COMPACT SPACES, ELEMENTARY SUBMODELS, AND THE COUNTABLE CHAIN CONDITION

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Dedicated to Jim Baumgartner on the occasion of his 60th birthday

ABSTRACT. Given a space $\langle X, \mathcal{J}\rangle$ in an elementary submodel M of $H(\theta)$, define X_M to be $X \cap M$ with the topology generated by $\{U \cap M : U \in \mathcal{J} \cap M\}$. It is established, using anti-large-cardinals assumptions, that if X_M is compact and its regular open algebra is isomorphic to that of a continuous image of some power of the two-point discrete space, then $X = X_M$. Assuming CH+SCH (the Singular Cardinals Hypothesis) in addition, the result holds for any compact X_M satisfying the countable chain condition.

1. INTRODUCTION

This paper continues the line of research of [10], [11], [6], [8] and [12], in which the question of which topological spaces are determined by their compact reflections in elementary submodels is investigated. A minor technical obstacle results from the fact that we cannot take elementary submodels of the entire universe, but we want our models to be elementary in structures much larger than the spaces we are considering. So, we adopt the following convention: whenever X is a topological space, the elementary submodels we consider have X as an element and are elementary in $H(\theta)$ for some regular cardinal θ of cardinality greater than all finite iterations of the power-set function starting with X.

Given a space $\langle X, \mathcal{T}\rangle$ in an elementary submodel M of $H(\theta)$, we define X_M to be $X \cap M$ with the topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$ [5]. If X_M is compact T_2 (in fact, we shall assume all spaces are T_2), this constrains X to the point that simple additional topological hypotheses on X_M ensure that $X_M = X$ [6]. When powers of the two-point discrete space D are considered, the situation is more complicated: roughly, for κ below very large cardinals, X_M homeomorphic to D^κ implies $X_M = X$, but this is not the case above such large cardinals [8], [11], [6]. This was generalized to continuous images of powers of D in [12]. In Section 5, we generalize the positive results to compact spaces co-absolute with such spaces. Yet a further generalization is to compact spaces satisfying the countable chain condition. However, for this we need to assume $CH + SCH$ (where SCH stands for the Singular Cardinals Hypothesis).

Key words and phrases. compact, countable chain condition, reflection, elementary submodel, co-absolute with dyadic compact space, squashable.

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In [6] we also considered the contrasting situation of when X compact implies X_M compact. This turned out to be related to whether X is *scattered*, i.e. each subspace has an isolated point. Generalizations of scattering play a key role in the two new theorems mentioned above; as well, we explore in general the relationships between various forms of scattering and the question of "squashing" a compact space X to a compact X_M. Formally,

Definition 1.1. [8] A compact space X is *squashable* if for some elementary submodel M containing X, X_M is compact but not equal to X.

Kunen [8] noted that squashability does not depend on θ. He also showed:

Lemma 1.2. *If D^λ is squashable, λ is greater than the first 1-extendible cardinal.*

1-extendible cardinals are reasonably large; in particular, if κ is 1-extendible, κ is the κth measurable cardinal. For the definition and more on such cardinals, see [7] or [8].

In previous papers, [10], [11], [6], [12], we have used the anti-large-cardinal assumption "$0^\#$ does not exist" or, rather, its consequence that "$|M| \geq \kappa$ implies $M \supseteq \kappa$" to limit the types of elementary submodels that can exist. Here we introduce a weaker assumption that will serve our purposes.

Definition 1.3. (B): if θ is a regular cardinal, M is an elementary submodel of $H(\theta)$, and γ is a cardinal in M such that $2^\gamma \in M$, then $|M \cap \gamma| \subseteq M$.

Note that since M may not satisfy the power set axiom, the condition on γ in the statement of (B) is not vacuous.

Theorem 1.4. *Axiom (B) implies no D^κ is squashable.*

Proof. We show the contrapositive. Suppose that κ is a cardinal and M is an elementary submodel of some $H(\theta)$ such that κ and 2^κ are in M and $(D^\kappa)_M$ is compact but not equal to D^κ. Then κ is not included in M, since $\kappa \subseteq M$ and $(D^\kappa)_M$ compact imply $(D^\kappa)_M = D^\kappa$ [11], [6]. Thus we can fix $\alpha < \kappa$, the least ordinal not in M. By our convention, since $D^\kappa \in M$, 2^{2^κ} is also in M. Since $(D^\kappa)_M$ is compact, $|2^\kappa \cap M| = 2^{|\kappa \cap M|}$. $|\kappa \cap M| \geq |\alpha|$, so $2^{|\kappa \cap M|} > \alpha$. But (B) would imply $|2^\kappa \cap M| \subseteq M$, so $\alpha \in M$, contradiction. $\square$

For any elementary submodel M of any $H(\theta)$, we define o_M to be the least cardinal κ such that $\kappa^+ \not\subseteq M$ (note that even if o_M is a limit cardinal, M must include it). Equivalently, o_M is the cardinality of the least ordinal not in M. If $o_M \neq \theta$ (which must be the case if M is part of a counterexample to (B)) one of the following must hold:

- $o_M \in M$, in which case $o_M^+ \cap M = \eta$, where η is the least ordinal not in M (since for every ordinal α in M of cardinality o_M there is a bijection between o_M and α in M);
- $o_M \notin M$, in which case o_M is a limit cardinal, and the least ordinal in M greater than o_M is a cardinal (as M is closed under the function $\alpha \mapsto |\alpha|$).

