# Real games and strategically selective coideals

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**Abstract**

We introduce a notion of strategically selective coideal, and show that tall strategically selective coideals do not exist under $\mathrm{AD}_{\mathbb{R}}$, generalizing a classical theorem of Mathias. We discuss some issues involved in generalizing this result to semiselective coideals.

A set $\mathcal{C} \subseteq \mathcal{P}(\omega)$ is a *coideal* if $\mathcal{P}(\omega) \setminus \mathcal{C}$ is an ideal containing Fin, the collection of finite subsets of $\omega$. A *fast diagonalization* of a sequence $\langle A_i : i < \omega \rangle \in \mathcal{P}(\omega)^\omega$ is a set $E = \{e_i : i \in \omega\}$ (listed in increasing order) such that $e_0 \in A_0$ and $e_{i+1} \in A_{e_i}$ for all $i \in \omega$. A coideal $\mathcal{C}$ is

- *tall* if for no infinite $A \subseteq \omega$ is $\mathcal{C} \cap \mathcal{P}(A) = \mathcal{P}(A) \setminus \mathrm{Fin}$, and
- *selective* if every $\subseteq$-decreasing sequence $\langle A_i : i < \omega \rangle \in \mathcal{C}^\omega$ has a fast diagonalization in $\mathcal{C}$.

The axiom $\mathrm{AD}_{\mathbb{R}}$ asserts the determinacy of all games of length $\omega$ where the players play subsets of $\omega$. Woodin has shown the consistency of $\mathrm{AD}_{\mathbb{R}}$ relative to large cardinals (see Theorem 9.3 of [10], for instance). The axiom DC is a weak form of the Axiom of Choice, asserting that each tree of height $\omega$ without terminal nodes has an infinite branch. The axiom $\mathrm{DC}_{\mathbb{R}}$ is the restriction of of DC to trees on $\mathbb{R}$; $\mathrm{DC}_{\mathbb{R}}$ is easily seen to be a consequence of $\mathrm{AD}_{\mathbb{R}}$. Mathias [8] showed that $\mathrm{AD}_{\mathbb{R}}$ implies that there are no tall selective coideals on $\omega$. In this note we give a generalization of this fact. Our proof consists of combining Mathias's proof with results of Solovay and Woodin.

Given a coideal $\mathcal{C}$, we let $\mathcal{G}_{\mathcal{C}}$ be the game of length $\omega$ in which players $I$ and $II$ choose the members of a $\subseteq$-decreasing sequence of elements of $\mathcal{C}$, with $II$ winning if the sequence constructed has a fast diagonalization in $\mathcal{C}$.

**0.1 Definition.** A coideal $\mathcal{C}$ is *strategically selective* if player $I$ does not have a winning strategy in $\mathcal{G}_{\mathcal{C}}$.

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It follows almost immediately from the definitions that a selective coideal is strategically selective, as in this case all runs of the game $\mathcal{G}_{\mathcal{C}}$ are won by $II$. As defined by Farah [3], a coideal $\mathcal{C}$ is *semiselective* if whenever $A \in \mathcal{C}$ and $\mathcal{D}_i$ $(i \in \omega)$ are dense open subsets of the partial order $(\mathcal{C}, \subseteq)$ below $A$, there exist sets $A_i \in \mathcal{D}_i$ $(i \in \omega)$ and a fast diagonalization of $\langle A_i : i \in \omega \rangle$ in $\mathcal{C}$. As we shall see, the Axiom of Choice implies that the notions of strategically selective and semiselective are equivalent.

Assuming $\mathrm{AD}_{\mathbb{R}}$, strategically selective coideals are easily seen to be semiselective; we don't know if the reverse implication holds. Uniformization is the statement that for each $U \subseteq \mathbb{R} \times \mathbb{R}$, there is a partial function $f \colon \mathbb{R} \to \mathbb{R}$ with the property that for each $x \in \mathbb{R}$, if there is a $y$ such that $(x, y)$ is in $U$, then $x$ is in the domain of $f$ and $(x, f(x))$ is in $U$. Uniformization an easy consequence of $\mathrm{AD}_{\mathbb{R}}$, via a game in which each player plays once.

