# An introduction to $\mathbb{P}_{\max}$ forcing

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### 0.1 Pre-introduction

These notes are an account of a six-hour lecture series I presented at Carnegie Mellon University on September 9, 2006, as the inaugural meeting of the Appalachian Set Theory series. My remarks (occasionally improvised) were faithfully transcribed by Peter LeFanu Lumsdaine and Yimu Yin, who then presented me with a rough draft of this article. Their account aimed to capture the feel of the discussions (including some direct quotations), and I've tried to preserve that as much as possible.

## 1 Introduction

The forcing construction $\mathbb{P}_{\max}$ was invented by W. Hugh Woodin in the early 1990's in the wake of his result that the saturation of the nonstationary ideal on $\omega_1$ ($\mathrm{NS}_{\omega_1}$) plus the existence of a measurable cardinal implies that there is a definable counterexample to the Continuum Hypothesis (in particular, it implies that $\boldsymbol{\delta}^1_2 = \omega_2$, which implies $\neg$CH). These notes outline a proof of the $\Pi_2$ maximality of the $\mathbb{P}_{\max}$ extension, which we can state as follows.

**Theorem 1** ([9]). *Suppose that there exist proper class many Woodin cardinals, $A \subseteq \mathbb{R}$, $A \in L(\mathbb{R})$, $\varphi$ is $\Pi_2$ in the extended language containing two additional unary predicates, and in some set forcing extension*

$$\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}, A^* \rangle \models \varphi$$

*(where $A^*$ is the reinterpretation of $A$ in this extension). Then*

$$L(\mathbb{R})^{\mathbb{P}_{\max}} \models [\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}, A \rangle \models \varphi] .$$

Forcing with $\mathbb{P}_{\max}$ does not add reals, so there is no need to reinterpret $A$ in the last line of the theorem. The theorem says that any such $\Pi_2$ statement that we can force in any extension must hold in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$,

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so $H(\omega_2)$ of $L(\mathbb{R})^{\mathbb{P}_{\max}}$ is maximal or complete, in a certain sense; among other things, it models ZFC, Martin's Axiom, certain fragments of Martin's Maximum [1], and the negation of the Continuum Hypothesis. The reinterpretation $A^*$ will be defined later, in terms of tree representations for sets of reals. We will not give the definition of Woodin cardinals (but see [4]).

We have reworked the standard proof of Theorem 1 in order to minimize the prerequisites. In particular, the need for (mentioning) sharps has been eliminated. However, they and much more will need to be reintroduced to go any further than the material presented here.

"Woodin's book on $\mathbb{P}_{\max}$, *The axiom of determinacy, forcing axioms and the nonstationary ideal* [9] runs to around 1000 pages. My article for the Handbook of Set Theory [5], introducing $\mathbb{P}_{\max}$, has about 65. The advance notes for these lectures are about 30 pages, and previous lecture courses have taken about 12-15 hours to cover $\mathbb{P}_{\max}$; so today will, of course, have to be brief..."

## 2 Setup: iterations and the definition of $\mathbb{P}_{\max}$

Suppose that $M \models$ ZFC and $I \in M$ a normal ideal on $\omega_1^M$ in $M$. Force over $M$ with $((\mathcal{P}(\omega_1) \setminus I)/I)^M$. The resulting generic $G$ is now an $M$-normal ultrafilter on ${\omega_1}^M$; so we may form the corresponding ultrapower and elementary embedding $j : M \to \mathit{Ult}(M, G) := \{f : {\omega_1}^M \to M \mid f \in M\}/=_G$. ("We'll use this a thousand times today.") Note that $\mathrm{crit}(j) = {\omega_1}^M$, $j({\omega_1}^M) \geq {\omega_2}^M$, $\mathrm{Ord}^{\mathit{Ult}(M,G)} = \mathrm{Ord}^M$, and for $A \in \mathcal{P}(\omega_1)^M$, $A \in G \leftrightarrow {\omega_1}^M \in j(A)$. There is no need to assume that $A$ is transitive, though it will be in the cases were are interested in. When an ultrapower is well-founded, we identify it with its transitive collapse.

**Definition 2.** *$I$ is* precipitous *if $\mathit{Ult}(M, G)$ thus constructed is well-founded from the point of view of $M[G]$, for all $M$-generic $G$. (N.B. this is definable in $M$ via forcing.)*

We need a hierarchy of theories satisfying the following conditions.

- $T_0$, a theory consistent with ZFC and strong enough to make sense of the generic ultrapower construction above and prove that $j : M \to \mathit{Ult}(M, G)$ is elementary.
- $T_1$, a theory consistent with ZFC and at least as strong as $T_0$ + "every set lies in some $H(\kappa) \models T_0$."
- $T_2$, a theory consistent with ZFC and at least as strong as $T_1$ + "every set lies in some $H(\kappa) \models T_1$."

In [5], we take $\mathrm{T}_0$ (which we call ZFC°) to be ZFC - Replacement - Powerset plus "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the scheme saying that definable trees of height $\omega_1$ have maximal branches. Then we let $T_1 = T_0$ + Powerset + Choice + $\Sigma_1$-Replacement and $T_2 = ZFC$ (though we don't express it in these terms). In [9], Woodin has an even weaker fragment of ZFC (which he calls ZFC$^*$) playing the role of $T_0$. Today we may as well let $T_0$ be ZFC, $T_1$ be ZFC plus the existence of a proper class of strongly inaccessible cardinals and $T_2$ be ZFC plus the existence of a proper class of Woodin cardinals. From now on we will just use the terms $T_0$, $T_1$ and $T_2$.

Our basic construction is the generic ultrapower. We now extend to iterated ultrapowers. Suppose we have $(M_0, I_0)$, $G_0 \subseteq (\mathcal{P}(\omega_1)/I_0)^{M_0}$, $j_0 : (M_0, I_0) \to \mathit{Ult}(M_0, G_0)$, all as before; let $M_1 = \mathit{Ult}(M_0, G_0)$, $I_1 = j_0(I_0)$. Now we can take the generic ultrapower of $M_1$ by $I_1$, and iterate. At limit stages, we have a directed system of elementary embeddings, so can just take the direct limit, so we can keep going up to length $\omega_1$. (No further, as if we force again there, we collapse ${\omega_1}^V$, so are back to countable length!) Note that the final model of the iteration, $M_{\omega_1}$, is an element of $H(\omega_2)$.