The following reformulation of (B) in terms of o_M is immediate.

Theorem 1.5. *Axiom (B) is equivalent to the assertion that if θ is a regular cardinal and M is an elementary submodel of $H(\theta)$, then $|M \cap \gamma| = o_M$ for every $\gamma \geq o_M$ in M such that $2^\gamma \in M$.*

Axiom (B) is essentially the principle $|M| = o_M$ weakened so that, while it suffices for all of our applications, its failure implies a certain form of Chang's Conjecture (it is for this that we require $2^\gamma \in M$). Recall that for cardinals $\lambda > \kappa$ and $\delta > \eta$, the expression $\langle \lambda, \kappa\rangle \to \langle \delta, \eta\rangle$ says that the set of $Z \subseteq \lambda$ of cardinality δ with $|Z \cap \kappa| = \eta$ is stationary in $\mathcal{P}(\lambda)$ (equivalently, that for every function F from the finite subsets of λ to λ there exists a $Z \subseteq \lambda$ of cardinality δ closed under F with $|Z \cap \kappa| = \eta$). Another equivalent form of $\langle \lambda, \kappa\rangle \to \langle \delta, \eta\rangle$ is the following: there is an elementary substructure M of $H(\lambda^+)$ of cardinality δ with $|M \cap \kappa| = \eta$. *Chang's Conjecture* is the statement $\langle \omega_2, \omega_1\rangle \to \langle \omega_1, \omega\rangle$.

Supposing that (B) fails, fix the smallest θ for which there exists a counterexample, and let ζ be the least cardinal for which there exists an $M \prec H(\theta)$ witnessing the failure of (B) with $o_M = \zeta$. Let γ be the least ordinal such that there exists an $M \prec H(\theta)$ with

- $\gamma, 2^\gamma \in M$,
- $\gamma \geq \zeta$,
- $|M \cap \gamma| > o_M$,

and fix such an M. Now, let η be the least cardinal in M greater than o_M. Either $\eta = o_M^+$ or $\eta \cap M = o_M$, so in particular $|\eta \cap M| = o_M$. By the minimality of γ, $|\gamma \cap M| = o_M^+$. Since $2^\gamma \in M$, if $o_M \in M$ then we have the following version of Chang's Conjecture: $\langle \gamma, o_M^+\rangle \to \langle o_M^+, o_M\rangle$. To see this, note that otherwise there would be a function $F\colon [\gamma]^{<\omega} \to \gamma$ for which there exists no set $Z \subseteq \gamma$ closed under F with $|Z| = o_M^+$ and $|Z \cap o_M^+| = o_M$. Then since γ, o_M and o_M^+ are in M there must be such a function in M, but since $M \cap \gamma$ is closed under any function $F'\colon [\gamma]^{<\omega} \to \gamma$ in M, we have a contradiction. If $o_M \notin M$, we have the weaker statement that for every function $F\colon [\gamma]^{<\omega} \to \gamma$ there exists a $Z \subseteq \gamma$ closed under F such that $\eta \geq |Z \cap \gamma| = |Z \cap \eta|^+$. This weaker statement is implied by the version of Chang's Conjecture from the first case, and in fact is equivalent to the failure of (B). The failure of (B) then has consistency strength somewhere in between Chang's Conjecture and the existence of the sharp of every real (see [7]). In particular, (B) is weaker than the assumption "$0^\#$ does not exist", which has been used in other papers in the references, and (B) suffices for those arguments.

We will frequently be using the following consequence of (B):

Lemma 1.6. *Assume (B) and suppose ϕ is a cardinal function on topological spaces such that $\phi(X)$ is bounded by some finite iteration of the exponential function applied to $|X|$, and $|\phi(X) \cap M| \geq \phi(X_M)$. Then $M \supseteq \phi(X_M)$.*

Proof. Suppose $\phi(X) \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^{|X|}}}}}}$. By (B), it suffices to show $|\phi(X) \cap M| \geq \phi(X_M)$ which we have assumed. $\square$

2. Examples

The following example shows that we can have, modulo an inaccessible, two different elementary submodels M and N of the same size and a compact space X such that X_M is compact but X_N is not compact. We do not think the inaccessible is necessary, but we do not have another example.

Example 2.1. *A space X and elementary submodels M and N containing X such that $|M| = |N|$, X_M is compact, but X_N is not.*

Let κ be an inaccessible cardinal smaller than the first 1-extendible. Take X to be the one-point compactification of the disjoint sum of D^γ, for $\gamma < \kappa$. Let M be an elementary submodel of a suitable $H(\theta)$ with the property that $M \cap \kappa$ is an ordinal less than κ, and such that all subsets of γ are in M whenever $\gamma \in M \cap \kappa$. Then X_M is compact. To get such an M, build an increasing sequence of elementary submodels M_n ($n < \omega$) of cardinality less than κ such that for each even n, $M_n \cap \kappa$ is an ordinal, and such that for each odd n, M_n contains the powerset of γ whenever $\gamma \in M_k \cap \kappa$, for $k < n$. Then $\bigcup_{n\in\omega} M_n$ will be as desired. Note that for this construction to work we need κ to be a strong limit, but we also need κ to be regular: if not, we would have $cf\ \kappa \in M$, and therefore $cf\ \kappa \subseteq M$; but then a cofinal subset of κ would be in M, so $M \cap \kappa$ would have to be κ.

