PFA and the definability of the nonstationary ideal

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Abstract

We produce, relative to a ZFC model with a supercompact cardinal, a ZFC model of the Proper Forcing Axiom in which the nonstationary ideal on ω_1 is Π_1-definable in a parameter from $H_{\aleph_2}$.

1 Introduction

A subset of ω_1 is said to be nonstationary if there exists a club $C \subseteq \omega_1$ disjoint from it. It follows that the ideal NS_{ω_1} of nonstationary subsets of ω_1 is Σ_1-definable in the parameter ω_1. Theorem 1.3 of [3] shows that in the presence of BPFA (the Bounded Proper Forcing Axiom) NS_{ω_1} may also be Π_1-definable in the parameter ω_1. On the other hand, Martin's Maximum and Woodin's axiom (*) each imply that NS_{ω_1} cannot be Π_1-definable over $H_{\aleph_2}$ in any parameter from $H_{\aleph_2}$, cf. [3, Theorem 2.3] and [11]. The current paper strengthens Theorem 1.3 of [3] by proving the following theorem.

Theorem 1.1. *If there exists a supercompact cardinal, then there exists a proper forcing extension in which* PFA *holds and* NS_{ω_1} *is* Π_1*-definable in a parameter from* $H_{\aleph_2}$*.*

Theorem 1.1 follows from Theorem 3.5 below. The parameter cannot be removed from Theorem 1.1, as Corollary 4.13 of [9] shows that under PFA, NS_{ω_1} is not Π_1-definable in the parameter ω_1.

The overall strategy of the proof is as follows. First the parameter A (a partition of ω_1 into $\aleph_1$-many pieces) is added generically by countable approximations. This is followed by a countable support iteration which interleaves a certification forcing at successor stages with the standard PFA iteration at limit stages (due to Baumgartner; see [1] for the corresponding iteration for Martin's

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Maximum). The entire iteration (including the first step adding A) is proper, although the tails of the iteration are only semi-proper over the nontrivial initial extensions. One potentially novel aspect of the construction is that the bookkeeping used in the certification part of the iteration is chosen with some care, in order to enable the construction of suitably generic conditions. The certification mechanism used is not tied closely to the nonstationary ideal, and could be used to code other ideals.

In the model produced, NS_{ω_1} is not saturated (and in fact Canonical Function Bounding fails, see Theorem 3.5). It remains open whether PFA plus the saturation of NS_{ω_1} (or Canonical Function Bounding) is consistent with the Π_1-definability of NS_{ω_1} relative to a subset of ω_1.

2 A coding machinery

We write $\mathrm{otp}(a)$ for the ordertype of a set a of ordinals. Given a set $A \subseteq \omega_1$, $\tilde{A}$ ("A-tilde") is the set of $\gamma \in [\omega_1, \omega_2)$ for which there is a bijection $b \colon \omega_1 \to \gamma$ such that $\{\alpha < \omega_1 : \mathrm{otp}(b[\alpha]) \in B\}$ contains a club (see [13, 6]).

Let A be a subset of ω_1, and let $\gamma \geq \omega_1$ be an ordinal. We let $Q(A,\gamma)$ denote the natural partial order to force γ into $\tilde{A}$. That is, conditions in $Q(A,\gamma)$ are sequences

$$p = \langle a_\alpha : \alpha \leq \zeta \rangle,$$

for some countable ordinal ζ, such that

- for all $\alpha \leq \zeta$, $a_\alpha \in [\gamma]^\omega$ and $\mathrm{otp}(a_\alpha) \in S$,
- $a_\alpha \subsetneq a_\beta$ when $\alpha < \beta \leq \zeta$ and
- $a_\beta = \bigcup_{\alpha<\beta} a_\alpha$ when $\beta \leq \zeta$ is a limit ordinal.

The order on $Q(A,\gamma)$ is defined by setting $p \leq q$ to hold if p end-extends q.

We will use partial orders of the form $Q(A,\gamma)$ only the in the case where γ is a measurable cardinal. In doing so, we will be using the following standard fact, which appears in many places, including Lemma 1.1.21 of [7]. We include a proof for the convenience of the reader.