**Definition 3.** *An* iteration *of $(M, I)$ (as above; $M$ countable) of length $\gamma$ consists of $M_\alpha$, $I_\alpha$ $(\alpha \leq \gamma)$, $G_\eta$ $(\eta < \gamma)$, and $j_{\alpha,\beta}$ $(\alpha \leq \beta \leq \gamma)$, satisfying*

- $M_0 = M$, $I_0 = I$
- *$G_\eta$ is $M_\eta$-generic for $(P(\omega_1)/I_\eta)^{M_\eta}$*
- *$j_{\eta,\eta+1}$ is the canonical embedding of $M_\eta$ into $\mathit{Ult}(M_\eta, G_\eta) = M_{\eta+1}$*
- *$j_{\alpha,\beta} : M_\alpha \to M_\beta$ are a commuting family of elementary embeddings*
- $I_\beta = j_{0,\beta}(I)$
- *For limit $\beta$, $M_\beta$ is the direct limit of $\{M_\alpha \mid \alpha < \beta\}$ under the embeddings $j_{\alpha,\eta}$ $(\alpha \leq \eta < \beta)$.*

In practice, we almost always have $\gamma = {\omega_1}^N$ for some larger $N \supseteq M$. We will generally write $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma \rangle$ for the iteration, or just "$j$ is an iteration" to mean that $j$ is the $j_{0,\gamma}$ of an iteration, with $j(M)$ for $M_\gamma$. (As we will see, in some circumstances, if we know $M_0$, $M_\gamma$, $j_{0,\gamma}$, we can (with slight assumptions) recover the full iteration.) We say that the $M_\alpha$'s are *iterates* of $(M, I)$; $(M, I)$ is *iterable* if all iterates are well-founded; and $(M, I)$ is an *iterable pair* if $M$ is a countable transitive model of $T_0$, $I$ a normal ideal on $\mathcal{P}(\omega_1)$ in $M$, and $(M, I)$ is iterable.

If $M$ is well-founded and $M \models$ "$I$ is precipitous," then certainly $(M, I)$ is finitely iterable (i.e., its finite-length iterations produce wellfounded models); and in fact, we will show that in this case $(M, I)$ is iterable to any $\alpha \in \mathrm{Ord}^M$.

The proof of the following lemma is left an exercise (the proof is by induction on the length of the iteration). In a typical application, $M$ is $H(\kappa)^N$ for some suitable $\kappa$.

**Lemma 4.** *Suppose that $M \in N$ are models of $T_0$, $M$ is closed under $\omega_1$-sequences from $N$, and $\mathcal{P}(\mathcal{P}(\omega_1))^M = \mathcal{P}(\mathcal{P}(\omega_1))^N$. Let $I \in M$ be an $M$-normal ideal on ${\omega_1}^M$. Then the following hold.*

- *for each iteration* $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* $(M, I)$ *there is a unique iteration* $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* $(N, I)$ *such that* $\forall\beta \leq \gamma$, $j^*_{0,\beta}(M) = M_\beta$, $M_\beta$ *is closed under* $\omega_1$*-sequences from* $N_\beta$, $\mathcal{P}(\mathcal{P}(\omega_1))^{M_\beta} = \mathcal{P}(\mathcal{P}(\omega_1))^{N_\beta}$, *and* $j^*_{\alpha,\beta}\restriction_{M_\alpha} = j_{\alpha,\beta}$.
- *for each iteration* $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* $(N, I)$ *there is a unique iteration* $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma\rangle$ *of* $(M, I)$ *such that* $\forall\beta \leq \gamma$, $j^*_{0,\beta}(M) = M_\beta$, $M_\beta$ *is closed under* $\omega_1$*-sequences from* $N_\beta$, $\mathcal{P}(\mathcal{P}(\omega_1))^{M_\beta} = \mathcal{P}(\mathcal{P}(\omega_1))^{N_\beta}$, *and* $j^*_{\alpha,\beta}\restriction_{M_\alpha} = j_{\alpha,\beta}$.

**Corollary 5.** *In the context of Lemma 4, if* $(M, I)$ *has an ill-founded iterate by an iteration of length* $\alpha$*, then so does* $(N, I)$*.*

Lemma 6 below then shows that $(M, I)$ is iterable if $N$ contains $\omega_1$ (recall that iterations can have length at most $\omega_1$, and note that an illfounded iteration of length $\omega_1$ must be illfounded at some countable stage).

First we fix a coding of elements of $H(\omega_1)$ by reals. Fix a recursive bijection $\pi : \omega \times \omega \to \omega$, and say $X \subseteq \omega$ *codes* $a \in H(\omega_1)$ if

$$(\mathrm{tc}(\{a\}), \in) \cong (\omega, \{(i, j) \mid \pi(i, j) \in X\},$$

where $tc(b)$ denotes the transitive closure of $b$. Then $\in$ and $=$ are $\Sigma^1_1$ (as permutations of $\omega$ induce different codes for the same object).

**Lemma 6.** *Suppose that* $N$ *is a transitive model of* $T_1$, $\gamma \in \mathrm{Ord}^N$*, and* $I$ *is a normal precipitous ideal on* ${\omega_1}^N$ *in* $N$*. Then any iterate of* $(N, I)$ *by an iteration of length* $\gamma$ *is well-founded.*

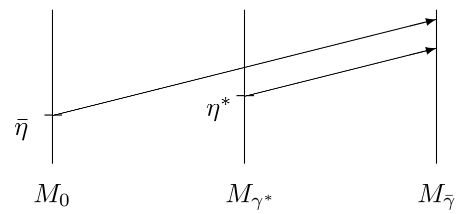
*Proof.* It suffices to prove that iterations of the form $(H(\kappa)^N, I)$ produce well-founded models for all $\kappa \in N$ such that $H(\kappa)^N \models T_0$; for if any iterate of $N$ is ill-founded, then some ordinal in $N$ is large enough to witness this (i.e. $\sup(\mathrm{rge}(f))$, where $f$ witnesses ill-foundedness) and by assumption (as $N \models T_1$) this is contained in some $H(\kappa)^N$ that models $T_0$.

Let $(\bar\gamma, \bar\kappa, \bar\eta)$ be the lexicographically minimal triple $(\gamma, \kappa, \eta)$ satisfying (with $N$) the formula $\varphi(N, \gamma, \kappa, \eta)$ defined by "$H(\kappa)^N \models T_0$ and there exists an iteration of $(H(\kappa)^N, I)$ of length $\gamma$ which is ill-founded below the image of $\eta$".

Using our fixed coding of elements of $H(\omega_1)$ by reals there is a $\Sigma^1_1$ formula $\varphi'(x, y, z)$ saying "$x$ codes a model of $T_0$ and a normal ideal in the model on the $\omega_1$ of the model and there exists an iteration of this pair whose length is coded by $y$ and which is illfounded below the image of the element of this model coded by $z$."

For all cardinals $\kappa, \rho \in N$ and all ordinals $\gamma, \eta \in N$, if $\rho \in N$ is larger than $|H(\kappa)|^N$, $|\eta|^N$ and $|\gamma|^N$, then there exist reals coding $H(\kappa)^N$, $\eta$, and $\gamma$ in any forcing extension of $N$ by $\mathrm{Coll}(\omega, \rho)$. Such an extension is correct about whether these reals satisfy $\varphi'$. However, this is a homogeneous forcing extension of $N$; so there is a formula $\psi(\gamma, \kappa, \eta)$ saying that in every forcing extension in which $H(\kappa)$ (of the ground model), $\eta$ and $\gamma$ are all countable there exist reals coding $H(\kappa)$, $\eta$ and $\gamma$ which satisfy $\varphi'$. It follows that that $N \models \psi(\gamma, \kappa, \eta)$ if and only if $\varphi(N,\gamma,\kappa,\eta)$ holds, and furthermore, for all well-founded iterates $N^*$ of $N$, and all $\gamma,\kappa,\eta\in N^*$, $N^*\models\psi(\gamma,\kappa,\eta)$ if and only if $\varphi(N^*,\gamma,\kappa,\eta)$ holds.

