ) such that $\mathsf{rk}_{\vec{p}}(y) < \zeta$ for every $y \in \mathsf{obj}(T)$ responding to (v, δ). To finish the proof it suffices to find one such y, and a sequence $\bar{\alpha}_y$ as in item (4) above such that $(y, \zeta, \bar{\alpha}_y)$ is suitable.

Let $\gamma^\star$ be the δ-th element of C_{α_v} and $\gamma^{\star\star}$ the $(\delta+1)$-th element of C_{α_v}. Let Z be the collection of all pairs (γ', α') such that

(1) $\alpha' \in \mathsf{gd}'(E_\zeta) \cap S_{z^\star}$,
(2) $\gamma^\star \in C_{\alpha'}$,
(3) γ' is the $(\delta+1)$-th element of $C_{\alpha'}$,
(4) $v \subseteq y_{\alpha'}$.

Then

(1) for all $(\gamma', \alpha') \in Y$, $C_{\alpha'} \cap \gamma^\star = C_{\alpha_v} \cap \gamma^\star = C_{\gamma^\star}$,
(2) $(\gamma^{\star\star}, \alpha_v) \in Z$.

Since $\alpha_v \in \mathsf{gd}'(E_\xi)$ it follows that there is $\bar{\gamma} \in (\gamma^\star, \gamma^{\star\star}) \cap E_\xi$. Then all parameters from the definition of Z are in $N_{\xi,\bar{\gamma}}$.

Lemma 56. *Let $F = \{\gamma' : \exists\alpha'\ (\gamma', \alpha') \in Z\}$. Then F is unbounded in λ^+.*

Proof. If F were bounded it would be the same set in $N_{\zeta,\bar{\gamma}}$ as in $\mathbf{H}(\lambda^{++})$. However, $\gamma^{\star\star} \in F$, and $\gamma^{\star\star} \notin N_{\zeta,\bar{\gamma}}$. □

Fix $\gamma' \in F$ such that $\gamma' \neq \gamma^{\star\star}$, and let α' be such that $(\gamma', \alpha') \in Y$. Since $\gamma^\star \in C_{\alpha'} \cap C_{\alpha_v}$ it follows that $C_{\alpha'} \cap \gamma^\star = C_{\alpha_v} \cap \gamma^\star$ and since $\gamma^\star$ is the δ-th element of C_{α_v} the first δ elements of $C_{\alpha'}$ and C_{α_v} are the same. Recall that each y_α was defined to be $F(\langle x_\beta : \beta \in C_\alpha\rangle)$. Consequently,

$$\bigcup\{\mathsf{proj}_\eta(y_{\alpha'} \restriction n) : n \in \omega\} = \bigcup\{\mathsf{proj}_\eta(y_{\alpha_v} \restriction n) : n \in \omega\}$$

for $\eta \leq \delta$. On the other hand since the $(\delta+1)$-th elements of C_α and C_{α_v} are different

$$\bigcup\{\mathsf{proj}_{\delta+1}(y_{\alpha'} \restriction n) : n \in \omega\} \neq \bigcup\{\mathsf{proj}_{\delta+1}(y_{\alpha_v} \restriction n) : n \in \omega\}.$$

Define $y \geq x$ as follows. First find $n_y \in \omega$ such that

$$\mathsf{proj}_{\delta+1}(y_{\alpha'}) \restriction n_y \neq \mathsf{proj}_{\delta+1}(y_{\alpha_v}) \restriction n_y.$$

Next let $s_y = z^\star{\restriction}n_y$. Let $t_y = \{y'_\alpha{\restriction}n_y\} \cup \{y_{\alpha_w}{\restriction}n_y : w \in t_x \cap 2^{n_x}\}$. Finally, let $\bar{\alpha}_y = \{\alpha_w : w \in t_y \cap 2^{n_y}\}$ be defined as follows:

$$\alpha_w = \begin{cases} \alpha' & \text{if } w = y_{\alpha'}{\restriction}n_y \\ \alpha_v & \text{if } w = y_{\alpha_v}{\restriction}n_y \\ \alpha_s & \text{if } w = y_{\alpha_s}{\restriction}n_y \text{ for } s \in t_x \cap 2^{n_x} \setminus \{v\} \end{cases}$$

By the choice of n_y, the node v gets two distinct extensions, $y'_\alpha{\restriction}n_y$ and $y_{\alpha_v}{\restriction}n_y$, and one is assigned α' and the other α_v. All other nodes follow appropriate reals and have the same ordinals assigned to them. □

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