Parametrizing the Ramsey theory of block sequences

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Joint Mathematics Meetings
Special Session on Choiceless Set Theory and Related Areas
Denver, CO
January 16, 2020

This talk is about (preliminary) results in the Ramsey theory of countable infinite-dimensional vector spaces.

But first, let's review the situation on $\mathbb{N}$.

The Galvin-Prikry/Silver Theorem

The prototypical infinite-dimensional Ramsey theorem is:

Theorem (Galvin-Prikry/Silver, 1970)

If $A \subseteq [\mathbb{N}]^\infty$ is analytic, then there is an $x \in [\mathbb{N}]^\infty$ such that either $[x]^\infty \subseteq A$ or $[x]^\infty \cap A = \emptyset$.

- $[x]^\infty = \{y \subseteq x : |y| = \infty\}$ for $x \subseteq \mathbb{N}$.
- This is a generalization of the (1-dimensional) pigeonhole principle and the (finite-dimensional) Ramsey's Theorem.

Localization

The Galvin-Prikry/Silver Theorem can be "localized":

Theorem (Mathias, 1977)

Let $\mathcal{U}$ be a selective ultrafilter on $\mathbb{N}$. If $A \subseteq [\mathbb{N}]^\infty$ is analytic, then there is an $x \in \mathcal{U}$ such that either $[x]^\infty \subseteq A$ or $[x]^\infty \cap A = \emptyset$.

- A (non-principle) ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is selective if it is closed under certain diagonalizations: whenever $x_0 \supseteq x_1 \supseteq x_2 \supseteq \cdots$ in $\mathcal{U}$, there is an $x \in \mathcal{U}$ such that $x/n \subseteq x_n$ for all $n \in x$.

Preservation

Selective ultrafilters are robust under “mild” forcing:

Theorem (Baumgartner-Laver, 1979)

Let $\mathcal{U}$ be a selective ultrafilter. If g if $\mathbf{V}$-generic for $\mathbb{S}$, then $\mathcal{U}$ generates a selective ultrafilter in $\mathbf{V}[g]$.

- $\mathbb{S}$ is Sacks forcing, the collection of all perfect trees $p \subseteq 2^{<\infty}$ ordered by containment.
- Moreover: If CH holds in $\mathbf{V}$, then $\mathcal{U}$ generates a selective ultrafilter in the extension by an ω_2-length countable support iteration of Sacks forcing.

Parametrization

The preservation of selective ultrafilters by $\mathbb{S}$ is closely related to the following "parametrized" Galvin-Prikry/Silver Theorem:

Theorem (Miller-Todorcevic, 1989)

If $A \subseteq 2^{\mathbb{N}} \times [\mathbb{N}]^{\infty}$ is analytic, then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ and an $x \in [\mathbb{N}]^{\infty}$ such that either $P \times [x]^{\infty} \subseteq A$ or $(P \times [x]^{\infty}) \cap A = \emptyset$.

- A partitions a family of copies of $[\mathbb{N}]^{\infty}$ parametrized by $2^{\mathbb{N}}$. The result says that, by passing to a "large" subset P of $2^{\mathbb{N}}$, we can find a single x homogeneous for all of the slices indexed by P.
- A localized version holds as well, à la Mathias.

Parametrization (cont'd)

We'll sketch a proof when $A \subseteq 2^{\mathbb{N}} \times [\mathbb{N}]^\infty$ is Borel.

Proof sketch.

- WLOG assume the existence of a selective ultrafilter $\mathcal{U}$.
- Let M be a sufficient ctm containing $\mathcal{U} \cap M$ and the relevant data.
- By Baumgartner-Laver, if g is M-generic for $\mathbb{S}$, then $\mathcal{U} \cap M$ generates a selective ultrafilter in $M[g]$.
- In $M[g]$, apply Mathias to get an $x \in \mathcal{U} \cap M$ such that $\{g\} \times [x]^\infty \subseteq A$ or $(\{g\} \times [x]^\infty) \cap A = \emptyset$. (Note: this is Π^1_1 in g and x.)
- Take $p \in \mathbb{S}$ forcing this.
- Use fusion to find $p_\infty \leq p$ such that every $g \in [p_\infty]$ is M-generic.
- By absoluteness, x and $P = [p_\infty]$ are as claimed.