Since $I$ is precipitous in $N$, $\bar\gamma$ is a limit ordinal, and clearly $\bar\eta$ is a limit ordinal as well. Fix an iteration $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta<\alpha\le\delta\le\bar\gamma\rangle$ of $(H(\bar\kappa)^N, I)$ such that $j_{0\bar\gamma}(\bar\eta)$ is not wellfounded, and let $\langle N_\alpha, G_\beta, j'_{\alpha\delta} : \beta<\alpha\le\delta\le\bar\gamma\rangle$ be the corresponding iteration of $N$ as in Lemma 4. By the minimality of $\bar\gamma$ we have that $N_\alpha$ is wellfounded for all $\alpha<\bar\gamma$. Since $M_{\bar\gamma}$ is the direct limit of the iteration leading up to it, we may fix $\gamma^*<\bar\gamma$ and $\eta^*<j_{0\gamma^*}(\bar\eta)$ such that $j_{\gamma^*\bar\gamma}(\eta^*)$ is not wellfounded. By Lemma 4, $j'_{\gamma^*,\bar\gamma}(\eta^*)=j_{\gamma^*,\bar\gamma}(\eta^*)$ and $j'_{\gamma^*,\bar\gamma}(\bar\eta)=j_{\gamma^*,\bar\gamma}(\bar\eta)$.




But now, $N_{\gamma^*}\models\psi(\bar\gamma-\gamma^*, j_{0,\gamma^*}(\bar\kappa),\eta^*)$, $\bar\gamma-\gamma^*\le\bar\gamma$, and $\eta^*<j_{0,\gamma^*}(\bar\eta)$, contradicting minimality of $(j_{0,\gamma^*}(\bar\gamma), j_{0,\gamma^*}((\bar\kappa), j_{0,\gamma^*}(\bar\eta))$ in $N_{\gamma^*}$. □

We note that ZFC does not imply the existence of iterable pairs. However, by Lemma 6, if there exist a normal, precipitous ideal $J$ on $\omega_1$, and a measurable cardinal $\kappa$ with a $\kappa$ complete ultrafilter $\mu$, then there exist iterable pairs. The main point here is that if $\theta>\kappa$ is a regular cardinal and $X$ is a countable elementary submodel of $H(\theta)$ with $\kappa, J\in X$, then $X$ can be end-extended below $\kappa$ by taking $\gamma$ to be the least member of $\bigcap(X\cap\mu)$, and letting $X[\gamma]$ be the set of values $f(\gamma)$ for all functions $f$ in $X$ with domain $\kappa$. Applying this fact $\omega_1$ many times, we get that the transitive collapse $M$ of $X\cap V_\kappa$ is a countable model which is a rank initial segment of a model containing $\omega_1$. Letting $I$ be the image of $J$ under the transitive collapse, then, $(M,I)$ is an iterable pair. This is a special case of the proof of Lemma 22, and a key point in Woodin's proof (which appears in Chapter 3 of [9]) that if there exists a measurable cardinal and the nonstationary ideal on $\omega_1$ is saturated, then CH fails.

If there is a precipitous ideal on $\omega_1$, then sharps exist for subsets of $\omega_1$, and a countable iterable model will be correct about these sharps. We will work around this today to avoid having to talk about sharps.

**Lemma 7.** *If $(M,I)$ is an iterable pair and $A$ is an element of $\mathcal{P}(\omega_1)^M$, then $(\omega_1^{L[A]})^M=\omega_1^{L[A]}$.*

*Proof.* Let $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta}\mid \alpha\le\beta\le\omega_1,\ \eta<\omega_1\rangle$ be an iteration of $(M,I)$. The ordinals of $M$ and $M_1$ are the same, so $L[A]^M=L[A]^{M_1}$. The critical point of $j_{1\omega_1}$ is greater than $\omega_1^M$, and thus greater than the $\omega_1$ of $L[A]^M$. The restriction of $j_{1\omega_1}$ to $L[A]^M$ embeds $L[A]^M$ elementarily into $L[A]^{M_{\omega_1}}$, which means that $L[A]^M$ and $L[A]^{M_{\omega_1}}$ have the same $\omega_1$. Since $\omega_1\subset M_{\omega_1}$, $(\omega_1^{L[A]})^{M_{\omega_1}}=\omega_1^{L[A]}$. □

Now we can define $\mathbb{P}_{\max}$.

**Definition 8.** *The partial order $\mathbb{P}_{\max}$ is the set of pairs $\langle (M,I), a\rangle$ such that*

1. *$M$ is a countable transitive model of $T_0$ + $\mathrm{MA}_{\aleph_1}$*
2. *$(M,I)$ is an iterable pair*
3. *$a \in \mathcal{P}(\omega_1)^M$ and $\exists x \in \mathcal{P}(\omega)^M$ such that $\omega_1^{L[x,a]} = \omega_1^M$*

*ordered by: $p < q$ (where $p = \langle (M,I), a\rangle$, $q = \langle (N,J), b\rangle$) if there is some iteration $j : (N,J) \to (N^*, J^*)$ such that*

1. *$j \in M$*
2. *$j(b) = a$*
3. *$J^* = N^* \cap I$ ( and hence $j(\omega_1^N) = \omega_1^M$ )*
4. *$q \in H(\omega_1)^M$*

Note that since $j \in M$ in definition of the $\mathbb{P}_{\max}$ order above, $N$ and $N^*$ are both in $M$ as well.

**Definition 9.** *We say that $(M,I)$ is a $\mathbb{P}_{\max}$* precondition *if there exists an $a$ such that $\langle (M,I), a\rangle \in \mathbb{P}_{\max}$, or equivalently just if $(M,I)$ satisfies conditions 1 and 2 in the definition of $\mathbb{P}_{\max}$ conditions above.*

Suppose that $(M,I)$ is an iterable pair, and $j\colon (M,I) \to (M',I')$ is an iteration. Then for any $A \in \mathcal{P}(\omega_1)^M$ which is bounded in $\omega_1^M$, $j(A) = A$. By Lemma 7, it follows then that $\omega_1^{L[A]} < \omega_1^M$, since $j(\omega_1^M) > \omega_1^M$ if $j$ is nontrivial. Therefore, the set $a$ from a $\mathbb{P}_{\max}$ condition $\langle (M,I), a\rangle$ must always be unbounded in $\omega_1^M$ to make ${\omega_1}^{L[x,a]} = {\omega_1}^M$ possible.

If $p_0 < p_1 < p_2$ ($p_i = \langle (M_i, I_i), a_i\rangle$), and these are witnessed by $j_{1,0}$, $j_{2,1}$, then $p_0 < p_2$ is witnessed by $j_{1,0}(j_{2,1})$: $j_{1,0} \in H(\omega_2)^{M_0}$, $j_{2,1} \in H(\omega_2)^{M_1}$; $j_{2,1}$ is an iteration of $(M_2, I_2)$, and $j_{1,0}((M_2, I_2)) = (M_2, I_2)$.

Under our fixed coding, "$(M,I)$ is iterable" is $\Pi^1_2$ in a code for $(M,I)$ : roughly, "for anything satisfying the first-order properties of being an iteration, either there is no infinite descending sequence in the ordinals of the final model, or there is an infinite descending sequence in the indices of the iteration." Since iterable models embed elementarily into models containing $\omega_1$, they are $\mathbf{\Pi}^1_2$-correct. It follows that "$(M,I)$ is iterable" is absolute to iterable models containing a code for $(M,I)$.