**Theorem 0.2.** *If Uniformization holds, then every strategically selective coideal is semiselective.*

*Proof.* Suppose that $\mathcal{C}$ is a coideal which is not semiselective, and let $\mathcal{D}_i$ $(i \in \omega)$ and $A \in \mathcal{C}$ witness this. Fix a strategy $\Sigma$ in $\mathcal{G}_{\mathcal{C}}$ for player $I$ where $I$ plays a subset of $A$ in $\mathcal{D}_0$ as his first move, and, for any $i > 0$, $I$ plays in response to any sequence of length $2i$ an element of $\mathcal{D}_i$ contained in the last move made by player $II$. Uniformization implies that there exists such a strategy. Now suppose that $\bar{A} = \langle A_i : i < \omega \rangle$ is a run of $\mathcal{G}_{\mathcal{C}}$ where $I$ has played according to $\Sigma$, and suppose toward a contradiction that $\{e_i : i \in \omega\}$ is a fast diagonalization of $\bar{A}$ in $\mathcal{C}$. Then $\{e_{2i+1} : i \in \omega\}$ and $\{e_{2i} : i \in \omega\}$ are each fast diagonalizations of $\langle A_{2i} : i \in \omega \rangle$, and at least one of these two sets is in $\mathcal{C}$, giving a contradiction. $\square$

Given a coideal $\mathcal{C}$, we let $P_{\mathcal{C}}$ denote the partial order of mod-$I$ containment on $\mathcal{C}$, where $I = \mathcal{P}(\omega) \setminus \mathcal{C}$. Farah [3] has shown that when $\mathcal{C}$ is semiselective, forcing with $P_{\mathcal{C}}$ adds a selective ultrafilter (see Theorem 1.1 below).

The following is our main theorem.

**Theorem 0.3.** *If* $\mathrm{AD}_{\mathbb{R}}$ *holds, then there are no tall strategically selective coideals on* $\omega$.

Question 0.4 below is open, as far as we know. Corollary 3.4 of [2] states a result which would imply a positive answer to the question. However, there appears to be a gap in the proof, corresponding roughly to the issue of obtaining functions $a$ and $b$ as in the statement of Proposition 1.4 below in the context of $\mathrm{AD}_{\mathbb{R}}$ (i.e., without using the Axiom of Choice).

**0.4 Question.** Does $\mathrm{AD}_{\mathbb{R}}$ imply that there are no tall semiselective coideals on $\omega$?

We begin with Solovay's result on the existence of a normal fine measure on $\mathcal{P}_{\aleph_1}(\mathbb{R})$ under $\mathrm{AD}_{\mathbb{R}}$. Here *normality* of a measure $\mu$ on $\mathcal{P}_{\aleph_1}(\mathbb{R})$ means that if $A \in \mu$ and $f$ is a function on $A$ such that $f(\sigma)$ is a nonempty subset of $\sigma$ for each $\sigma \in A$, then there is a real $x$ which is in $f(\sigma)$ for $\mu$-many $\sigma$.

**Lemma 0.5** (Solovay [9]). *If* $\mathrm{AD}_{\mathbb{R}}$ *holds, then there is normal fine measure on* $\mathcal{P}_{\aleph_1}(\mathbb{R})$.

*Proof.* Given $A \subseteq \mathcal{P}_{\aleph_1}(\mathbb{R})$, consider the game $\mathcal{G}_m(A)$ where players $I$ and $II$ pick alternately pick finite sets of reals $s_i$ ($i \in \omega$) and $I$ wins if $\bigcup\{s_i : i \in \omega\} \in A$. Let $\mu$ be the set of $A$ for which $I$ has a winning strategy in $\mathcal{G}_m(A)$. That $\mu$ is fine (i.e., contains the set of supersets of each countable set of reals) is immediate. That it is an ultrafilter follows from running two strategies against one another. Normality follows the fact that given a family of games $\mathcal{G}_x$ indexed by reals for which $II$ has a winning strategy, there is a function picking such a strategy for each game, induced by the game where $I$ first picks $x$ and then $I$ and $II$ play $\mathcal{G}_x$. Fixing $f$ as in the statement of normality and supposing that player $II$ has a winning strategy for each payoff set of the form $F_x = \{\sigma \mid x \in f(\sigma)\}$, we can build a countable set of reals $\sigma \in \mathrm{dom}(f)$ which results from a run of $\mathcal{G}_m(F_x)$ according to a winning strategy for player $II$, for each $x \in \sigma$, giving a contradiction. $\square$