Now take N to be another elementary submodel such that $|N| = \kappa$, $X \in M$ and $\kappa \in N$. Then $2^\kappa \in N$ but $2^\kappa \not\subseteq N$. Therefore, X_N is not compact, since (by Lemma 1.2) D^κ is not squashable and is not included in N, yet $(D^\kappa)_N$ is a closed subspace of X_N.

We know that compact scattered spaces are squashable [6]. It is easy to get examples of non-scattered spaces that are squashable, like the previous one, taking perfect pre-images of scattered spaces in the correct way. (A map is *perfect* if it is continuous, closed, and points have compact inverses.) However not all squashable spaces are like that, even assuming there are no large cardinals.

Example 2.2. Let X be the long closed interval of length $\kappa + 1$, $\kappa > 2^{\aleph_0}$. Then X is a connected squashable space – just pick M countably closed such that $|M| < \kappa$. Since X is connected, X cannot be a perfect pre-image of a scattered space.

Problem. Assuming say (B), characterize topologically the class of squashable spaces.

3. κ-scattered and strongly κ-scattered spaces

We shall look at two generalizations of scattered spaces:

Definition 3.1. For $p \in F \subseteq X$, $\chi(p, F)$ is the least cardinality of a neighbourhood base for p in the subspace F. $\chi(X) = \sup\ \{\chi(p, X) : p \in X\}$. $\chi(F, X)$ is the least cardinality of a neighbourhood base about F in X. $\pi\chi(x, X)$ is the least cardinality of a collection of non-empty open sets such that every open set about x includes one. $\pi\chi(X) = \sup\ \{\pi\chi(x, X) : x \in X\}$. Clearly $\pi\chi(X) \leq \chi(X)$.

Definition 3.2. [2]. A space X is *κ-scattered* if for every closed subset F of X, there is a $p \in F$ such that $\chi(p, F) < \kappa$.

Definition 3.3. A space X is *strongly κ-scattered* if for every closed subset F of X, there is a $p \in F$ and an neighbourhood V of p such that $|V \cap F| < \kappa$.

In [6] it is shown that if X is compact and scattered, then X_M is compact. One could hope to generalize this result to κ-scattered, assuming maybe that $\kappa \subseteq M$. But this is consistently not true:

Example 3.4. Let κ be a cardinal and suppose $\kappa^+ < 2^\kappa$. Let θ be a regular cardinal greater than 2^κ, and let M be an elementary submodel of $H(\theta)$ of cardinality κ^+ including $\kappa^+ + 1$. Then $(D^\kappa)_M$ is not compact. Let X be the one-point compactification of κ^+ disjoint copies of D^κ. Then X is κ^+-scattered, but X_M is not compact.

However we can still improve some results in [6].

If X is any topological space and $\leq_X$ is a well-ordering of a basis for X, then the *minimal decomposition* of X according to $\leq_X$ is the sequence $\langle P_\alpha, O_\alpha, X_\alpha : \alpha < \gamma\rangle$ defined as follows. For each $\alpha < \gamma$, let $X_\alpha = X \setminus \bigcup\{O_\beta : \beta < \alpha\}$ (so $X_0 = X$; the construction continues as long as the X_α's are non-empty), let O_α be the $\leq_X$-least member of the basis for X such that $O_\alpha \cap X_\alpha$ is non-empty and has the smallest possible cardinality, and let P_α be $O_\alpha \cap X_\alpha$. If X is compact, this construction must end at a successor stage (and so we write $\langle P_\alpha, O_\alpha, X_\alpha : \alpha \leq \gamma\rangle$).

Theorem 3.5. *Let X be a compact space, let θ be a regular cardinal greater than $2^{|X|}$ and let M be an elementary submodel of $H(\theta)$. Let $\leq_X$ be a well-ordering in M of a basis for X, and let $\langle P_\alpha, O_\alpha, X_\alpha : \alpha \leq \gamma\rangle$ be the minimal decomposition of X according to $\leq_X$. If M is closed under sequences of length $|P_\alpha|$ for each $\alpha \in (\gamma+1)\cap M$, then X_M is compact.*

Proof. Let D be an open cover of X_M, and let γ^* be the least $\alpha \in (\gamma+1)\cap M$ such that $(X_{\gamma^*})_M$ can be covered by a finite subcover D_0 of D. Then $X \setminus \bigcup D_0$ is compact, and D_0 is in M, so γ^* must be a successor ordinal or 0. Towards a contradiction, suppose that $\gamma^* = \eta^*+1$. Now, $X_{\eta^*} \setminus \bigcup D_0$ is compact, and since $X_{\eta^*} \setminus \bigcup D_0 \subseteq P_{\eta^*}$, $X_{\eta^*} \setminus \bigcup D_0 = (X_{\eta^*} \setminus \bigcup D_0)_M$. Therefore, some finite subcover D_1 of D covers $(X_{\eta^*} \setminus \bigcup D_0)_M$. Then $D_0 \cup D_1$ is a finite subcover of D covering $(X_{\eta^*})_M$, giving a contradiction. $\square$

Theorem 3.5 has the following corollary.