So now we see that $\mathbb{P}_{\max} \in L(\mathbb{R})$ — all constructions involved are nicely codable.

## 3 First properties of $\mathbb{P}_{\max}$

The requirement that the models in $\mathbb{P}_{\max}$ conditions satisfy $\mathrm{MA}_{\aleph_1}$ is used for a particular consequence of $\mathrm{MA}_{\aleph_1}$ known as *almost disjoint coding* [2]. That is, it follows from $\mathrm{MA}_{\aleph_1}$ that if $Z = \{z_\alpha : \alpha < \omega_1\}$ is a collection of infinite subsets of $\omega$ whose pairwise intersections are finite (i.e., $Z$ is an *almost disjoint family*), then for each $B \subseteq \omega_1$ there exists a $y \subseteq \omega$ such that for all $\alpha < \omega_1$, $\alpha \in B$ if and only if $y \cap z_\alpha$ is infinite. This is used to show that if $\langle (M, I), a\rangle$ is a $\mathbb{P}_{\max}$ condition, then any iteration of $(M, I)$ is uniquely determined by the image of $a$ (Lemma 10 below), and so the order on each comparable pair of conditions is witnessed by a unique iteration.

**Lemma 10.** *Let $\langle (M, I), a\rangle$ be a $\mathbb{P}_{\max}$ condition and let $A$ be a subset of $\omega_1$. Then there is at most one iteration of $(M, I)$ for which $A$ is the image of $a$.*

*Proof.* Fix a real $x$ in $M$ such that $\omega_1^M = \omega_1^{L[a,x]}$, and let $Z = \langle z_\alpha : \alpha < \omega_1^M\rangle$ be the almost disjoint family defined recursively from the constructibility order in $L[a, x]$ on $\mathcal{P}(\omega)^{L[a,x]}$ (using $a$ and $x$ as parameters) by letting $\langle z_i : i < \omega\rangle$ be the constructibly least partition of $\omega$ into infinite pieces, and, for each $\alpha \in [\omega, \omega_1^M)$, letting $z_\alpha$ be the constructibly least infinite $z \subset \omega$ almost disjoint from each $z_\beta$ ($\beta < \alpha$). Suppose that

$$\mathcal{I} = \langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

and

$$\mathcal{I}' = \langle M'_\alpha, G'_\beta, j'_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma'\rangle$$

are two iterations of $(M, I)$ such that $j_{0\gamma}(a) = A = j'_{0\gamma'}(a)$. Then $j_{0\gamma}(Z) = j'_{0\gamma'}(Z)$ (this uses Remark 7 to see that the constructibility order on reals in $L[A, x]$ is computed correctly in $M_\gamma$ and $M'_{\gamma'}$). Let $\langle z_\alpha : \alpha < j_{0\gamma}(\omega_1^M)\rangle$ enumerate $j_{0\gamma}(Z)$.

Without loss of generality, $\gamma \leq \gamma'$. We show by induction on $\alpha < \gamma$ that, for each such $\alpha$, $G_\alpha = G'_\alpha$. This will suffice. Fix $\alpha$ and suppose that

$$\{G_\beta : \beta < \alpha\} = \{G'_\beta : \beta < \alpha\}.$$

Then $M_\alpha = M'_\alpha$. For each $B \in \mathcal{P}(\omega_1)^{M_\alpha}$, $B \in G_\alpha$ if and only if $\omega_1^{M_\alpha} \in j_{\alpha(\alpha+1)}(B)$, and $B \in G'_\alpha$ if and only if $\omega_1^{M_\alpha} \in j'_{\alpha(\alpha+1)}(B)$. Applying almost disjoint coding, fix $x \in \mathcal{P}(\omega)^{M_\alpha}$ such that for all $\eta < \omega_1^{M_\alpha}$, $\eta \in B$ if and only if $x \cap z_\eta$ is infinite. Then $B \in G_\alpha$ if and only if $x \cap z_{\omega_1^{M_\alpha}}$ is infinite if and only if $B \in G'_\alpha$. $\square$

**Lemma 11.** *($T_0$) Suppose that $(M, I)$ is an iterable pair, and $J$ is a normal ideal on $\omega_1$. Then there exists an iteration $j\colon (M, I) \to (M^*, I^*)$ of length $\omega_1$ such that $I^* = M^* \cap J$.*

*Proof.* Note that $I^* \subseteq M^* \cap J$ holds for any such $\omega_1$-length iteration. To see this, first note that the critical sequence of an iteration of length $\omega_1$ is a club.

Every element $B$ of $I^*$ is $j_{\alpha,\omega_1}(b)$ for some $\alpha < \omega_1$ and $b \in I_\alpha$. Then for all $\beta \in [\alpha, \omega_1)$, $j_{\alpha,\beta}(b) \notin G_\beta$, so ${\omega_1}^{M_\beta} \notin j_{\alpha,\omega_1}(b) = B$; thus $B \in \mathrm{NS}_{\omega_1}$, but $J$ is normal, so $NS_{\omega_1} \subseteq J$.

Conversely, for $\supseteq$: as $J$ is normal, we may let $\langle E_i^\alpha \mid \alpha < \omega_1, i < \omega\rangle$ be a partition of $\omega_1$ into $J$-positive pieces. Now, as we construct an iteration, let $\{e_i^\alpha \mid i < \omega\}$ enumerate $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$, and build each $G_\beta$ in such a way that if ${\omega_1}^{N_\beta} \in E_i^\alpha$ for some $\alpha \leq \beta$ and $i < \omega$, then $j_{\alpha,\beta}(e_i^\alpha)$ is in $G_\beta$.

Now, for all $B \in \mathcal{P}(\omega_1)^{M_\alpha} \setminus I_{\omega_1}$, $\exists \alpha < \omega_1$, $i < \omega$ such that $B = j_{\alpha,\omega_1}(e_i^\alpha)$ and for all $\beta \in [\alpha, \omega_1)$, ${\omega_1}^{M_\beta} \in E_i^\alpha \Rightarrow {\omega_1}^{M_\beta} \in j_{0,\beta+1}(e_i^\alpha) \Rightarrow {\omega_1}^{M_\beta} \in B$.

So in particular, we have a club $C \subseteq \omega_1$ such that $C \cap E_i^\alpha \subseteq B$, so $B \notin J$. □

We may consider this as an *iteration game* $G((M, I), J, B)$: two players collaborate on building an iteration of $(M, I)$, and play is as follows at each round $\alpha$:

- if $\alpha \notin B$, player I does nothing, and player II chooses $G_\alpha$;
- if $\alpha \in B$, player I specifies some element for $G_\alpha$, and player II must choose some $G_\alpha$ containing it.

Player I wins if $I_{\omega_1} = M_{\omega_1} \cap J$. The above proof shows that player I has a winning strategy iff $B \notin J$. (More precisely, it shows $\Leftarrow$; $\Rightarrow$ is because if $B \in J$ then II may choose some $I$-positive set and keep its images out of every $G_\alpha$.)

