Discontinuous homomorphisms, selectors and automorphisms of the complex field (in ZF) *

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June 2, 2020

Abstract

We show, in Zermelo-Fraenkel set theory without the Axiom of Choice, that the existence of a discontinuous homomorphism between separable Banach spaces induces a selector for the Vitali equivalence relation $\mathbb{R}/\mathbb{Q}$. In conjunction with a result of Di Prisco and Todorcevic, this shows that a nonprincipal ultrafilter on the integers is not sufficient to construct a discontinuous automorphism of the complex field, confirming a conjecture of Simon Thomas. This is an improved version of [11], which used a weak version of the Axiom of Choice for the same result.

Assuming the Zermelo-Fraenkel axioms for set theory (ZF), the Axiom of Choice (AC) implies that every vector space has a basis (in fact the two statements are equivalent over ZF [1]). The existence of a basis for the vector space $\mathbb{R}$ over the field of scalars $\mathbb{Q}$ in turn implies, in ZF, the existence of a selector for the Vitali equivalence relation $\mathbb{R}/\mathbb{Q}$ (the equivalence relation on $\mathbb{R}$ defined by the formula $x - y \in \mathbb{Q}$) and the existence of a discontinuous homomorphism from the group $(\mathbb{R}, +)$ to itself (see [8, 9, 6], for instance). We show that the existence of a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself implies the existence of a selector for $\mathbb{R}/\mathbb{Q}$. We do this without using the axiom $\mathsf{CC}_{\mathbb{R}}$, which asserts the existence of Choice function for each countable set of subsets of $\mathbb{R}$, which we did use to prove the same result in [11]. Our result applies to the additive group of any separable Banach space in place of $(\mathbb{R}, +)$.

A *selector* for an equivalence relation E on a set X is a subset of X meeting each E-equivalence class in exactly one point. The classical construction of a nonmeasurable Vitali set begins by using AC to find a selector for $\mathbb{R}/\mathbb{Q}$. Instead of $\mathbb{R}/\mathbb{Q}$ however we will work with the equivalence relation E_0 of mod-finite equivalence for subsets of ω; our introduction of $\mathbb{R}/\mathbb{Q}$ is only for the expository

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benefit of readers who are less familiar with E_0. The equivalence relations $\mathbb{R}/\mathbb{Q}$ and $\mathcal{P}(\omega)/E_0$ are both hyperfinite and nonsmooth, so Borel bi-embeddable (see [4]), which implies among other things that the existence of a selector for either of these equivalence relations implies the existence of one for the other.

The result in this paper confirms a conjecture of Simon Thomas saying that the existence of a nonprincipal ultrafilter on the integers is consistent with the nonexistence of a discontinuous automorphism of the complex field. We briefly give some background information connecting our result to his conjecture. Let P be the set of primes, and for each $p \in P$ let $\bar{\mathbb{F}}_p$ be the algebraic closure of the field $\mathbb{F}_p$ of size p. Given a nonprincipal ultrafilter U on P, the U-ultraproduct $\prod_U \bar{\mathbb{F}}_p$ is an algebraically closed field of characteristic 0 and cardinality $2^{\aleph_0}$. It follows that if AC holds (or just if there is a wellordering of $\mathcal{P}(\omega)$) this ultraproduct is isomorphic to the complex field $(\mathbb{C}, +, \cdot)$ (see [2], for instance). Even without AC, this ultraproduct has $2^{\aleph_0}$ many automorphisms induced by the powers of the Frobenius automorphisms of the fields $\bar{\mathbb{F}}_p$ (see [5, 13]).

Di Prisco and Todorcevic proved in [3] that a certain strong Ramsey principle for countable products of finite sets holds in Solovay's model $L(\mathbb{R})$ from [15]. This principle has implications for forcing extensions of $L(\mathbb{R})$ via the partial order $\mathcal{P}(\omega)/\mathrm{Fin}$ (such an extension has the form $L(\mathbb{R})[U]$, where U is a nonprincipal ultrafilter on ω). For instance [3], it implies that there is no E_0-selector in this model. Thomas observed that this Ramsey principle also precludes the existence of an injection from $\prod_U \bar{\mathbb{F}}_p$ into $\mathbb{C}$ in $L(\mathbb{R})[U]$. He then conjectured that there are no discontinuous automorphisms of $(\mathbb{C}, +, \cdot)$ in this model, i.e., that the only automorphisms are the identity function and complex conjugation. Our result confirms this conjecture, as the restriction of such an automorphism to $(\mathbb{C}, +)$ would be a discontinuous homomorphism. We state this formally in Theorem 0.2 and Corollary 0.5 below. We note that $\mathsf{CC}_\mathbb{R}$ holds in $L(\mathbb{R})[U]$, as it is an inner model of a model AC with the same set of real numbers.