Corollary 3.6. *If M is* κ-closed *(i.e. subsets of M of size $\leq \kappa$ are in M) and X is strongly κ^+-scattered and compact, then X_M is compact.*

We next relate characters to squashability. A key concept in Kunen's work [8] is the following:

Definition 3.7. A *λ-Čech-Pospíšil tree in a space X is a tree $\mathcal{K} = \{K_s : s \in {}^{\leq\lambda}2\}$ satisfying:*

i) *Each K_s is non-empty and closed in X;*
ii) *$s \subseteq t$ implies $K_s \supseteq K_t$;*
iii) *$K_{s0} \cap K_{s1} = \emptyset$;*
iv) *if the length of s is γ, a limit ordinal, then $K_s = \bigcap_{\alpha<\gamma} K_{s\restriction\alpha}$.*

Čech and Pospíšil proved (see e. g. [3, 3.16]):

Lemma 3.8. *If X is compact and for each $x \in X$, $\chi(x, X) \geq \lambda$, then there is a λ-Čech-Pospíšil tree in X, and hence $|X| \geq 2^\lambda$.*

We have the following results. We first use a proof from [8] to show:

Lemma 3.9. *If X_M is compact, $\kappa+1 \subseteq M$ and $\chi(x, X) \geq \kappa$, for every $x \in X$, then $2^\kappa \subseteq M$.*

Proof. Let $\{K_s : s \in {}^{\leq\kappa}2\}$ be a κ-Čech-Pospíšil tree in X. By elementarity, we can suppose it is in M. We will prove by induction that $2^\gamma \subseteq M$, for every ordinal $\gamma \leq \kappa$. If γ is a sucessor ordinal, this is immediate. So suppose γ is a limit ordinal. Note that $\gamma \in M$ since $\kappa + 1 \subseteq M$. Fix $s \in 2^\gamma$. By the induction hypothesis, we have $s\restriction\alpha \in M$, for every $\alpha < \gamma$. Thus $K_{s\restriction\alpha} \in M$. Also, by elementarity, $(K_{s\restriction\alpha})_M$ is closed in X_M, for every $\alpha < \gamma$. Since X_M

is compact and $\{(K_{s\restriction\alpha})_M : \alpha < \gamma\}$ is centered, we have that there is an $x \in \bigcap_{\alpha<\gamma}(K_{s\restriction\alpha})_M$. Note that $x \in M$, and therefore

$$s = \bigcup\{t : x \in K_t \text{ and length } (t) < \gamma\} \in M,$$

and we are done. $\square$

By induction, we can now get as far as the first inaccessible, without assuming $\kappa \subseteq M$.

Theorem 3.10. *If X_M is compact, $\kappa \in M$ is less than the first inaccessible, and $\chi(x, X) \geq \kappa$ for every $x \in X$, then $2^\kappa \subseteq M$.*

Proof. Let κ be the least counterexample. By the previous lemma, κ cannot be $\aleph_0$. Also, κ cannot be λ^+, else $\lambda \in M$ and hence $2^\lambda \subseteq M$, so $\kappa \subseteq M$ and we can apply the previous lemma. A similar argument gives us that κ cannot be a limit cardinal that is not a strong limit. Finally, if $cf(\kappa) = \lambda < \kappa$, then $\lambda \in M$ and there is a sequence $\{\lambda_\alpha\}_{\alpha<\lambda}$ of cardinals in M with supremum κ. By the minimality of κ, we have that λ is included in M and that each $2^{\lambda_\alpha} \subseteq M$. Since κ is the supremum of the 2^{λ_α}'s, we have that $\kappa \subseteq M$, so by the previous lemma, $2^\kappa \subseteq M$. $\square$

We suspect that 3.10 can be improved, replacing "inaccessible" by "1-extendible", but we have been unable to prove that.

The following result from [6] will be useful:

Lemma 3.11. *If $\chi(X) \leq \kappa \subseteq M$ and X_M is compact, then $X = X_M$.*

From this we deduce:

Theorem 3.12. *Assume (B). Suppose that a topological space X is squashed by an elementary submodel M such that $|\mathcal{P}(X) \cap M| \in M$. Then there is an $x \in X$ such that $\chi(x, X) < |\mathcal{P}(X) \cap M|$ and there is a $y \in X$ such that $\chi(y, X) > |\mathcal{P}(X) \cap M|$.*

Proof. By (B), $|\mathcal{P}(X) \cap M| = |X \cap M| = o_M$. If $\chi(x, X) \geq o_M$, for every $x \in X$, then by the definition of o_M, $o_M \subseteq M$ and thus by Lemma 3.9, we would have $2^{o_M} \subseteq M$, a contradiction. If $\chi(x, X) \leq o_M$, for every $x \in X$, by Lemma 3.11, we would have that X_M is not compact, also a contradiction. $\square$