Given a set of ordinals $S$ and a formula $\phi$, let us write $A_{S,\phi}$ for the set $\{x \in (\omega^\omega)^{<\omega} : L[S,x] \models \phi(S,x)\}$. Formally extending a definition due to Woodin, we say that the pair $(S,\phi)$ is an $\infty$-*Borel code* for the set $A_{S,\phi}$, and we say that a set $B \subseteq (\omega^\omega)^{<\omega}$ is $\infty$-*Borel* if there exists such a pair $(S,\phi)$ with $A_{S,\phi} = B$ (i.e., if $B$ has an $\infty$-Borel code). A tree on the ordinals projecting to a subset of $\omega^\omega$ is an example of an $\infty$-Borel code, but the assumption that every set of reals is $\infty$-Borel is weaker than the assumption that every set of reals is the projection of a tree on the ordinals. The statement that every subset of $(\omega^\omega)^{<\omega}$ is $\infty$-Borel is easily seen to be equivalent to the assertion that every subset of $\omega^\omega$ is $\infty$-Borel, which in turn is part of Woodin's axiom $AD^+$ (see [4]).

The following theorem is unpublished.

**Theorem 0.6** (Woodin). *If* $\mathrm{AD}_{\mathbb{R}}$ *holds, then every set of reals is* $\infty$*-Borel.*

Recall that *Mathias forcing* $Q_U$ relative to an ultrafilter $U$ consists of pairs $(s,A)$, where $s$ is finite subset of $\omega$ and $A \in U$, with the order $(s,A) \geq (t,B)$ if $s \subseteq t$, $B \subseteq A$ and $t \setminus s \subseteq A$. The following is due to Mathias ([8], Theorem 2.0).

**Theorem 0.7** (Mathias). *Suppose that* $M$ *is a model of ZF +* $\mathrm{DC}_{\mathbb{R}}$ *and that* $U$ *is a selective ultrafilter in* $M$*. Then a set* $x \subseteq \omega$ *is* $M$*-generic for* $Q_U$ *if and only if* $x \setminus y \in \mathrm{Fin}$ *for all* $y \in U$.

The following proof puts together the facts listed above. The ultraproduct construction in the proof is taken from the proof of Theorem 9.39 from [11], except that we use $\infty$-Borel codes instead of trees on the ordinals.

*Proof of Theorem 0.3.* Suppose that $\mathrm{AD}_{\mathbb{R}}$ holds, and that $\mathcal{C}$ is a tall strategically selective coideal. Fix a winning strategy $\Sigma$ for player $II$ in $\mathcal{G}_{\mathcal{C}}$. By Theorem 0.6, there are formulas $\phi$ and $\psi$, and sets of ordinals $S$ and $T$ such that

$$\mathcal{C} \times \Sigma = A_{S,\phi}$$

and

$$(\omega^\omega)^{<\omega} \setminus (\mathcal{C} \times \Sigma) = A_{T,\psi}.$$

Let $\mu$ be a normal fine measure on $\mathcal{P}_{\aleph_1}(\mathbb{R})$. Let $(M, E)$ be the ultraproduct $\prod_{\sigma \in \mathcal{P}_{\aleph_1}(\mathbb{R})} L(\sigma, S, T)/\mu$ constructed inside $L(\mathbb{R})[S, T, \mu]$. Then

- elements of $M$ are represented by functions $f$ in $L(\mathbb{R})[S, T, \mu]$, with domain $\mathcal{P}_{\aleph_1}(\mathbb{R})$, such that $f(\sigma)$ is in $L(\sigma, S, T)$, for each $\sigma \in \mathcal{P}_{\aleph_1}(\mathbb{R})$ (let $\mathcal{F}$ be the class of such functions),
- the members of $M$ are the equivalence classes of functions in $\mathcal{F}$, under the relation of mod-$\mu$ equivalence, and
- given two such equivalence classes $[f]_\mu$ and $[g]_\mu$, $[f]_\mu E [g]_\mu$ if and only if

$$\{\sigma \in \mathcal{P}_{\aleph_1}(\mathbb{R}) : f(\sigma) \in g(\sigma)\} \in \mu.$$