Lemma 2.1. *Suppose that κ is a measurable cardinal, $\theta > 2^\kappa$ is a regular cardinal, $A \subseteq \omega_1$ is stationary and $X \prec H_\theta$ is countable with $\kappa \in X$. Then there exists a countable $Y \prec H_\theta$ such that*

- *$X \subseteq Y$,*
- *$X \cap \omega_1 = Y \cap \omega_1$ and*
- *$\mathrm{otp}(Y \cap \kappa) \in A$.*

Proof. Let $U \in X$ be a normal measure on κ. For any countable $Z \prec H_\theta$ with $U \in Z$, if

$$\eta \in \bigcap (U \cap Z)$$

and if Z' is the set of values $f(\eta)$ for f a function in Z with domain κ, then $Z' \prec H_\theta$, $Z \subseteq Z'$ and $Z' \cap \kappa$ end-extends $Z \cap \kappa$, i.e., $Z \cap \kappa = Z' \cap \sup(Z \cap \kappa)$.

We may then form a continuous chain $\langle X_\alpha \colon \alpha < \omega_1 \rangle$ of countable elementary substructures of H_θ such that $X_0 = X$ and $X_\alpha \cap \kappa = X_\beta \cap \sup(X_\alpha \cap \kappa)$ for all $\alpha \leq \beta < \omega_1$. It follows that $\{\mathrm{otp}(X_\alpha \cap \kappa) : \alpha < \omega_1\}$ is a club subset of ω_1. As A is stationary, there is then some $\alpha < \omega_1$ such that setting $Y = X_\alpha$, Y is as desired. □

Our first application of Lemma 2.1 is the following.

Lemma 2.2. *Let A be a stationary subset of ω_1 and let κ be a measurable cardinal. Then $Q(A, \kappa)$ is semi-proper and forces the statement "$\kappa \in \tilde{A}$".*

Proof. To see that $Q(A, \kappa)$ is semi-proper, let $\theta > 2^\kappa$ be a regular cardinal and let $X \prec H_\theta$ be countable with $A, \kappa \in X$. Applying Lemma 2.1, let $Y \prec H_\theta$ contain X, with $X \cap \omega_1 = Y \cap \omega_1$, and $\mathrm{otp}(Y \cap \kappa) \in A$. Then the union of any $(Q(A, \kappa), Y)$-generic filter is a condition in $Q(A, \kappa)$. That $Q(A, \kappa)$ forces κ into $\tilde{A}$ follows by a standard genericity argument. □

Let us say that $\vec{A} = \langle A_\alpha \colon \alpha < \omega_1 \rangle$ *splits ω_1 into stationary sets* if

- each A_α is a stationary subset of ω_1,
- $\omega_1 = \bigcup_{\alpha < \omega_1} A_\alpha$, and
- $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha < \beta < \omega_1$.

Given such a $\vec{A}$, and a set $S \subseteq \omega_1$, we will write $\vec{A}(S)$ for $\bigcup_{\alpha \in S} A_\alpha$. Given an ordinal γ, we will say that S is *certified at* γ (*modulo* $\vec{A}$) if γ is in the tilde of $\vec{A}(S)$, i.e., if there exists a sequence $\langle a_\alpha \colon \alpha < \omega_1 \rangle$ such that

- for all $\alpha < \omega_1$ $a_\alpha \in [\gamma]^\omega$ and $\mathrm{otp}(a_\alpha) \in S$,
- $a_\alpha \subsetneq a_\beta$ for all $\alpha < \beta < \omega_1$,
- $a_\beta = \bigcup_{\alpha < \beta} a_\alpha$ when $\beta < \omega_1$ is a limit ordinal, and
- $\gamma = \bigcup_{\alpha < \omega_1} a_\alpha$.

Note that if S is certified at γ then so is every superset of S.