Using 3.9, we can also show:

Theorem 3.13. *Assume (B). Suppose that a topological space X is squashed by an elementary submodel M such that $|\mathcal{P}(X) \cap M| \in M$. Then X is $|\mathcal{P}(X) \cap M|$-scattered.*

Proof. Fix X and M as in the statement of the theorem. Let κ denote $|\mathcal{P}(X) \cap M|$. By (B), $o_M = \kappa \subseteq M$. Suppose that X is not κ-scattered. Then there is a closed subset F of X such that $\chi(x, F) \geq \kappa$, for every $x \in F$. By elementarity we can take $F \in M$ and we will also have that F_M is a closed subspace of X_M. Since X_M is compact, F_M will also be compact. Now using Lemma 3.9 for F, we would have $2^\kappa \subseteq M$, a contradiction. $\square$

By Example 2.2, the hypothesis $|\mathcal{P}(X) \cap M| \in M$ cannot be removed in the previous theorems. Since ω is included in every elementary submodel of every $H(\theta)$, the same proof shows the following result from [6]:

Corollary 3.14. *If X is compact and squashed by a countable elementary submodel, then X is scattered.*

The following cardinal functions will be useful now and later:

Definition 3.15. $c(X)$ is the sup of cardinalities of disjoint collections of open sets. $d(X)$ is the least cardinality of a dense subset of X. $\pi w(X)$ is the least cardinality of a collection $\mathcal{P}$ of non-empty open sets such that each non-empty open set includes a member of $\mathcal{P}$. $w(X)$ is the least cardinality of a basis for X. Clearly $c(X) \leq d(X) \leq \pi w(X) \leq w(X)$.

Now we quote the following result from [12]:

Lemma 3.16. *Suppose X_M is compact and either $\chi(X_M) \leq \lambda$ or $d(X_M) \leq \lambda$. If $2^\lambda \subseteq M$, then $X = X_M$.*

We will also need the following result from [4]:

Lemma 3.17. *If X_M is compact, then there is a perfect map from X onto X_M, and hence X is compact.*

In investigating whether or not a compact space is squashable, a natural dichotomy occurs between the κ-scattered and non-κ-scattered cases. We will first consider the non-κ-scattered case.

Theorem 3.18. *Suppose $\kappa \in M$ is less than the first inaccessible cardinal or suppose $\kappa+1 \subseteq M$. Suppose X_M is compact and X (or X_M) is not κ-scattered. If $\chi(X_M) \leq \kappa$ or $d(X_M) \leq \kappa$, then $X_M = X$.*

Proof. We first deal with the case when X is not κ-scattered. Then there is a closed $F \subseteq X$, $F \in M$, such that $\chi(p, F) \geq \kappa$ for every $p \in F$. By 3.10 or by 3.9 , $2^\kappa \subseteq M$, and by 3.16, $X = X_M$.

Now suppose instead that X_M is not κ-scattered. Then by 3.8, there is a κ-Čech-Pospíšil tree in X_M. Pulling back via the perfect map, we get a κ-Čech-Pospíšil tree in X, hence by the proof of 3.9 and 3.10, we again get $2^\kappa \subseteq M$. □

If we assume (B), we do not have to worry about inaccessibility provided our knowledge of X_M's cardinal functions is sharp:

Theorem 3.19. *Assume (B). Suppose X_M is compact and $\chi(X_M) = \kappa$ or $d(X_M) = \kappa$ or $\pi w(X_M) = \kappa$, for some $\kappa \in M$. If either X or X_M is not κ-scattered, then $X = X_M$.*

Proof. Assuming (B), by the previous theorem, we just have to show that $\kappa \subseteq M$. To see this, it suffices to note that since X is compact, $\chi(X) \leq |X|$, so we can apply 1.6, since $|\chi(X) \cap M| \geq \chi(X_M) = \kappa$. A similar argument works for d or for πw. □

The following structure lemma for κ-scattered compact spaces will be used in the next two sections. It slightly strengthens a result of Efimov [2].

Lemma 3.20. *Suppose X is a κ-scattered compact space with $\pi w(X) = \kappa$, $cf(\kappa) = \omega$, $\kappa > \omega$. Then for any increasing sequence of cardinals $\{\kappa_n\}_{n<\omega}$, with $sup_{n<\omega}\, \kappa_n = \kappa$, κ_n regular, there exist regular closed subspaces $\{X_n\}_{n<\omega}$ of X such that:*

(a) $\bigcup\limits_{n<\omega} X_n$ *is dense in* X;

(b) $\{y \in X_n : \chi(y, X) < \kappa_n\}$ *is dense in* X_n;

(c) $\pi w(X) = \sum_{n\in\omega} \pi w(X_n)$.