For each set $x \in L(\mathbb{R})[S, T, \mu]$, let $c_x$ be the constant function from $\mathcal{P}_{\aleph_1}(\mathbb{R})$ to $\{x\}$. Since $\mu$ normal (and thus countably complete), and $L(\mathbb{R})[S, T, \mu]$ satisfies DC (as $\mathrm{AD}_{\mathbb{R}}$ implies $\mathrm{DC}_{\mathbb{R}}$), $(M, E)$ is wellfounded. A standard argument by induction on subformulas, using the normality of $\mu$ for the step corresponding to existential quantifiers, shows that for any finite set of functions $f_1, \ldots, f_n$ from $\mathcal{F}$, and any $n$-ary formula $\phi$,

$$(M, E) \models \phi([f_1]_\mu, \ldots, [f_n]_\mu)$$

if and only if

$$L(\sigma, S, T) \models \phi(f_1(\sigma), \ldots, f_n(\sigma))$$

for $\mu$-many $\sigma$. Let us call this fact the *elementarity* of the ultraproduct. One consequence of this fact (and the wellfoundedness of $(M, E)$) is that there is an isomorphism $\pi$ from $(M, E)$ to an inner model of the form $L(\mathbb{R}, S^*, T^*)$, where $S^* = \pi([c_S]_\mu)$ and $T^* = \pi([c_T]_\mu)$. Then $S^*$ and $T^*$ are sets of ordinals.

By the elementarity of the ultraproduct, $A_{S,\phi} \subseteq A_{S^*,\phi}$ and $A_{T,\psi} \subseteq A_{T^*,\psi}$. Since

$$A_{S,\phi} = (\omega^\omega)^{<\omega} \setminus A_{T,\psi},$$

it follows again by elementarity that $A_{S,\phi} = A_{S^*,\phi}$ and $A_{T,\psi} = A_{T^*,\psi}$. By the normality of $\mu$ (and elementarity once again), it follows that for $\mu$-many $\sigma$, $\sigma = \mathbb{R} \cap L(\sigma, S, T)$ and $A^{L(\sigma,S,T)}_{S,\phi}$ (which is $(\mathcal{C} \times \Sigma) \cap L(\sigma, S, T)$) is the product of a tall strategically selective coideal and a strategy witnessing that it is strategically selective, in $L(\sigma, S, T)$. Fixing one such $\sigma$, there is an $L(\sigma, S, T)$-generic filter $H$ for $P^{L(\sigma,S,T)}_{\mathcal{C}}$ which is generated by a run of $\mathcal{G}_{\mathcal{C}}$ according to $\Sigma$. One can build such a run of $\mathcal{G}_{\mathcal{C}}$ by letting $II$ play according to $\Sigma$ and having $I$ play to meet each dense set in $L(\sigma, S, T)$ from $P^{L(\sigma,S,T)}_{\mathcal{C}}$. Note that $L(\sigma, S, T)$ is closed under $\Sigma$. Furthermore, since $\sigma$ is a countable, $L(\sigma, S, T)$ is contained in a model of Choice, which implies that $\mathcal{P}(\mathcal{P}(\mathbb{R})) \cap L(\sigma, S, T)$ is countable, so there exists (in $V$) an enumeration of the dense subsets of $P^{L(\sigma,S,T)}_{\mathcal{C}}$ in $L(\sigma, S, T)$ in ordertype $\omega$.

Let $U$ be the selective ultrafilter in $L(\sigma, S, T)[H]$ given by $H$. By Theorem 0.7, since $\Sigma$ is a winning strategy for player $II$, there is an $x \in \mathcal{C}$ which is $L(\sigma, S, T)[H]$-generic for the Mathias forcing $Q_U$. Some condition $(s, B)$ in the corresponding generic filter then forces that the generic real will be the left coordinate of a pair in $A_{S,\phi}$. However, this is a contradiction, as some infinite subset of $x$ containing $s$ is not in $\mathcal{C}$, and any such set is still generic below $(s, B)$, by Theorem 0.7. $\square$