$\square$

- Analytic sets require an extra step using Mathias-Prikry forcing.

Block sequences in vector spaces

The setting for our results is a countably infinite-dimensional vector space E over a countable field F, with a basis given by $(e_n)_{n\in\mathbb{N}}$.

- E.g., $E = \bigoplus_n F$ and e_n is the nth unit coordinate vector.

A sequence of nonzero vectors $X = (x_n)_{n\in\mathbb{N}}$ in E is a block sequence if

$$\max(\operatorname{supp}(x_n)) < \min(\operatorname{supp}(x_{n+1}))$$

for all n, where $\operatorname{supp}(v) = \{i : v = \sum a_j e_j \Rightarrow a_i \neq 0\}$.

- The (Polish) space of all block sequences is denoted by $\mathrm{bb}^\infty(E)$.
- Block sequences are ordered $\preceq$ by containment of their spans.
- Every infinite-dimensional subspace of E contains a block sequence. (Why? Row reduce and thin out.)

Asymptotic pairs

The main obstacle to Ramsey theorems on vector spaces is the failure of a natural pigeonhole principle:

Proposition

If $|F| > 2$, then there exists a pair of disjoint, scalar invariant subsets of E which intersect every infinite-dimensional subspace of E. (This is called an asymptotic pair.)

- If $|F| = 2$, then a pigeonhole principle *does* hold; this is essentially Hindman's Theorem (1974).

Games with vectors

The relevant Ramsey-theoretic dichotomy is stated in terms of games. Given $X \in \mathrm{bb}^\infty(E)$:

- $F[X]$ denotes the infinite asymptotic game below X: Players I and II alternate with I going first
  - I plays $n_k \in \mathbb{N}$,
  - II responds with a vector $y_k \in \langle X \rangle$ such that $n_k < y_k < y_{k+1}$.
- $G[X]$ denotes the Gowers game below X: Players I and II alternate with I going first.
  - I plays $X_k \preceq Y$,
  - II responds with a vector $y_k \in \langle X_k \rangle$ such that $y_k < y_{k+1}$.
- In both games, the outcome is the sequence of II's moves $(y_k)_{k\in\mathbb{N}}$. Players will have strategies for playing in to or out of a set.

A Ramsey theorem for block sequences

The analogue of the Galvin-Prikry/Silver theorem is:

Theorem (Rosendal, 2010)

If $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ be analytic, then there is an $X \in \mathrm{bb}^\infty(E)$ such that either

- *I has a strategy in $F[X]$ for playing out of $\mathbb{A}$, or*
- *II has a strategy in $G[X]$ for playing in to $\mathbb{A}$.*

- I.e., below some X, either $\mathbb{A}^c$ or $\mathbb{A}$ is "large".
- This result is based on a dichotomy for block sequences in Banach spaces due to Gowers (1996/2002) and implies it.

A Ramsey theorem for block sequences (cont'd)

Rosendal's result can be localized:

Theorem (S., 2018)

Let $\mathcal{U} \subseteq \mathrm{bb}^\infty(E)$ be a (p^+)-filter. If $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic, then there is an $X \in \mathcal{U}$ such that either

- *I has a strategy in $F[X]$ for playing out of $\mathbb{A}$, or*
- *II has a strategy in $G[X]$ for playing in to $\mathbb{A}$.*

- A filter $\mathcal{U}$ in $(\mathrm{bb}^\infty(E), \preceq)$ has the (p)-property if it is closed under diagonalizations: whenever $X_0 \succeq X_1 \succeq X_2 \succeq \cdots$ in $\mathcal{U}$, there is an $X \in \mathcal{U}$ such that $X \preceq^* X_n$ for all n.
- $\mathcal{U}$ is full if whenever $D \subseteq E$ is such that for every $Y \in \mathcal{U} \upharpoonright X$, there is a $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, there is such a $Z \in \mathcal{U} \upharpoonright X$.
- (p^+) = (p) + full.