Proof. Fix an increasing sequence of cardinals $\{\kappa_n : n \in \omega\}$ as in the hypothesis and define

$$E_n = \{x \in X : \chi(x, X) < \kappa_n\}.$$

First note that since X is κ-scattered, then $D = \{x \in X : \chi(x, X) < \kappa\}$ is G_δ-dense in X (see e.g. [12]). We give the proof here for completeness. Let V be a non-empty G_δ set. Then there is a non-empty closed G_δ set $F \subseteq V$. Since X is κ-scattered, there is an $x \in F$ such that $\chi(x, F) < \kappa$. But then

$$\chi(x, X) \leq \chi(x, F) \cdot \chi(F, X) \leq \chi(x, F) \cdot \omega < \kappa.$$

Thus $x \in D \cap V$ and we are done.

Let $F_n = \overline{E_n}$. Since D is G_δ-dense in X, we have that $X = \bigcup_{n\in\omega} F_n$. Indeed, if $X \setminus \bigcup_{n\in\omega} F_n = \bigcap_{n\in\omega} X \setminus F_n \neq \emptyset$, then it would have to intersect D, a contradiction. Define $X_n = \overline{intF_n}$. Note that $X_n \supseteq \overline{intX_n} \supseteq \overline{intF_n} = X_n$, so X_n is a regular closed subspace of X. By the Baire Category Theorem, some F_n, hence all F_n from some n_0 onward have non-empty interiors. It follows that $\bigcup_{n\in\omega} intF_n$ and, a fortiori, $\bigcup_{n\in\omega} X_n$ is dense in X. To see this, suppose there were a non-empty open $V \subseteq X - \bigcup_{n<\omega} intF_n$. Then $V = \bigcup_{n<\omega} F_n \cap V$. Again by Baire Category, for some n, $F_n \cap V$ has non-empty interior in V and hence in X. But $int(F_n \cap V) \subseteq intF_n \subseteq X - V$, contradiction. Also, since π-weight is inherited by and from dense sets, $\pi w(X) = \pi w(\bigcup_{n\in\omega} X_n)$, so $\pi w(X) \leq \sum_{n\in\omega} \pi w(X_n)$. On the other hand, since $\pi w(intF_n) \leq \pi w(X)$, we have $\pi w(X) = \sum_{n\in\omega} \pi w(X_n)$.

Note that since E_n is dense in F_n, $E_n \cap intF_n$ is dense in $intF_n$ and hence in $\overline{intF_n} = X_n$. Thus X_n has a dense subspace of points of character $< \kappa_n$. $\square$

4. The countable chain condition case

In [4], [6], [8], [10], [11], [12] it is shown that if X_M is compact and satisfies various properties stronger than the countable chain condition, then $X = X_M$, if $|X|$ is small or anti-large-cardinal assumptions are made. It is therefore a natural question whether assuming X_M is compact and satisfies the countable chain condition is enough to get that X is not squashable. We will show this question is undecidable. First we have a consistent example.

Example 4.1. "Ψ-space" is any space obtained by taking a maximal almost disjoint family $\mathcal{A}$ of subsets of ω, and putting a topology on $\mathcal{A} \cup \omega$ as follows. Each point in ω is isolated. For each $A \in \mathcal{A}$, a neighbourhood of A is $\{A\}$ union a cofinite piece of A.

Assume $\neg CH$. Then the one-point compactification X of Ψ-space satisfies the countable chain condition and is scattered and compact. Then X_M is compact, but it is different from X if $|M| < |X|$. So X is squashable.

Problem. Is there a consistent example without assuming large cardinals which is not scattered?

On the way to proving a consistent theorem, we first show a partial positive result:

Theorem 4.2. *Assume (B) and $\kappa \in M$. Suppose X_M is compact and satisfies the countable chain condition, and $d(X_M) = \kappa$ or $\chi(X_M) = \kappa$, where $\kappa^\omega = \kappa$. Then $X_M = X$.*

The following lemma of Šapirovskiĭ (see e.g. [3]) will be useful.

Lemma 4.3. $w(X) \leq \pi\chi(X)^{c(X)}$.

We also need the following lemma proved in [6, Theorem 4.9] (there we have the assumption $2^\omega \subseteq M$, but just $\omega_1 \subseteq M$ is used for this fact):

Lemma 4.4. *If $c(X_M) = \aleph_0$ and $\omega_1 \subseteq M$, then $c(X) = \aleph_0$.*

Proof of 4.2. By 1.6 we have $\kappa \subseteq M$. Since $\kappa^\omega = \kappa$, κ is uncountable, so by 4.4, X satisfies the countable chain condition. If X is not κ-scattered we are done by Theorem 3.19, so we can suppose X is κ-scattered. But then as we saw before,

$$D = \{x \in X : \pi\chi(x, X) < \kappa\}$$

is dense in X.

We will have that $w(D) \leq \kappa^\omega = \kappa$, since $\pi\chi(D) \leq \kappa$ and D by density also satisfies the countable chain condition. But then $\pi\chi(X) \leq \pi w(X) = \pi w(D) \leq w(D) \leq \kappa$ (since X is regular), whence by 4.3 again, $w(X) \leq \kappa^\omega = \kappa$.

Since $\kappa \subseteq M$, this implies that $X_M = X$. □

We can now show a consistent positive result. Let *SCH* stand for the *Singular Cardinals Hypothesis*, namely that for every singular cardinal κ, if $2^{cf(\kappa)} < \kappa$, then $\kappa^{cf(\kappa)} = \kappa^+$.