**0.8 Remark.** A *one-point diagonalization* of a sequence $\langle A_i : i < \omega \rangle \in \mathcal{P}(\omega)^\omega$ is a set $E = \{e_i : i \in \omega\}$ (listed in increasing order) such that $e_i \in A_i$ for all $i \in \omega$. In an earlier version of this paper we used one-point diagonalizations instead of fast diagonalizations in the definitions of selective, semiselective and strategically selective. Example 0.9 below shows that these definitions are not equivalent. We note that [1] uses one-point diagonalizations in the definition of Ramsey (i.e., selective) ultrafilters; Example 0.9 shows that there is a gap in the proof of Theorem 4.5.2 there claiming to show that this definition is equivalent to the standard one. Similarly, Example 0.9 shows that the corresponding version of Theorem 0.3 using one-point diagonalizations in place of fast diagonalizations is false. The paper [7] cites (the earlier vesion of) this paper for proving this false version of Theorem 0.3. Modifying that paper to use the correct definitions requires making minor changes.

**0.9 Example.** Let $\bar{F} = \{F_n : n \in \omega\}$ be a partition of $\omega$ into finite sets, such that $\{|F_n| : n \in \omega\}$ is infinite and $F_n \cap n = \emptyset$ for all $n \in \omega$. Let $\mathcal{I}_{\bar{F}}$ be the ideal of sets $x \subseteq \omega$ for which there exists an $m \in \omega$ such that $|x \cap F_n| < m$ for all $n \in \omega$. Then $\mathcal{I}_{\bar{F}}$ is $F_\sigma$, so Borel. Let $\mathcal{C}_{\bar{F}}$ be the corresponding coideal. If, for each $n \in \omega$, $A_n = \omega \setminus \bigcup_{m<n} F_m$, then each $A_n$ is in $\mathcal{C}_{\bar{F}}$ and each fast diagonalization of the sequence $\langle A_n : n \in \omega \rangle$ intersects each $F_n$ at most 2 points, so is not in $\mathcal{C}_{\bar{F}}$. On the other hand, if $\bar{B} = \langle B_n : n \in \omega \rangle$ is any $\subseteq$-decreasing sequence of members of $\mathcal{C}_{\bar{F}}$, $\bar{B}$ has a one-point diagonalization in $\mathcal{C}_{\bar{F}}$. This shows that changing "fast" to "one-point" in the definition of selective coideal gives a weaker notion (and similarly for semiselective and strategically selective). Assuming that the Continuum Hypothesis holds, one can easily construct an ultrafilter contained in $\mathcal{C}_{\bar{F}}$ with the property that each $\subseteq$-descending $\omega$-sequence from $U$ has a one-point diagonalization in $U$.

# 1 Semiselective coideals

In this section we discuss some issues related to the question of whether $\mathrm{AD}_{\mathbb{R}}$ implies the nonexistence of tall semiselective coideals. First we note two alternate characterizations of semiselectivity shown by Farah in [3]. Theorem 1.1 can be proved in ZF.

**Theorem 1.1** (Farah [3])**.** *The following statements are equivalent, for a coideal $\mathcal{C}$ on $\omega$.*

1. *$\mathcal{C}$ is semiselective.*

2. *The generic filter added by forcing with $P_{\mathcal{C}}$ is a selective ultrafilter.*

3. *Forcing with $P_{\mathcal{C}}$ does not add reals, and whenever $\{E\} \cup \{B_i : i \in \omega\} \subseteq \mathcal{C}$ and for all $i \in \omega$, $E \setminus B_i \notin \mathcal{C}$, there exists an $E' \subseteq E$ in $\mathcal{C}$ such that for each $i \in \omega$, $E' \setminus (i+1) \subseteq B_i$.*