Given an ordinal γ in the interval $[\omega_1, \omega_2)$, a *canonical function* for γ is a function $f \colon \omega_1 \to \omega_1$ such that, for some bijection $\pi \colon \omega_1 \to \gamma$, $f(\alpha)$ is the ordertype of $\pi[\alpha]$ for each $\alpha < \omega_1$. Fixing such an f, and S and $\vec{A}$ as above, we say that S is *coded at* γ (*modulo* $\vec{A}$) if for all $\alpha < \omega_1$,

$$\alpha \in S \Leftrightarrow \{\beta < \omega_1 \colon f(\beta) \in A_\alpha\} \text{ is stationary.}$$

Since any two canonical functions for γ agree on a club subset of ω_1, this definition does not depend on the choice of f.

Canonical Function Bounding (CFB) is the statement that for each function $f \colon \omega_1 \to \omega_1$ there is a canonical function $g \colon \omega_1 \to \omega_1$ for some $\gamma \in [\omega_1, \omega_2)$

such that the set $\{\alpha < \omega_1 : f(\alpha) < g(\alpha)\}$ contains a club. It is a standard fact that CFB follows from the saturation NS_{ω_1} (the proof starts by noting that if NS_{ω_1} is saturated, then forcing with $\mathcal{P}(\omega_1)/\mathsf{NS}_{\omega_1}$ cannot collapse ω_2, which means that the identity function on ω_1 must represent ω_2^V in any induced generic ultrapower).

Lemma 2.3. *Suppose that $\vec{A} = \langle A_\alpha : \alpha < \omega_1\rangle$ splits ω_1 into stationary sets, and that $S \subseteq \omega_1$ is nonempty. Let κ be a measurable cardinal. Suppose that $\vec{a} = \langle a_\alpha \colon \alpha < \omega_1\rangle$ is $Q(\vec{A}(S), \kappa)$-generic over V. Then, in $V[\vec{a}]$,*

1. *S is certified at κ via $\vec{A}$, and*
2. *S is coded at κ, modulo $\vec{A}$.*

Proof. The first part of the lemma follows from the second part of Lemma 2.2. The second follows similarly, by a genericity argument, as follows. The function $f\colon \omega_1 \to \omega_1$ defined by letting $f(\beta)$ be $\mathrm{otp}(a_\beta)$ is a canonical function for κ. If $\alpha \in \omega_1 \setminus S$, then $\{\beta < \omega_1 \colon \mathrm{otp}(a_\beta) \in A_\alpha\} = \emptyset$. We thus only need to see that if $\alpha \in S$, then $\{\beta < \omega_1 \colon \mathrm{otp}(a_\beta) \in A_\alpha\}$ is stationary in $V[g]$.

Let $p \in \mathbb{P}$ and $\dot{C} \in V^{\mathbb{P}}$ be such that $p \Vdash \dot{C}$ is a club subset of ω_1. Let $\theta > 2^\kappa$ be a regular cardinal and let X be a countable elementary submodel of $H(\theta)$ with p and $\dot{C}$ as members. Applying Lemma 2.1, let $Y \prec H_\theta$ contain X, with $X \cap \omega_1 = Y \cap \omega_1$, and $\mathrm{otp}(Y \cap \kappa) \in A_\alpha$. Then the union of any $(Q(\vec{A}(S), \kappa), Y)$-generic filter is a condition in $Q(\vec{A}(S), \kappa)$ forcing that $X \cap \omega_1 \in \dot{C}$ and $a_{X\cap\omega_1} = Y \cap \kappa$, so $\mathrm{otp}(a_{X\cap\omega_1}) \in A_\alpha$. □

Note that being certified via $\vec{A}$ at a given ordinal is Σ_1 in ω_1, so absolute to outer models. The property of being coded modulo $\vec{A}$ is Σ_1 in NS_{ω_1} and $\vec{A}$, and therefore absolute to models preserving stationary subsets of ω_1.

In what follows, we write $\mathrm{Col}(\omega_1, \omega_1)$ for the partial order consisting of all functions $p\colon \zeta \to \omega_1$, for some countable ordinal ζ, ordered by end-extension. If g is $\mathrm{Col}(\omega_1, \omega_1)$-generic over V, then we confuse g with $\bigcup g$, a function from ω_1 to ω_1, and also write it as g.

Given a function $g\colon \omega_1 \to \omega_1$, we let the *partition of ω_1 induced by g* be the sequence $\langle A_\alpha : \alpha < \omega\rangle$ such that each A_α is the set $\{\beta < \omega_1 : g(\beta) = \alpha\}$. The proof of the following standard fact is elementary.