Let us say that an abelian topological group $(G, +)$ is *suitable* if there is an invariant metric d inducing the topology on G such that

- G is complete with respect to d;

- letting $\mathbf{0}$ be the identity element of G, $d(\mathbf{0}, n \cdot x) = n \cdot d(\mathbf{0}, x)$ holds for all $x \in G$ and $n \in \omega$ (where $n \cdot x$ denotes the result of adding x to itself n times).

The additive group of a Banach space is suitable, under the metric given by the norm. Moreover, Theorem 1.2 of [14] shows that a group is suitable if and only if it is isomorphic to closed subset of real Banach space under its addition operation. Note that the second condition above implies that a bounded metric cannot witness suitability. When working with a fixed suitable group $(G, +)$ and a witnessing metric d_G, we will write $\mathbf{0}_G$ for the identity element of G, $|x|_G$ for $d_G(\mathbf{0}_G, x)$ and $B_G(x, \epsilon)$ for $\{y \in G : d(x, y) < \epsilon\}$.

Lemma 0.1 (ZF). *Suppose that $(G, +)$ and $(K, +)$ are suitable topological groups, and that $h \colon (G, +) \to (K, +)$ is a homomorphism. If there exists a*

convergent sequence $\langle x_i : i \in \omega \rangle$ in G such that $\langle h(x_i) : i \in \omega \rangle$ does not converge to $h(\lim_{n \in \omega} x_i)$, then there is a selector for E_0.

Proof. Let d_G and d_K be metrics witnessing the respective suitability of $(G, +)$ and $(K, +)$, and let h and $\langle x_i : i \in \omega \rangle$ be as in the statement of the theorem. By the invariance of d_G and d_K, it suffices to consider the case where $\langle x_i : i \in \omega \rangle$ converges to $\mathbf{0}_G$. Since $h(\mathbf{0}_G) = \mathbf{0}_K$, we have that $\langle h(x_i) :\in \omega \rangle$ does not converge to $\mathbf{0}_K$, which means that for some $\epsilon > 0$ the set of $i \in \omega$ with $|h(x_i)|_K \geq \epsilon$ is infinite.

We may now find a sequence $\langle y_i : i \in \omega \rangle$ of elements of G such that

1. for each $i \in \omega$ there exist $k \in \omega$ and $n \in \omega \setminus \{0\}$ such that $y_i = n \cdot x_k$;

2. for all $i < j$ in ω, $|y_j|_G < |y_i|_G/3$;

3. for all $i \in \omega$, $|h(y_i)|_K > i + \sum_{j<i} |h(y_j)|_K$.

To see this, let y_0 be any element of $\{x_i : i \in \omega\} \setminus \{\mathbf{0}_G\}$. Given $j \in \omega$ and $\{y_i : i \leq j\}$, let $n \in \omega \setminus \{0\}$ be such that

$$n \cdot \epsilon > (j+1) + \sum_{i < j+1} |h(y_i)|_K.$$

There exists then a $k \in \omega$ such that $|x_k|_G < |y_i|_G/3n$ for all $i \leq j$ and such that $|h(x_k)|_K \geq \epsilon$. Then $y_{j+1} = n \cdot x_k$ is as desired.

Condition (2) on $\langle y_i : i \in \omega \rangle$ implies that each value $|y_i|_G$ is more than $\sum\{|y_j|_G : j > i\}$. This in turn, along with the completeness of G, implies that $\sum_{i \in A} y_i$ converges for each $A \subseteq \omega$. Let $Y = \{y_i : i \in \omega\}$ and let Y^+ be the set of elements of G which are sums of (finite or infinite) subsets of Y. By condition (2) on Y, each $y \in Y^+$ is equal to $\sum\{y_i : i \in S_y\}$ for a unique subset S_y of ω. Let F be the equivalence relation on Y^+ where $y_0 F y_1$ if and only if S_{y_0} and S_{y_1} have finite symmetric difference (i.e., $S_{y_0} E_0 S_{y_1}$). By condition (3) on $\langle y_i : i < \omega \rangle$, if yFy' and i is the maximum point of disagreement between S_y and $S_{y'}$, then $d_K(h(y), h(y')) > i$. It follows that the h-preimage of each set of the form $B_K(\mathbf{0}_K, M)$ (for $M \in \mathbb{R}^+$) intersects each F-equivalence class in only finitely many points (since if $2M \leq i$, then for every y in this intersection the set $S_y \setminus i$ is the same). It follows from this (and the fact that there is a Borel linear order $<$ on Y^+ induced by the natural lexicographic order on $\mathcal{P}(\omega)$) that there is an F-selector : for each F-equivalence class, let $M \in \mathbb{Z}^+$ be minimal so that the h-preimage of $B_K(\mathbf{0}_K, M)$ intersects the class, and then pick the $<$-least element of this intersection. Since Y^+/F is isomorphic to $\mathcal{P}(\omega)/E_0$ via the map $y \mapsto S_y$, there is then an E_0-selector. $\square$

Theorems 0.2 and 0.4 are each applications of Lemma 0.1. A *choice function* for a set A is a function c with domain $A \setminus \{0\}$ such that $c(a) \in a$ for all $a \in A \setminus \{0\}$. If D is dense subset of G, a choice function for the powerset of D can be used to find convergent sequences from D. Recall that choice functions exist for the powerset of any wellorderable set, and therefore the powerset of any countable set, so the following theorem includes the case where G is separable.