Given any partial order $P$, let $\mathcal{G}_{ds}(P)$ be the game where players $I$ and $II$ alternately choose the members of a descending sequence of conditions in $P$, with $I$ winning if the sequence does not have a lower bound in $P$. For any coideal $\mathcal{C}$, a winning strategy for $I$ in $\mathcal{G}_{ds}(P_{\mathcal{C}})$ is a winning strategy for $I$ in $\mathcal{G}_{\mathcal{C}}$. By the second part of statement (3) of Theorem 1.1, if $\mathcal{C}$ is semiselective, then a winning strategy for $II$ in $\mathcal{G}_{ds}(P_{\mathcal{C}})$ is a winning strategy for $II$ in $\mathcal{G}_{\mathcal{C}}$. Theorem 0.3 shows that, assuming $\mathrm{AD}_{\mathbb{R}}$, player $I$ has a winning strategy in $\mathcal{G}_{\mathcal{C}}$ for each tall coideal $\mathcal{C}$ on $\omega$. It follows that if $\mathcal{C}$ is a tall semiselective coideal on $\omega$, and $\mathrm{AD}_{\mathbb{R}}$ holds, then $I$ has a winning strategy in $\mathcal{G}_{ds}(P_{\mathcal{C}})$. The first part of the following proposition then shows that forcing with $P_{\mathcal{C}}$ must make $\mathrm{DC}_{\mathbb{R}}$ fail.

**Proposition 1.2** (ZF)**.** *Suppose that $\mathcal{C}$ is a coideal on $\omega$ such that forcing with $P_{\mathcal{C}}$ does not add subsets of $\omega$.*

1. *If $I$ has a winning strategy in $\mathcal{G}_{ds}(P_{\mathcal{C}})$, then $\mathrm{DC}_{\mathbb{R}}$ fails after forcing with with $P_{\mathcal{C}}$.*

2. *If $II$ has a winning strategy in $\mathcal{G}_{ds}(P_{\mathcal{C}})$, and $\mathrm{DC}_{\mathbb{R}}$ holds, then it holds after forcing with $P_{\mathcal{C}}$.*

*Proof.* For the first part of the proposition, let $\tau$ be a winning strategy for $I$ in $\mathcal{G}_{ds}(P_{\mathcal{C}})$, and let $G \subseteq \mathcal{C}$ be generic for $P_{\mathcal{C}}$. In $V[G]$, consider the set of finite sequences from $G$ of odd length which are partial plays of $\mathcal{G}_{ds}(P_{\mathcal{C}})$ according to $\tau$. Every such sequence is extended by a longer one, but no infinite play of $\mathcal{G}_{ds}(P_{\mathcal{C}})$ according to $\tau$ can be forced by any condition in $P_{\mathcal{C}}$ to be a subset of the generic filter.

For the second part, fix $A \in \mathcal{C}$, a winning strategy $\tau$ for $II$ in $\mathcal{G}_{ds}(P_{\mathcal{C}})$ and a $P_{\mathcal{C}}$-name $\sigma$ for a set of finite sequences of reals with the property that every sequence in the set is extended by another sequence in the set. Consider the set of finite sequences

$$A_0, x_0, A_1, A_2, x_2, A_3, \ldots, A_{2n}, x_{2n}, A_{2n+1}$$

for which $A_0 \subseteq A$, $\langle A_0, \ldots, A_{2n+1} \rangle$ is a partial play of $\mathcal{G}_{ds}(P_{\mathcal{C}})$ according to $\tau$, and each $A_{2i}$ forces that $\langle x_0, x_2, \ldots, x_{2i} \rangle$ is a member of the realization of $\sigma$. Then $\mathrm{DC}_{\mathbb{R}}$, plus the fact that $\tau$ is a winning strategy for $II$, gives a condition below $A$ forcing some infinite sequence to be a path through the realization of $\sigma$. □

A positive answer to any part of the following question would show that no tall semiselective coideals exist, assuming $\mathrm{AD}_{\mathbb{R}}$.

**1.3 Question.** Suppose that $\mathrm{AD}_{\mathbb{R}}$ holds, and let $\mathcal{C}$ be a coideal on $\omega$. Must $P_{\mathcal{C}}$ preserve $\mathrm{DC}_{\mathbb{R}}$? What if $\mathcal{C}$ is tall, or if $P_{\mathcal{C}}$ is assumed not to add subsets of $\omega$, or to be semiselective, or tall and semiselective?

Proposition 1.4 shows that if $\mathcal{C}$ is a semiselective coideal, and there exist suitable choice functions, then $\mathcal{C}$ is strategically selective (from which it follows that under AC the two notions are equivalent). Given a function $a$ as in the statement of Proposition 1.4, the existence of a function $b$ follows from Uniformization.