The following lemma shows that $\mathbb{P}_{\max}$ satisfies a homogeneity property strong enough to imply that the theory of the generic extension can be computed in the ground model. Since large cardinals (a proper class of Woodin cardinals suffices) make the theory of $L(\mathbb{R})$ generically absolute, they make the theory of the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ generically absolute as well.

**Lemma 12.** *Suppose for each $x \in H(\omega_1)$ there exists a $\mathbb{P}_{\max}$ precondition $(M, I)$ such that $x \in M$. Then $\forall p_0, p_1 \in \mathbb{P}_{\max}$, $\exists q_0, q_1 \in \mathbb{P}_{\max}$ such that each $q_i \leq p_i$, and $\mathbb{P}_{\max}\restriction_{q_0} \cong \mathbb{P}_{\max}\restriction_{q_1}$.*

*Proof.* Take $p_i = \langle (M_i, I_i), a_i\rangle$. Then let $(N, J)$ be a $\mathbb{P}_{\max}$ precondition such that $p_0, p_1 \in H(\omega_1)^N$. Now take $j_i\colon (M_i, I_i) \to (M_i^*, I_i^*)$ to be iterations in $N$ such that $I_i^* = M_i^* \cap J$ (we may do so, by the previous theorem applied in $N$).

Now set $q_i = \langle (N, J), j_i(a_i)\rangle \in \mathbb{P}_{\max}$. Certainly these satisfy $q_i < p_i$ as desired. (To see that these $q_i$ are indeed conditions, note that the witnessing $x_i$ for $p_i$ (i.e., the $x \in \mathcal{P}(\omega)^{M_i}$ such that $\omega_1^{L[x,a_i]} = \omega_1^{M_i}$ correctly computing $\omega_1$ still works for $q_i$.) But now $\mathbb{P}_{\max}\restriction_{q_0} \cong \mathbb{P}_{\max}\restriction_{q_1}$, for given any $r_0 = \langle (N', I'), b\rangle <$ $q_0$, there is unique $j : (N, J) \to (N^*, J^*)$ witnessing this (and we have $b =$ $j(j_0(a_0)))$; now take $r_0$ to $r_1 := \langle (N', J'), j(j_1(a_1))\rangle < q$, also witnessed by $j$. □

Given $\gamma \in [\omega_1, \omega_2)$, a *canonical function* for $\gamma$ is a function $f\colon \omega_1 \to \omega_1$ such that for some (equivalently, every) bijection $\pi : \omega_1 \to \gamma$, $\{\alpha < \omega_1 \mid \mathrm{ot}(\pi[\alpha]) = f(\alpha)\}$ contains a club. In a normal ultrapower context, we then have: $f\colon \omega_1 \to \mathrm{Ord}$, $[f]_{Ult} = j(f)(\omega_1) = \mathrm{ot}(\pi[\omega_1]) = \gamma$.

Suppose that $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \omega_1,\ \eta < \omega_1\rangle$ is an iteration of length $\omega_1$ of some $(M, I)$, and let $\pi : \omega_1 \to \mathrm{Ord}^{M_{\omega_1}}$ be a bijection; then $\pi$ induces a canonical function $g$. For club-many $\alpha < \omega_1$, ${\omega_1}^{M_\alpha} = \alpha$, and also $\pi[\alpha] = j_{\alpha,\omega_1}[\mathrm{Ord}^{M_\alpha}]$ (recall that we take direct limits at limit stages of iterations); so there is a club $C \subset \omega_1$ such that for all $\alpha \in C$, $\mathrm{ot}(\pi[\alpha]) = \mathrm{Ord}^{M_\alpha}$. If $f \in ({\omega_1}^{\omega_1})^{M_\beta}$, for some $\beta < \alpha$, then for any $\alpha \in [\beta, \omega_1)$,

$$j_{\beta,\alpha+1}(f)({\omega_1}^{M_\alpha}) < {\omega_1}^{M_{\alpha+1}} < \mathrm{Ord}^{M_{\alpha+1}} = \mathrm{Ord}^{M_\alpha},$$

which equals $g(\alpha)$ if $\alpha \in C$. It follows that for any $f \in ({\omega_1}^{\omega_1})^{M_{\omega_1}}$, $g(\alpha) > f(\alpha)$ for club-many $\alpha$. We will use this fact to show that $\mathbb{P}_{\max}$ $\sigma$-closed (this is an alternate proof avoiding sharps; some of what follows can be done more easily and in more generality with sharps).

**Lemma 13.** *Suppose that for each $x \in H(\omega_1)$ there exists a $\mathbb{P}_{\max}$ precondition $(M, I)$ such that $x \in M$. If $p_i \in \mathbb{P}_{\max}$ $(i < \omega)$ are such that $\forall i\ p_{i+1} < p_i$, then $\exists q \in \mathbb{P}_{\max}$ such that $\forall i\ q < p_i$.*

The proof of Lemma 13 involves some new notions. Say that $p_i = \langle (M_i, I_i), a_i\rangle$, and for each $j < i < \omega$, let $k_{i,j} : M_j \to M_j^* \in M_i$ be the unique witness for $p_i < p_j$. By uniqueness of witnesses, the $k_{i,j}$ commute, so let $N_i$, $J_i$ be the images of $M_i$, $I_i$ in the limit of the directed system given by the embeddings $k_{i,j}$. Each $(N_i, J_i)$ is an iterate of the corresponding $(M_i, I_i)$ by an iteration of length $\sup\{\omega_1^{M_i} : i < \omega\}$, so each $N_i$ is wellfounded. Let $b = \bigcup_{i<\omega} a_i$. Then:

1. Each $(N_i, J_i)$ is iterable;
2. for all $i$, ${\omega_1}^{N_i} = \sup_{j<\omega} {\omega_1}^{M_j}$;
3. $i < j \Rightarrow N_i \in H(\omega_2)^{N_j}$;
4. for all $i$, $J_i = J_{i+1} \cap N_i$;
5. for each $i$ there exists some iteration $j_i : (M_i, I_i) \to (N_i, J_i)$ in $N_{i+1}$ such that $j_1(a_i) = b$ (and so in $N_{i+1}$, there is a canonical function for $\mathrm{Ord}^{N_i}$ that dominates on a club every member of $(\omega_1^{\omega_1})^{N_i}$).

We call a sequence $\langle (N_i, J_i) : i < \omega\rangle$ satisfying (1) to (5) above a $\mathbb{P}_{\max}$ *limit sequence*. An $\langle (N_i, J_i) \mid i < \omega\rangle$*-normal ultrafilter* is a filter $G \subseteq \bigcup_i (\mathcal{P}(\omega_1)^{N_i} \setminus J_i)$ such that for all $i < \omega$, and for all regressive $f \in ({\omega_1}^{\omega_1})^{N_i}$ $\exists e \in G$ such that $f$ is constant on $e$. Then we have $\mathit{Ult}(\langle (N_i, J_i) : i < \omega\rangle, G)$, i.e. a sequence of models whose $j$th model $[\mathit{Ult}(\langle (N_i, J_i) : i < \omega\rangle, G)]_j$ is

$$\left(\{f : {\omega_1}^{N_0} \to N_j \mid f \in \bigcup\{N_i \mid i < \omega\}\}/ =_G\right).$$

Now we will iterate this operation.