Parametrization?

Can we parametrize these Ramsey theorems for block sequences? A natural first attempt is the following:

$(*)$ If $\mathbb{A} \subseteq 2^{\mathbb{N}} \times \mathrm{bb}^{\infty}(E)$ is analytic, then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ and an $X \in \mathrm{bb}^{\infty}(E)$ such that either

- I has a strategy σ in $F[X]$ such that $(P \times [\sigma]) \cap \mathbb{A} = \emptyset$, or
- II has a strategy α in $G[X]$ such that $P \times [\alpha] \subseteq \mathbb{A}$.

But, $(*)$ is false if $|F| > 2$: The idea is to use an asymptotic pair to code the first coordinate into the second coordinate.

Parametrization (cont'd)

There is too much uniformity in $(*)$, so we weaken it:

Theorem (S.)

If $\mathbb{B} \subseteq 2^{\mathbb{N}} \times \mathrm{bb}^{\infty}(E)$ is Borel, then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ and an $X \in \mathrm{bb}^{\infty}(E)$ such that either

- *for all $f \in P$, I has a strategy in $F[X]$ to play out of $\mathbb{B}_f$, or*
- *for all $f \in P$, II has a strategy in $G[X]$ to play in to $\mathbb{B}_f$.*

- Our results (at least in ZFC) are only for Borel sets, though we suspect that this can be improved to analytic sets.

An outline of the proof

The general outline of our proof is:

- Prove the result for clopen $\mathbb{B} \subseteq 2^{\mathbb{N}} \times \mathrm{bb}^{\infty}(E)$.
    - The argument uses MA and Shoenfield Absoluteness. It derives from an observation of Christian Rosendal that the parametrized result holds for all definable sets under sufficient large cardinals.
- Prove the result localized to a strategic (p^+)-filter, again, for clopen sets.
    - $\mathcal{U}$ is strategic if whenever α is a strategy for II in $G[X]$, $X \in \mathcal{U}$, there is an outcome of α in $\mathcal{U}$.
- Prove that strategic (p^+)-filters are preserved* by Sacks forcing.
- Mimic the above proof of the Miller-Todorcevic Theorem to get the result for all Borel sets.

Preservation*

A key step is the analogue of the Baumgartner-Laver theorem:

Theorem (S.)

Let $\mathcal{U}$ be a strategic (p^+)-filter. If g if $\mathbf{V}$-generic for $\mathbb{S}$, then $\mathcal{U}$ generates a (p^+)-filter in $\mathbf{V}[g]$.

- What happened to “strategic”?
- If $|F| > 2$, being strategic does not survive when adding a Sacks real. This answers a question implicit in the author’s thesis, namely whether (p^+) implied strategic, and presents a potential obstacle to iteration.

Choiceless?

The proofs described in this talk seem to use choice and other non-effective methods to an almost excessive degree:

- Countable models
- Forcing
- Martin's Axiom
- Shoenfield and Mostowski Absoluteness
- Ultrafilters/(p^+)-filters (that need not exist in ZFC)...

Choiceless? (cont'd)

But, our main theorem makes good sense in a choiceless context:

Theorem (S.)

If $\mathbb{B} \subseteq 2^{\mathbb{N}} \times \mathrm{bb}^{\infty}(E)$ is Borel, then there is a perfect set $P \subseteq 2^{\mathbb{N}}$ and an $X \in \mathrm{bb}^{\infty}(E)$ such that either

- *for all $f \in P$, I has a strategy in $F[X]$ to play out of $\mathbb{B}_f$, or*
- *for all $f \in P$, II has a strategy in $G[X]$ to play in to $\mathbb{B}_f$.*

- What's missing is a more direct, combinatorial, "Ellentuck-style" argument (in ZF $+$ DC); this exists for the Miller-Todorcevic Theorem (due to Pawlikowski, 1990).
- Such an argument *should* more easily extend to analytic sets.

Thanks for listening!