Lemma 2.4. *If $g\colon \omega_1 \to \omega_1$ is $\mathrm{Col}(\omega_1, \omega_1)$-generic over V then in $V[g]$ the partition of ω_1 induced by g splits ω_1 into stationary sets.*

In the rest of the paper we will write $P(S, \gamma, g)$ for the partial order $Q(\vec{A}(S), \gamma)$, where S is a subset of ω_1, $\gamma \geq \omega_1$ is an ordinal, and $\vec{A}$ is the partition of ω_1 induced by g.

3 The forcing iteration

We are now ready to define our forcing iteration. In order to facilitate the discussion, let us introduce an ad hoc term for the kind of iterations which we will be interested in.

Definition 3.1. *Let* $\langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi \colon \eta \leq \delta, \xi < \delta\rangle$ *be a countable support iteration of forcings. We call this iteration* appropriate *if* $\mathbb{P}_0 = \mathrm{Col}(\omega_1, \omega_1)$ *and there exists a sequence*

$$\langle \dot{S}_\xi, \alpha_\xi, \kappa_\xi : \xi < \delta\rangle$$

such that, for all $\xi > 0$*, either*

1. $\dot{\mathbb{Q}}_\xi$ *is a* $\mathbb{P}_\xi$*-name for a proper forcing or*
2. $\dot{S}_\xi$ *is a* $\mathbb{P}_\xi$*-name for a stationary subset of* ω_1 *with* α_ξ *as a member,* κ_ξ *is a measurable cardinal greater than* $|\mathbb{P}_\xi|$ *and, letting* $\dot{g}_0$ *be a* $\mathbb{P}_0$*-name for the generic function from* ω_1 *to* ω_1 *added by* $\mathbb{P}_0$*,* $\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi = P(\dot{S}_\xi, \check{\kappa}_\xi, \dot{g}_0)$*.*

In what follows we let $\dot{g}_\xi$ (for some ordinal ξ, relative to an appropriate iteration) denote the canonical name for the generic filter for $\mathbb{P}_\xi$, and when talking of a particular generic filter $g \subseteq \mathbb{P}_\delta$, let g_ξ denote the restriction of g to $\mathbb{P}_\xi$.

Lemma 3.2. *Let* $\bar{\mathbb{P}} = \langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi \colon \eta \leq \delta, \xi < \delta\rangle$ *be an appropriate iteration, as witnessed by* $W = \langle \dot{S}_\xi, \alpha_\xi, \kappa_\xi : \xi < \beta\rangle$*. Let* $\rho < \delta$ *be nonzero, let* $\theta > 2^{|\mathbb{P}|}$ *be a regular cardinal and let* X *be a countable elementary substructure of* H_θ *with* $\bar{\mathbb{P}}$*,* W *and* ρ *in* X*. Suppose that* q*,* $\dot{p}$ *are such that*

1. $q \in \mathbb{P}_\rho$ *is* $(X, \mathbb{P}_\rho)$*-generic,*
2. *for each* $\xi \in X \cap \delta$*,* $q(0)(\mathrm{otp}(X \cap \kappa_\xi)) = \alpha_\xi$*,*
3. $\dot{p} \in V^{\mathbb{P}_\rho}$*, and*
4. $q \Vdash_{\mathbb{P}_\rho} \dot{p} \in \mathbb{P}_\delta \cap X \wedge \dot{p} \restriction \rho \in \dot{G}_\rho$*.*

Then there is a condition $r \in \mathbb{P}_\delta$ *such that*

- r *is* $(X, \mathbb{P}_\delta)$*-generic,*
- $r \restriction \rho = q$*, and*
- $r \Vdash_{\mathbb{P}_\delta} \dot{p} \in \dot{G}_\delta$*.*

Proof. The proof is by induction on δ. If δ is a limit ordinal, then this is by the usual proper forcing argument, see e.g. the proof of [4, Lemma 31.17]. If δ is a successor, then we may assume that $\delta = \rho + 1$. If $\Vdash_{\mathbb{P}_\rho} \dot{\mathbb{Q}}_\rho$ is proper, then this is again by the usual proper forcing argument, see e.g. [4, Lemma 31.18]. Let us thus assume that $\dot{S}_\rho$, α_ρ and κ_ρ are as in the second case of Definition 3.1.