**Definition 14.** *An iteration of $\langle (N_i, J_i) : i < \omega\rangle$ of length $\gamma$ is some*

$$\langle\langle (N_i^\alpha, J_i^\alpha) : i < \omega\rangle, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma, \eta < \gamma\rangle$$

*in which:*

- *each $G_\eta$ is a $\bigcup_{i<\omega} N_i^\eta$-normal ultrafilter $\subseteq \bigcup_i (\mathcal{P}(\omega_1)^{N_i^\eta} \setminus J_i^\eta)$ such that for each $i < \omega$, $j_{\eta,\eta+1} \upharpoonright N_i^\eta \to N_i^{\eta+1}$ is the induced ultrapower*

- *the $j_{\alpha,\beta}$ commute, and for limit $\beta$, $N_i^\beta$ is the direct limit of $N_i^\alpha$ $(\alpha < \beta)$ under $j_{\alpha,\rho} \upharpoonright N_i^\alpha$ $(\alpha \leq \rho < \beta)$.*

Note that in an iteration of this form, for $i, \alpha$, there is in $N_{i+1}^\alpha$ an iteration of $(M_i, I_i)$ of length ${\omega_1}^{N_0^\alpha}$, with final model $N_i^\alpha$. Since each $(M_i, I_i)$ is iterable, the wellfoundedness of each $N_i^\alpha$ will follow from the wellfoundedness of $\omega_1^{N_0^\alpha}$.

For each $\alpha < \gamma$, $\omega_1^{N_0^{\alpha+1}} = \sup\{\mathrm{Ord}^{N_i^\alpha} \mid i < \omega\}$. To see this, fix canonical functions $g_i \in N_{i+1}^\alpha$ for $\mathrm{Ord}^{N_i}$ such that each $g_i$ dominates on a club every member of $(\omega_1^{\omega_1})^{N_i^\alpha}$. The $g_i$ are cofinal under mod-$NS_{\omega_1}$ domination in $\bigcup_i ({\omega_1}^{\omega_1})^{N_i^\alpha}$, and each $g_i$ represents an ordinal in $N_{i+1}^\alpha$ in this ultrapower, which shows that $\omega_1^{N_0^{\alpha+1}} = \sup\{\mathrm{Ord}^{N_i^\alpha} \mid i < \omega\}$. For limit $\beta$, ${\omega_1}^{N_0^\beta} = \sup\{{\omega_1}^{N_0^\alpha} \mid \alpha < \beta\}$. It follows that each $N_0^\alpha$ is wellfounded.

We have shown the following.

**Fact 15.** *All iterations of $\mathbb{P}_{\max}$ pre-limit sequences give well-founded models.*

Again this can be rephrased in terms of games. Let $G_\omega(\langle (N_i, J_i) \mid i < \omega \rangle, I, B)$ be the game of length $\omega_1$, in which I and II collaborate to build an iteration of $\langle (N_i, J_i) : i < \omega \rangle$ of length $\omega_1$, in which at stage $\alpha$:

- if $\alpha \in B$, player I chooses $e \in \bigcup_i (\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha)$, and player II chooses $G_\alpha$ some $\langle (N_i^\alpha$-normal ultrafilter on $\bigcup_i (\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha)$ containing $e$;

- if $\alpha \notin B$, player I does nothing, and player II chooses any suitable $G_\alpha$.

Player I wins if $\forall i\ j_{0,\omega_1}(J_i) = I \cap M_i^{\omega_1}$. The argument just given (along with the argument for Lemma 11) shows the following.

**Lemma 16.** *Suppose $\langle (N_i, J_i) \mid i < \omega \rangle$ is a $\mathbb{P}_{\max}$ pre-limit-sequence, $I$ is a maximal ideal on $\omega_1$ and $B \subseteq \omega_1$. Then Player I has a winning strategy in $G_\omega(\langle (N_i, J_i) \mid i < \omega \rangle, I, B)$ if and only if $B$ is not in $I$.*

We now return to the proof of Lemma 13. We have that the limit sequence $\langle (N_i, J_i) : i < \omega \rangle$ induced by the descending sequence $p_i$ $(i \in \omega)$ is iterable. Fix $\mathbb{P}_{\max}$ precondition $(M', I')$ such that this sequence is in $H(\omega_1)^{M'}$. Apply a winning strategy for player I in $M'$ for $G_{\omega_1}(\langle (N_i, J_i) \mid i < \omega \rangle, I', \omega_1)$ to get an iteration $j$ of $\langle (N_i, J_i) : i < \omega \rangle$ of length $\omega_1^{M'}$. Then $\forall i < \omega$, $j(j_i)$ witnesses that $p_i > \langle (M', I'), j(b) \rangle$.

Thus $\mathbb{P}_{\max}$ forcing is $\sigma$-closed, so does not add any reals; so $L(\mathbb{R})^{V^{\mathbb{P}_{\max}}} = L(\mathbb{R})^V$.

# 4 Existence of $\mathbb{P}_{\max}$ conditions

**Definition 17.** *Given $A \subseteq \mathbb{R}$, and an iterable pair $(M, I)$, we say $(N, I)$ is $A$-*iterable *if $A \cap M \in M$ and for any iteration $j : (M, I) \to (M^*, I^*)$, $j(A \cap M) = A \cap M^*$.*

In this section we will work through a proof of the following existence theorem for $\mathbb{P}_{\max}$ conditions.

**Lemma 18** (Main existence lemma)**.** *Suppose there are infinitely many Woodin cardinals below some measurable cardinal, and let $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. Then there exists an $A$-iterable $\mathbb{P}_{\max}$ precondition $(M, I)$ such that for every set forcing extension $M^+$ of $M$ and every precipitous ideal $I^+ \in M^+$ on $\omega_1^{M^+}$, $(M^+, I^+)$ is $A$-iterable.*

We need to introduce towers of measures and homogeneous tree. See [8] for a detailed discussion of this material

**Definition 19.** *Given $Z \neq \phi$, a* tower of measures *on $Z$ is a sequence $\langle \mu_i \mid i < \omega \rangle$ such that each $\mu_i \subseteq \mathcal{P}(Z^i)$ is an ultrafilter, and $\forall k < i < j$, $\forall A \in \mu_i$, we have $\{b \in Z^j \mid b \restriction i \in A\} \in \mu_j$ and $\{b \restriction k \mid b \in A\} \in \mu_k$.*

*Such a tower is* countably complete *if whenever $\langle A_i \mid i < \omega \rangle$ is such that $\forall i$ $A_i \in \mu_i$, there is $a \in Z^\omega$ such that $\forall i$ $a \restriction i \in A_i$.*

We note briefly that countable completeness is equivalent to: the direct limit of $Ult(V, \mu_i)$ is well-founded.