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Theorem 0.2 (ZF). *Suppose that $(G, +)$ is a suitable group, D is a dense subset of G, and that there exists a choice function for the powerset of D. If there is a discontinuous homomorphism of $(G, +)$ to itself, then there is a selector for E_0.*

Proof. Let h be a discontinuous homomorphism from a suitable group $(G, +)$ to itself. Let D be a wellorderable dense subset of G. Since h is discontinuous, there exist a $x \in G$ and a sequence $\langle x_n : n \in \omega \rangle$ in D such that $\langle x_n : n \in \omega \rangle$ converges to x but $\langle h(x_n) : n \in \omega \rangle$ does not converge to $h(x)$. Now we may apply Theorem 0.1. $\qquad\square$

Rephrasing in terms of Banach spaces gives the following.

Corollary 0.3 (ZF). *If there is a discontinuous homomorphism between separable Banach spaces then there is a selector for E_0.*

Lemma 0.1 and Theorem 0.2 do not require the Axiom of Choice, but in general it may require some form of Choice to find a sequence $\langle x_i : i < \omega \rangle$ as in the statement of Lemma 0.1, given a discontinuous homomorphism on a suitable group. Theorem 0.4 applies to the case of (possibly nonseparable) groups of cardinality continuum.

Theorem 0.4 (ZF + $CC_{\mathbb{R}}$). *If there is a discontinuous homomorphism between suitable groups of cardinality $2^{\aleph_0}$ then there is a selector for E_0.*

Proof. Let $(G, +)$ and $(K, +)$ be suitable groups of cardinality $2^{\aleph_0}$, and let h be a discontinuous homomorphism from $(G, +)$ to $(K, +)$. Let d_G and d_K be metrics on G and K witnessing suitability. Since h is discontinuous, and d_G and d_K are invariant, there exists an $\epsilon > 0$ such that for each $\delta > 0$ there exists an $x \in B_G(\mathbf{0}_G, \delta)$ with $h(x) \notin B_K(\mathbf{0}_K, \epsilon)$. For each $i \in \omega$, let X_i be the set of $x \in B_G(\mathbf{0}_G, 1/(i+1))$ such that $h(x) \notin B_K(\mathbf{0}_K, \epsilon)$. Then each X_i is nonempty, and by $CC_{\mathbb{R}}$ there is a sequence $\langle x_i : i \in \omega \rangle$ with each x_i in the corresponding X_i. Now we may apply Lemma 0.1. $\qquad\square$

Combined with the results of Di Prisco and Todorcevic cited above, we have the following corollary, which says that the assumption of the existence of a nonprincipal ultrafilter on the integers is not sufficient to define a third automorphism of the complex field. The strongly inaccessible cardinal in the hypothesis (which we conjecture to be unnecessary) comes from the construction of the model $L(\mathbb{R})$ in [15].

Corollary 0.5. *If the theory ZF is consistent with the existence of a strongly inaccessible cardinal, then it is also consistent with the conjunction of the following three statements:*

- $CC_{\mathbb{R}}$ *holds;*

- *there is a nonprincipal ultrafilter on ω;*

- *there are exactly two automorphisms of the complex field.*

This paper is part of the project outlined in [10, 12], which studies fragments of the Axiom of Choice holding in certain generic extensions of models of the Axiom of Determinacy. The following are shown in [12], relative to the consistency of the existence of a strongly inaccessible cardinal.

1. The existence of an E_0 selector does not imply the existence of a discontinuous homomorphism on $(\mathbb{R}, +)$.

2. If we drop the second condition from the definition of suitability, Theorem 0.2 no longer holds. In particular, letting $(G, +)$ be the group induced by addition modulo 1 on the interval $[0, 1)$, the existence of a discontinuous homomorphism of $(G, +)$ does not imply the existence of an E_0-selector.

Our proof of Theorem 0.1 was discovered by adapting arguments from [10], with additional inspiration from [7].

We end with some related questions. The intended context for each question is the theory $\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$, although the versions for other forms of AC may be interesting.

1. Does the existence of a discontinuous homomorphism on $(\mathbb{R}, +)$ imply the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$?

2. Does the existence of a discontinuous homomorphism of $(\mathbb{R}, +)$ imply the existence of a discontinuous automorphism of $(\mathbb{C}, +, \cdot)$?

We thank Paul McKenney for reminding us of Question (1).

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