**Proposition 1.4** (ZF). *Let $\mathcal{C}$ be a coideal, and let $\Sigma$ be a winning strategy for $I$ in $\mathcal{G}_{\mathcal{C}}$. Let $S$ be the set of finite partial runs $\tau$ in $\mathcal{G}_{\mathcal{C}}$ according to $\Sigma$ for which it is $II$'s turn to move. Suppose that there exist functions $a$ on $S$ and $b$ on $S \times \mathcal{C}$ such that*

- *for each $\tau \in S$, $a(\tau)$ is a maximal antichain in $(\mathcal{C}, \subseteq)$ below the last member of $\tau$, contained in the set of members of $\mathcal{C}$ which are responses by $\Sigma$ to a move for $II$ following $\tau$;*
- *for each $\tau \in S$ and $B \in a(\tau)$, $b(\tau, B)$ is an element of $\mathcal{C}$ such that $\tau^\frown\langle b(\tau, B), B\rangle$ is a partial run of $\mathcal{G}_{\mathcal{C}}$ according to $\Sigma$.*

*Then $\mathcal{C}$ is not semiselective.*

*Proof.* Let $A_0$ be the first move made by $\Sigma$. Let $H$ be a generic filter for $P_{\mathcal{C}}$, with $A_0 \in H$. In $V[H]$, consider the collection $T$ consisting of those sequences of the form $\langle B_0, \ldots, B_{2n}\rangle$ contained in $H$, where for each even $i < 2n$, $B_{i+2}$ is in $a(\langle B_0, \ldots, B_i\rangle)$ and $B_{i+1} = b(\langle B_0, \ldots, B_i\rangle, B_{i+2})$. By genericity, each sequence in $T$ has an extension in $T$. Since each $a(\tau)$ ($\tau \in S$) is an antichain, the members of $T$ extend one another, and there exists a unique sequence $\bar{B} = \langle B_i : i < \omega\rangle$ whose finite initial segments are all in $T$.

Suppose now toward a contradiction that $\mathcal{C}$ is semiselective. Then forcing with $P_{\mathcal{C}}$ doesn't add reals, so $\bar{B}$ is in $V$. Since $\Sigma$ is a winning strategy for $I$ in $\mathcal{G}_{\mathcal{C}}$, $\bar{B}$ does not have a fast diagonalization in $\mathcal{C}$. By part (3) of Theorem 1.1, then, $\bar{B}$ does not have a lower bound in $\mathcal{C}$, so no element of $\mathcal{C}$ could force all of the elements of $\bar{B}$ to be in $H$, giving a contradiction. $\square$

Proposition 1.4 gives the following.

**Theorem 1.5** (ZF). *If there exists a wellordering of $\mathbb{R}$, and $M$ is a model of* ZF + $\mathrm{AD}_{\mathbb{R}}$ *containing $\mathbb{R}$, then there is no tall coideal in $M$ which is semiselective in $V$.*

Finally, we note that the following theorem of Todorcevic (a version of which appears in [3]; the form given here is proved in [5]) allows one to argue from the point of view of a model of the Axiom of Choice that certain inner models do not contain tall coideals which are semiselective in $V$. In the presence of suitably large cardinals (for instance, a measurable cardinal above infinitely many Woodin cardinals), typical inner models of determinacy (such as $L(\mathbb{R})$) have the property that all of their sets of reals are at least $\mathfrak{c}$-universally Baire in $V$ (see, for instance, Theorems 3.3.9 and 3.3.13 of [6]).

**Theorem 1.6** (Todorcevic)**.** *If $U$ is a selective ultrafilter and $I$ is a tall ideal on $\omega$ containing* Fin *which is $\mathfrak{c}$-universally Baire, then $U \cap I \neq \emptyset$.*

**Corollary 1.7.** *If $\mathcal{C}$ is a tall coideal in an inner model $M$ containing the reals, and every set of reals in $M$ is $\mathfrak{c}$-universally Baire, then $\mathcal{C}$ is not semiselective in $V$*

*Proof.* If $\mathcal{C}$ were semiselective in $V$, then forcing with $P_{\mathcal{C}}$ would produce a selective ultrafilter disjoint from $\mathcal{P}(\omega) \setminus \mathcal{C}$. $\square$

As with Theorem 1.5, the corollary to Todorcevic's result leaves open the possibility that there is a tall coideal which is semiselective in the model in question, but no longer semiselective in any outer model of Choice.

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