Suppose now that g_ρ is $\mathbb{P}_\rho$-generic over V with $q \in g_\rho$. We have that

- $\dot{\mathbb{Q}}_{\rho, g_\rho} = P(\dot{S}_{\rho, g_\rho}, \kappa_\alpha, g_0)$;
- $\dot{S}_{\rho, g_\rho}$ is in $X[g_\rho]$ and stationary in $V[g_\rho]$;
- κ_ρ is in X and measurable in $V[g_\rho]$;

- $g_0(\mathrm{otp}(X \cap \kappa_\rho)) = \alpha_\rho$ and $\alpha_\rho \in \dot{S}_{\rho, g_\rho}$;
- $\dot{p}_g \in X[g_\rho]$.

We may then produce in a standard fashion, in much the same way as in the proof of Lemma 2.2, some $s \in P(\dot{S}_{\rho, g_\rho}, \kappa_\rho, g_0)$ such that

- $s <_{P(\dot{S}_{\rho,g_\rho}, \kappa_\rho, g_0)} \dot{p}_{g_\rho}(\rho)$,
- $\mathrm{dom}(s) = (X \cap \omega_1) + 1 = (X[g] \cap \omega_1) + 1$,
- for each $D \in X[g_\rho]$ which is dense in $P(\dot{S}_{\rho, g_\rho}, \kappa_\rho, g_0)$ there is some

  $$\bar{s} >_{P(\dot{S}_{\rho,g_\rho}, \kappa_\rho, g_0)} s$$

  with $\bar{s} \in D \cap X[g]$, and
- $s(X \cap \omega_1) = X \cap \kappa_\rho$.

In particular, s is $(X[g_\rho], P(\dot{S}_{\rho,g_\rho}, \kappa_\rho, g_0))$-generic.

By fullness, there is then some $\mathbb{P}_\rho$-name $\dot{s}$ such that q forces that $\dot{s} \in \dot{\mathbb{Q}}_\rho$ is $(X[\dot{g}_\rho], \dot{\mathbb{Q}}_\rho)$-generic and $\dot{s} <_{\mathbb{Q}_\rho} \dot{p}(\rho)$. It follows that $r = q^\frown \dot{s}$ is as desired. $\square$

Lemma 3.2 (plus the ability to choose $(X, \mathbb{P}_0)$-generic q satisfying the hypothesis of the lemma in case $\rho = 0$ with respect to any given $\dot{p}$) gives the following.

Theorem 3.3. *If Let $\langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi : \eta \leq \delta,\ \xi < \delta \rangle$ is an appropriate iteration then for every $\eta \leq \delta$, $\mathbb{P}_\eta$ is proper.*

Remark 3.4. Since appropriate iterations are proper, they preserve the property of having uncountable cofinality. It follows that the tails of appropriate iterations also preserve uncountable cofinality. Each iterand in an appropriate iteration is semi proper (this follows from Lemma 2.2 in the second case). A Revised Countable Support (RCS) iteration of semi-proper forcings is semi-proper ([2, 10, 12]). However, an RCS iteration of partial orders is equivalent to the corresponding countable support iteration if it preserves uncountable cofinalities. It follows then that the tails of appropriate iterations are semi-proper, and in particular preserve stationary subsets of ω_1.

To complete the proof of the main theorem, suppose that δ is a supercompact cardinal. By a *Laver function* for δ we mean some $R\colon \delta \to V_\delta$ such that for all $X \in V$ there are $\bar{\delta} < \bar{\theta} < \delta$ and $\theta > \delta$ together with an elementary embedding

$$j\colon H_{\bar{\theta}} \to H_\theta$$

such that

- $\mathrm{crit}(j) = \bar{\delta}$,
- $j(\bar{\delta}) = \delta$,

- $X \in H_\theta$, and
- $j(R(\bar{\delta})) = X$.