**Definition 20.** *A* tree on $\omega \times Z$ *is some $T \subseteq (\omega \times Z)^{<\omega}$ such that $\forall i < \omega, t \in T$ we have $t \restriction i \in T$. The projection of $T$ is $p[T] := \{y \in \omega^\omega \mid \exists c \in Z^\omega\ \forall i < \omega\ (y \restriction i, c \restriction i) \in T\}$.*

*Such a tree is* weakly $\kappa$-homogeneous *(for $\kappa$ a cardinal) if there exist $\kappa$-complete ultrafilters $\mu_{a,b} \subseteq \mathcal{P}(Z^{|a|})$ such that $\forall a, b \in \omega^{<\omega}$ with $|a| = |b|$,*

$$\{c \in Z^{|a|} \mid (a, c) \in T\} \in \mu_{a,b},$$

*and such that for each $x \in p[T]$ there exists a $b \in \omega^\omega$ such that $\langle \mu_{x \restriction i, b \restriction i} \mid i < \omega \}$ is a countably complete tower.*

Weakly homogeneous trees originated from work of Kechris, Martin and Solovay. The following fact is due to Woodin. A proof appears in [6].

**Fact 21.** *If $\delta$ is a limit of Woodin cardinals and there is a measurable cardinal above $\delta$, then for each $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ and $\gamma < \delta$, there exists a $\gamma$-weakly-homogeneous tree $T$ such that $p[T] = A$.*

**Lemma 22.** *Suppose that $T \subset (\omega \times Z)^{<\omega}$ is a $\gamma^+$-weakly-homogeneous tree, $\theta > (2^{|T|})^+$ is regular, $X \prec H(\theta)$, $T, \gamma \in X$, $|X| < \gamma$, and $\bar{a} \in p[T]$. Then there exists $Y \prec H(\theta)$ with $X \subseteq Y$, $X \cap \gamma = Y \cap \gamma$, $|X| = |Y|$, and $\bar{a} \in p[T \cap Y]$.*

*Proof.* Fix $\langle \mu_{a,b} \mid a, b \in \omega^{<\omega} \rangle$ in $X$ witnessing the $\gamma^+$-weak-homogeneity of T. Since $\bar{a} \in p[T]$, $\exists \bar{b}$ such that $\langle \mu_{\bar{a} \restriction i, \bar{b} \restriction i} \mid i < \omega \rangle$ is a countably complete tower. Now let $A_i = \bigcap (X \cap \mu_{\bar{a} \restriction i, \bar{b} \restriction i})$. Each $A_i$ is in $\mu_{\bar{a} \restriction i, \bar{b} \restriction i}$; so there is $\bar{c} \in Z^\omega$ such that $\forall i$ $\bar{c} \restriction i \in A_i$. Take $Y = X[\{\bar{c} \restriction i \mid i < \omega\}] := \{f(\bar{c} \restriction i) \mid f \in X, \mathrm{dom}(f) = Z^{<\omega}, i < \omega\}$.

Elementarity of $Y$ follows from an argument similar to the proof of Łós's Theorem (see Theorem 1.1.13 of [6]). To see that $Y \cap \gamma = X \cap \gamma$, note that if $\alpha \in Y \cap \gamma$, then $\alpha = f(c \restriction i)$ for some $f \in X$, $\mathrm{dom} f = Z^i$; but $\mu_{a\restriction i, b\restriction i}$ is $\gamma^+$-complete, so $f$ is constant on a set in $\mu_{a\restriction i, b\restriction i}$. But then this constant value is in $X$, and $f(c \restriction i)$ is this value, since $c \restriction i = \bigcap(\mu_{a\restriction i, b\restriction i} \cap X)$. □

Note that $Y$ in the proof above is in some sense a limit ultrapower of the transitive collapse of $X$.

The following was first proved by Foreman, Magidor and Shelah from a supercompact cardinal, and later improved by Woodin.

**Fact 23.** *If $\delta$ is Woodin, then $\mathrm{Coll}(\omega_1, < \delta)$ forces that $\mathrm{NS}_{\omega_1}$ is presaturated, and hence precipitous.*

Recall that an ideal $I$ on $\omega_1$ is *presaturated* if for every sequence of maximal antichains $\{Q_i \mid i < \omega\} \subset \mathcal{P}(\omega_1) \setminus I$, $\forall A \in I^+$, $\exists B \subseteq A$, $B \in I^+$ such that for all $i < \omega$,

$$|\{E \in Q_i \mid E \cap B \in I^+\}| \leq \aleph_1.$$

The following was proved by Kakuda and Magidor independently [3, 7].

**Fact 24.** *Any c.c.c. forcing preserves that $\mathrm{NS}_{\omega_1}$ is precipitous.*

Recall the hypotheses of main existence lemma: $\delta$ is a limit of Woodin cardinals, there exists a measurable cardinal greater than $\delta$, and $A$ is in $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. To prove the lemma, let $\kappa$ be the least Woodin cardinal, and $\gamma$ the least strong inaccessible above $\kappa$. Fix $\gamma^+$-weakly-homogeneous trees $S, T$, with $p[S] = A$, $p[T] = \mathbb{R} \setminus A$. Fix a regular $\theta > (2^{|S|})^+, (2^{|T|})^+$. Let $X$ be a countable elementary submodel of $H(\theta)$, with $S, T, \gamma, \kappa \in X$. Repeatedly apply Lemma 22 above to obtain $Y \prec H(\theta)$ such that $X \subseteq Y$, $X \cap \gamma = Y \cap \gamma$, $A = p[S \cap Y]$, $\mathbb{R} \cap A = p[T \cap Y]$. (Then $|Y \cap \mathrm{Ord}| = 2^\omega$.) Now let $N$ be the transitive collapse of $Y$, and let $\bar{S}, \bar{T}, \bar{\gamma}, \bar{\kappa}$ be the images of $S, T, \gamma, \kappa$ therein. Let $h$ be $N$-generic for $\mathrm{Coll}(\omega_1, < \bar{\kappa})$ followed by a c.c.c. poset of size $2^{\omega_1}$ to make $\mathrm{MA}_{\aleph_1}$ hold. Then $N[h] \models \mathrm{MA}_{\aleph_1}$ + "$\mathrm{NS}_{\omega_1}$ is precipitous". Then $N[h]$ is iterable, by Lemma 6. Let $M$ be $(V_{\bar{\gamma}})^{N[h]}$, and let $j : (M, \mathrm{NS}^M_{\omega_1}) \to (M^*, \mathrm{NS}^*_{\omega_1})$ be an iteration. By Lemma 4, this induces an iteration of $(N[h], \mathrm{NS}^{N[h]}_{\omega_1})$ with final model $(N^*, I^*)$ (which we'll also call $j$). Now, $N^*$ is well-founded, and $p[\bar{S}] \subseteq p[j(\bar{S})]$ and $p[\bar{T}] \subseteq p[j(\bar{T})]$. But by elementarity, $N^* \models p[j(\bar{S})] \cap p[j(\bar{T})] = \emptyset$, and since $N^*$ is well-founded it is correct about this. Then $p[\bar{S}] = p[j(\bar{S})]$ and $p[\bar{T}] = p[j(\bar{T})]$, so $j(A \cap M) = p[j(\bar{S})] \cap M^* = A \cap M^*$.