Theorem 4.5. *Assume (B) and $CH + SCH$. Then if X_M is uncountable and compact, $\chi(X_M) \in M$ and X_M satisfies the countable chain condition, then $X_M = X$.*

Proof. Let $\kappa = \chi(X_M)$. The case when $\kappa \leq \aleph_0$ was done in [6] in ZFC, so we may assume $\kappa \geq \aleph_1$. Then, as before, since we are assuming (B), we have $\kappa \subseteq M$. Therefore, by lemma 4.4, X satisfies the countable chain condition.

If X is not κ-scattered, then, again, we are done by Theorem 3.19. If $cf(\kappa) > \omega$, then by $CH + SCH$ we have $\kappa^\omega = \kappa$ (CH takes care of $\kappa = \omega_1 = 2^\omega$ and and SCH of the others), and therefore, by the previous theorem, we are also done.

Suppose then that X is κ-scattered and that $cf(\kappa) = \omega$. The proof uses ideas from [12].

Fix $\{\kappa_n\}_{n\in\omega}$, E_n and X_n as in the proof of Lemma 3.20. It will then suffice to show each $(X_n)_M = X_n$. Note that each X_n satisfies the countable chain condition since they are regular closed subspaces of X. We can also assume that $\kappa_0 > \omega_1$.

Fix $n \in \omega$. To show that $(X_n)_M = X_n$ we can now repeat the same argument done in the proof of 4.2. We know that $E_n = \{x \in X : \chi(x, X) < \kappa_n\}$ is dense in X_n and that $\pi\chi(E_n) \leq \chi(E_n) \leq \kappa_n$. Since X_n satisfies the countable chain condition, so does E_n and therefore $w(E_n) \leq \kappa_n^\omega = \kappa_n$, by $CH + SCH$ (since κ_n is regular and $> \omega_1 = 2^\omega$). But then as before we can conclude that $\pi\chi(X_n) \leq \kappa_n$ and therefore $w(X_n) \leq \kappa_n^\omega = \kappa_n$. Since $\kappa \subseteq M$, we have $\kappa_n \subseteq M$ and we are done by 3.11. □

Problem. Is $CH + SCH$ necessary? Are there $ZFC + CH + SCH$ results below some large cardinal?

Note the hypothesis of X_M being uncountable is essential; otherwise just pick X to be any uncountable compact T_2 scattered space and M any countable elementary submodel. By [6], X_M will be compact. Also it satisfies the countable chain condition, but $X_M \neq X$.

5. X_M's co-absolute with dyadic spaces

Definition 5.1. Two spaces are *co-absolute* if they have isomorphic regular-open algebras. A compact space is *dyadic* if it is the continuous image of some power of the two-point discrete space.

In [2] it is shown that:

Theorem 5.2. *If λ is an infinite cardinal with $cf(\lambda) \geq \omega_1$ and X is a compact space co-absolute with a dyadic one with $\pi w(X) = \lambda$, then X is not λ-scattered.*

We then have:

Theorem 5.3. *Assume (B) or that $\kappa <$ the first inaccessible cardinal and $\kappa \in M$. If X_M is co-absolute with a dyadic compactum with $\pi w(X_M) = \kappa$, where $cf(\kappa) \geq \omega_1$, then $X = X_M$.*

Proof. If X_M is co-absolute with a dyadic compactum, then X_M is compact. By the previous theorem, X_M is not κ-scattered. Since $d(X_M) \leq \pi w(X_M) = \kappa$, by Theorem 3.18 or 3.19 we then have that $X = X_M$. □

To show the general result we will need the following lemmas:

Lemma 5.4. [9]. *If X is co-absolute with a dyadic compactum, then X is co-absolute with either a finite disjoint sum of powers of D or else with the one-point compactification of a countable disjoint sum of powers of D.*

Lemma 5.5. *Each regular closed subspace of a compact space co-absolute with a dyadic compactum is itself co-absolute with a dyadic compactum.*

Proof. Let X be a compact space co-absolute with K, a dyadic compactum. Let Z be a regular closed subspace of X. Let i be an isomorphism between the algebra of regular closed subspaces of X and the algebra of regular closed subspaces of K. Such an isomorphism exists, since the dual regular open algebras are isomorphic. Let $i(Z) = L$. By [1], L is dyadic, and clearly the algebra of regular closed subspaces of Z is isomorphic to the algebra of regular closed subspaces of L. It follows that Z and L are co-absolute. □

Lemma 5.6. [2]. *Suppose X is co-absolute to a dyadic compactum and X has a dense subspace of points of character, i.e. $\chi(p, X)$, less than λ, where λ is an uncountable regular cardinal. Then $\pi w(X) < \lambda$.*

Efimov [2] states this for "δ-character" rather than character, but the former does not exceed the latter.

We are now ready to show our main result:

Theorem 5.7. *Assume (B) or that κ is less than the first inaccessible cardinal. Suppose X_M is compact and $\pi w(X_M) = \kappa \in M$, and X_M is co-absolute with a dyadic compactum. Then $X_M = X$.*

Proof. It follows from Lemma 5.4 that compact spaces co-absolute with dyadic compacta have no isolated points, and thus that if X_M is such a space, then $2^{\aleph_0} \subseteq M$ [6]. Such X_M's satisfy the countable chain condition; since $\omega_1 \subseteq M$, it follows that X does as well.