As every supercompact cardinal has a Laver function [8], we fix a Laver function R for δ. We also fix a function $\pi\colon \delta \to \delta \times \omega_1$, with component functions π_0 and π_1, such that

- $\pi_1(\xi) \leq \xi$ for all $\xi < \delta$, and
- for each pair $(\eta, \alpha) \in \delta \times \omega_1$, $\pi^{-1}[\{(\eta, \alpha)\}]$ contains δ many successor ordinals.

We then define an appropriate iteration

$$\langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi \colon \eta \leq \delta, \xi < \delta \rangle$$

of length $\delta + 1$ having properties (1)-(3) below. In doing so we fix for each $\eta < \delta$ a wellordering $\leq_\eta$ of the set N_η consisting of the nice $\mathbb{P}_\eta$-names for stationary subsets of ω_1 (nice in the sense of [5]; there will be less than δ-many such names, and each stationary subset of ω_1 in the $\mathbb{P}_\eta$-extension will be the corresponding realization of one of them). Given $(\eta, \alpha) \in \delta \times \omega_1$, let $N_{\eta,\alpha}$ be the set of $\dot{S} \in N_\eta$ such that $\Vdash_{\mathbb{P}_\eta} \check{\alpha} \in \dot{S}$. Note then that if g_η is V-generic for $\mathbb{P}_\eta$, $S \in V[g_\eta]$ is a stationary subset of ω_1 and $\alpha \in S$, then S is the realization via g_η of some element of $N_{\eta,\alpha}$.

Now we fix the following iteration.

1. We let $\mathbb{P}_0$ be $\mathrm{Col}(\omega_1, \omega_1)$.
2. If $\xi \in (0, \delta)$ is a limit ordinal, then $\dot{\mathbb{Q}}_\xi = R(\xi)$, provided that $\Vdash_{\mathbb{P}_\xi}$ "$R(\xi)$ is a proper forcing"; $\dot{\mathbb{Q}}_\xi$ is trivial otherwise.
3. If $\xi < \delta$ is a successor ordinal, and κ is the least measurable cardinal strictly above $|\mathbb{P}_\xi|$, then $\dot{\mathbb{Q}}_\xi = P(\dot{S}, \check{\kappa}, \dot{g}_0)$ and $\dot{S}$ is the $\leq_{\pi_0(\xi)}$-least $\mathbb{P}_{\pi_0(\xi)}$-name such that
   - $\Vdash_{\mathbb{P}_{\pi_0(\xi)}} \pi_1(\xi) \in \dot{S}$ and
   - the realization of $\dot{S}$ by $g_{\pi_0(\xi)}$ is stationary and not certified at any member of $\omega_2^{V[g_\xi]}$ via the partition of ω_1 induced by the realization of $\dot{g}_0$,

   if such a $\dot{S}$ exists; otherwise we let $\dot{S}$ be $\check{\omega}_1$.

Note then that this iteration is appropriate, with each κ_ξ being the least measurable cardinal above $|\mathbb{P}_\xi|$, each α_ξ being $\pi_1(\xi)$ and each $\dot{S}_\xi$ being a name built from the two possibilities above for $\dot{S}$ for successor ξ (and $\check{\omega}_1$ when ξ is 0 or a limit ordinal).

Theorem 1.1 then follows from the following theorem.

Theorem 3.5. *Let $\langle \mathbb{P}_\eta, \dot{\mathbb{Q}}_\xi \colon \eta \leq \delta,\ \xi < \delta\rangle$ be the iteration defined above, let g be $\mathbb{P}_\delta$-generic over V, and let $\vec{A}$ be the partition of ω_1 induced by $g(0)$. Then the following hold in $V[g]$.*

1. *For all $S \subseteq \omega_1$ and $\gamma < \delta$, S is certified at γ via $\vec{A}$ if and only if there is a successor $\xi < \delta$ such that $\dot{S}_{\xi,g_\xi} \subseteq S$ and $\gamma = \kappa_\xi$.*
2. *For all $S \subseteq \omega_1$, S is stationary if and only if there is a $\gamma < \delta$ such that S is certified at γ via $\vec{A}$.*
3. $\mathsf{PFA} + \neg\mathsf{CFB} +$ *"NS_{ω_1} is Π_1-definable in a parameter from $H_{\aleph_2}$".*