**Remark 1.** *Instead of $\mathrm{Coll}(\omega_1, < \bar{\kappa})$, we could have taken $h$ to be $N$-generic for any poset in $V^N_{\bar{\gamma}}$ such that $N[h] \models$ "$\exists$ precipitous $I$ on $\omega_1$", and the rest of the proof would have still gone through.*

The main existence lemma gives not only $A$-iterable preconditions for any $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, but also $A$-iterable preconditions containing any given real $x$, for any $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, applying the lemma to the set $\{y \oplus x \mid y \in A\}$. Thus we have shown: if there exist infinitely many Woodin cardinals below a measurable, then $\forall x \in \mathbb{R}$, $\forall A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$, there exists some $\mathbb{P}_{\max}$ condition $\langle (M, I), a \rangle$, with $x \in M$, and $(M, I)$ $A$-iterable.

We didn't quite show $\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle$. We can do this using $A^\sharp$, or by using not just $S, T$ as above but similar trees for all sets projective in $A$. We omit this for now.

Given a filter $G \subset \mathbb{P}_{\max}$, $A_G$ denotes the set $\bigcup\{e \mid \exists\langle(N,J),e\rangle \in G\}$. We also omit a proof of the following:

**Fact 25** ("The combinatorial heart of the $\mathbb{P}_{\max}$ analysis")**.** *Suppose that for each $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ there exists an $A$-iterable $\mathbb{P}_{\max}$ precondition $(N,I)$ such that*

$$\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle,$$

*and suppose that $G \subseteq \mathbb{P}_{\max}$ is an $L(\mathbb{R})$-generic filter. Then $\forall B \in \mathcal{P}(\omega_1)^{L(\mathbb{R})[G]}$, $\exists\langle(M,I),a\rangle \in G$ such that $B = j(b)$ for some $b \in \mathcal{P}(\omega_1)^M$, where $j$ is the unique iteration of $(M,I)$ satisfying $j(a) = A_G$.*

In other words, all subsets of $\omega_1$ in extensions come from models in the conditions, and $L(\mathbb{R})[G] = L(\mathbb{R})[A_G]$.

**Corollary 26.** *Suppose that for each $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ there exists an $A$-iterable $\mathbb{P}_{\max}$ precondition $(N,I)$ such that*

$$\langle H(\omega_1)^M, \in, A \cap M\rangle \prec \langle H(\omega_1), \in, A\rangle,$$

*and suppose that $G \subseteq \mathbb{P}_{\max}$ is an $L(\mathbb{R})$-generic filter. Then $\mathrm{NS}_{\omega_1}^{L(\mathbb{R})[G]}$ is the collection of all sets of the form $j(e)$, where for some $\langle(M,I),a\rangle \in G$, $e \in I$, and $j$ is the iteration of $(M,I)$ sending $a$ to $A_G$.*

Woodin has shown that the hypotheses of Fact 25 are equivalent to the assertion that AD holds in $L(\mathbb{R})$.

# 5 $\Pi_2$ maximality

*Proof of Goal 1* So now fix some $\Pi_2$ sentence $\varphi = \forall x \exists y \psi(x,y)$ (in the extended language with two new unary predicates), and some $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$. To show that

$$\langle H(\omega_2), \in, A, \mathrm{NS}_{\omega_1}\rangle^{L(\mathbb{R})^{\mathbb{P}_{\max}}} \models \varphi,$$

it is sufficient to show that for each $\langle(M,I),a\rangle \in \mathbb{P}_{\max}$ and each $b \in H(\omega_2)^M$, there exist $\langle(N, \mathrm{NS}_{\omega_1}^N), e\rangle \in \mathbb{P}_{\max}$ and $j : (M,I) \to (M^*, I^*)$ in $N$ such that $j(a) = e$, $I^* = M^* \cap \mathrm{NS}_{\omega_1}^N$, and

$$\langle H(\omega_2)^N, \in, A \cap N, \mathrm{NS}_{\omega_1}^N\rangle \models \exists d\ \psi(j(b), d).$$

The argument is like the one for existence of conditions.

So suppose $\langle(M,I),a\rangle$ is given. Fix $P$ forcing $\varphi$. Let $\delta$ be the least Woodin cardinal with $p \in V_\delta$; let $\kappa$ be the least strong inaccessible above $\delta$. Let $S, T$ be $\kappa^+$-weakly-homogeneous trees projecting to $A$, $\mathbb{R} \setminus A$. Let $\theta > (2^{|S|})^+, (2^{|T|})^+$ be regular. Fix $Y \prec H(\theta)$ with $Y \cap \kappa$ countable, $p[S \cap Y] = A$, $p[T \cap Y] = \mathbb{R} \setminus A$ and $\langle(M,I),a\rangle \in Y$.

Let $N$ be the transitive collapse of $Y$, and let $\bar{P}$,$\bar{S}$,$\bar{\delta}$,$\bar{\kappa}$ be the respective images of $P$,$S$,$\delta$,$\kappa$ under this collapse. Let $h_0$ be $\bar{P}$-generic for $N$. Note that since $P \in V_\delta$, $\bar{\delta}$ remains Woodin in $N[h_0]$. The reinterpretation of $A$ is the projection of $\bar{S}$ in the extension. Thus

$$\langle H(\omega_2)^{N[h_0]}, \in, (p[\bar{S}])^{N[h_0]}, \mathrm{NS}_{\omega_1}^{N[h_0]} \rangle \models \varphi.$$

Pick an iteration $j$ of $(M, I)$ in $N$ such that $j(I) = j(M) \cap \mathrm{NS}_{\omega_1}^{N[h_0]}$. Then there exists a $d \in H(\omega_2)^{N[h_0]}$ such that

$$\langle H(\omega_2)^{N[h_0]}, \in, (p[\bar{S}])^{N[h_0]}, \mathrm{NS}_{\omega_1}^{N[h_0]} \rangle \models \psi(j(b), d).$$

Let $h_1$ be $N[h_0]$-generic for $\mathrm{Coll}(\omega_1, < \bar{\delta})^{N[h_0]}$ followed by some c.c.c. forcing making $\mathrm{MA}_{\aleph_1}$ hold. Now $\langle ((V_{\bar{\kappa}})^{N[h_0][h_1]}, \mathrm{NS}_{\omega_1}^{N[h_0][h_1]}), j(a) \rangle$ is the desired condition. ■

# 6 Discussion

**Question 1.** *You've shown that under these conditions, any forceable $\Pi_2$ statement must hold in the $\mathbb{P}_{\max}$ extension. Can you give us some cool examples?*

*Answer.* One example is $\varphi_{\mathrm{AC}}$: "For every stationary, costationary $A, B \subseteq \omega_1$, there is some $\gamma \in [\omega_1, \omega_2)$, some bijection $\pi : \omega_1 \to \gamma$ such that

$$\{\alpha < \omega_1 \mid \alpha \in A \leftrightarrow (\pi[\alpha]) \in B\}$$

contains a club." This can be used to get an injection $\mathcal{P}(\omega_1) \hookrightarrow \omega_2$, which shows that the Axiom of Choice holds in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$.

Also, in some cases one can use $\mathbb{P}_{\max}$ to get $\Pi_2$ maximality relative to a given $\Sigma_2$ statement; that is, for a given $\Sigma_2$ statement for $H(\omega_2)$, you can simultaneously get all $\Pi_2$ statements forceably consistent with it.

Another useful aspect: often, the combinatorics of forcing to kill off one thing while preserving another are not clear; the combinatorics of doing the same by an iteration may be much clearer. For instance, an analysis of iterations may help answer the question of whether there exists a Dowker space on $\omega_1$.

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