If X_M is not $\pi w(X_M)$-scattered, we are done by Theorem 3.19, so assume that it is. Then, by Theorem 5.2, $\kappa = \pi w(X_M)$ has countable cofinality. Similarly assume X is $\pi w(X_M)$-scattered. We also may assume that $\pi w(X_M)$ is uncountable, else $X = X_M$ by Lemma 3.16, since $2^{\aleph_0} \subseteq M$.

Applying Lemma 3.20 to X and letting $\{\kappa_n\}_{n<\omega}$ be a strictly increasing sequence of regular cardinals approaching κ, we can obtain a sequence of X_n's. Without loss of generality, assume the sequence as well as the sequence of κ_n's is in M. Then, by elementarity, we can get a sequence of X_n's satisfying the following conditions:

(d) each $(X_n)_M$ is a regular closed subspace of X_M;
(e) $\bigcup_{n<\omega} (X_n)_M$ is dense in X_M;
(f) $\pi w(X_M) = \sum_{n<\omega} \pi w((X_n)_M)$;
(g) $\pi w\big((X_n)_M\big) < \kappa$, for each $n \in \omega$.

To get (g), by Lemma 3.20 and elementarity, we have that for each $n \in \omega$, $\{y \in (X_n)_M : \chi(y, X_M) < \kappa_n\}$ is dense in $(X_n)_M$. Furthermore, note that for $y \in (X_n)_M$, $\chi(y,(X_n)_M) \leq \chi(y, X_n)$. Applying Lemmas 5.5 and 5.6, we see that $\pi w\big((X_n)_M\big) < \kappa_n < \kappa$.

We claim that we can also obtain the X_n's such that each $(X_n)_M$ is not $\pi w\big((X_n)_M\big)$-scattered. Since $\pi w\big((X_n)_M\big) < \kappa$, by 5.2 and 5.5 the claim holds for $\pi w(X_M) = \aleph_\omega$. If the claim fails, there is a counterexample X with $\pi w(X_M)$ minimal. Then none of the X_n's, obtained as above, are counterexamples. Thus for each n we can obtain a sequence $(Y_{nk})_{k\in\omega}$ satisfying all the conditions we want: for n such that X_n is already not $\pi w\big((X_n)_M\big)$-scattered, we just take $Y_{nk} = X_n$ for each k; for n such that X_n is $\pi w\big((X_n)_M\big)$-scattered, we apply Lemma 3.20 and our induction hypothesis. But then the set of all Y_{nk}'s for all X_n's is the desired countable collection of subspaces of X.

Now, since $(X_n)_M$ is not $\pi w\big((X_n)_M\big)$-scattered, for any $n \in \omega$, by the proofs of Theorem 3.18 and Theorem 3.19, we conclude that $2^{<\kappa} \subseteq M$, and so $\kappa \subseteq M$. Consider two sub-cases: κ is or is not a strong limit. In the former sub-case, the proof of Theorem 4.5 works without (B) and $CH + SCH$ to get that $X = X_M$. In the latter sub-case, there is a regular $\mu < \kappa$, such that $2^\mu = 2^\kappa$. Thus $2^\kappa \subseteq M$. Then by Lemma 3.16, $X_M = X$. □

6. Boolean algebras

One might expect that our topological results should have some implications for Boolean algebras, and they do. In [6] the following result was proved, where for a Boolean algebra A, "$S(A)$" denotes the Stone space of A:

Theorem 6.1. *Assume $0^\#$ does not exist. Let A be a Boolean algebra. If $A \cap M$ is complete and $(S(A))_M$ is compact, then $A = A \cap M$.*

Using our new topological results, we obtain:

Theorem 6.2. *Suppose (B) and that A is a Boolean algebra such that $(S(A))_M$ is compact. If the completion of $A \cap M$ is isomorphic to the completion of some Boolean algebra C homomorphically embedded in the clopen algebra K of some D^κ, then $A = A \cap M$. The conclusion holds in ZFC for A's with $|A|$ less than the first inaccessible.*

Proof. In [6] it was shown that if $(S(A))_M$ is compact, then $(S(A))_M = S(A \cap M)$. To say that C is homomorphically embedded in K says that $S(C)$ is a continuous image of D^κ. We are then assuming that $(S(A))_M$ is co-absolute with $S(C)$, and that the latter is dyadic. Therefore $(S(A))_M = S(A)$. But then $A = A \cap M$. $\square$

Corollary 6.3. *Suppose (B) and that A is a Boolean algebra such that $S(A)_M$ is compact. If the completion of $A \cap M$ is isomorphic to the algebra for adding κ many Cohen reals, then $A = A \cap M$. The conclusion holds in ZFC for A's with $|A|$ less than the first inaccessible.*

Assuming (B) and $CH + SCH$, we get a stronger result:

Theorem 6.4. *Suppose (B) and $CH + SCH$ and that C is a Boolean algebra such that $S(C)_M$ is compact and satisfies the countable chain condition. Then $C = C \cap M$.*

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