Proof. The reverse direction of part (1) follows from Lemma 2.2, and the fact that being certified at some γ via some $\vec{A}$ is upwards absolute. For the forward direction of (1), fix

- $\eta < \delta$
- a $\mathbb{P}_\delta$-name $\dot{S}$ for a subset of ω_1,
- $\gamma < \delta$,
- a condition $p \in \mathbb{P}_\delta$,
- a countable ordinal β which p forces not to be in $\dot{S}$,
- a $\mathbb{P}_\delta$-name $\dot{C}$ for a club subset of ω_1 and
- a $\mathbb{P}_\delta$-name $\dot{b}$ for a bijection between ω_1 and γ.

. Let θ be a regular cardinal greater than $2^{|\mathbb{P}_\delta|}$ and let X be a countable elementary submodel of H_θ with all the objects named above in X. Every $(X, \mathbb{P}_\delta)$-generic condition forces that $\omega_1 \cap X$ is in $\dot{C}$ and $\dot{b}[\omega_1 \cap X] = X \cap \gamma$. It suffices to find, under the assumption that either (Case 1) γ is not equal to any κ_ξ or (Case 2) γ is equal to some κ_ξ and p forces β to be in $\dot{S}_\xi$, an $(X, \mathbb{P}_\delta)$-generic condition $r \leq p$ forcing that $g(0)(\mathrm{otp}(X \cap \gamma)) \notin \dot{S}$. Since the proofs of the two cases are similar we do them simultaneously.

Fix $q <_{\mathbb{P}_0} p(0)$ such that

1. q is $(X, \mathbb{P}_0)$-generic,
2. $\mathrm{dom}(q) = \mathrm{otp}(X \cap \delta)$,
3. for each $\xi \in X \cap \delta$, if $\kappa_\xi \neq \gamma$ then $q(\mathrm{otp}(X \cap \kappa_\xi)) = \alpha_\xi$ and
4. $q(\mathrm{otp}(X \cap \gamma)) = \beta$.

Construing p as $(p(0), \dot{p})$, we have that hypotheses (a) through (d) of Lemma 3.2 are satisfied. If r is then as being given by the conclusion of Lemma 3.2, then r is as desired.

The reverse direction of part (2) follows from part (1) and the construction, and the fact that the tails of the iteration are semi-proper. (see Remark 3.4).

The forward direction is by our bookkeeping. For each stationary $S \subseteq \omega_1$ in $V[g]$ there exist $\eta_* < \delta$ and $\alpha_* < \omega_1$ (any member of S) such that S is the g_η-realization of some element $\dot{S}$ of N_{η_*,α_*}. Working by induction, it suffices to suppose that the realization of each member of N_{η_*,α_*} which is $\leq_{\eta_*}$-below $\dot{S}$ is certified in $V[g]$. We may then let $\rho \in [\xi_*, \delta)$ be such that all of these certifying sets exist in $V[g \restriction \mathbb{P}_\rho]$. For cofinally many successor ordinals $\xi < \delta$, $\pi(\xi) = (\eta_*, \alpha_*)$. For any such $\xi \geq \rho$, we have by Lemma 2.3 that S is certified at κ_ξ in $V[g \restriction (\xi + 1)]$.

For part (3) that $V[g]$ is a model of PFA follows by the standard consistency proof for PFA. The boldface $\Pi_1^{H_{\aleph_2}}$-definability of NS_{ω_1} in $V[g]$ follows from (2), the parameter being $g(0)$ (or the partition of ω_1 induced by $g(0)$). The failure of CFB follows by adding $q(X \cap \omega_1) > \mathrm{otp}(X \cap \delta)$ to the argument for the forward direction of (1) (we may assume that $\gamma \geq \omega_2$ there, since only club subsets of ω_1 have ω_1 in their tildes). □

The argument provided here isn't tied to the nonstationary ideal. We could code any subset of $\mathcal{P}(\omega_1)$ in the fashion above, as long as the subset is closed under supersets and has a definition which is absolute to stationary set-preserving extensions.

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