# The decomposability conjecture

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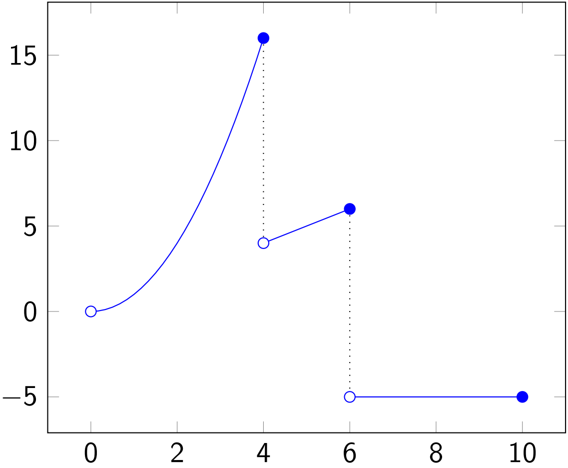
# What functions are piecewise continuous?



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$X$ and $Y$ will denote Polish spaces throughout the talk.
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Then if $A$ is is a $\mathbf{\Sigma}^0_n$ set, $f_i^{-1}(A)$ is relatively $\mathbf{\Sigma}^0_n$ in $\mathrm{dom}(f_i)$ (which is $\mathbf{\Delta}^0_n$), so it is $\mathbf{\Sigma}^0_n$. Thus, $f^{-1}(A) = \bigcup_i f_i^{-1}(A)$ is a countable union of $\mathbf{\Sigma}^0_n$ sets, and hence is $\mathbf{\Sigma}^0_n$.

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**Conjecture (2000's, various authors)**

*$f : X \to Y$ is a union of continuous functions with $\mathbf{\Delta}^0_n$ domains iff the preimage of every $\mathbf{\Sigma}^0_n$ set is $\mathbf{\Sigma}^0_n$.*

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**Theorem (Day-M.)**

*The decomposability conjecture is true assuming $\mathbf{\Sigma}^1_2$ determinacy.*

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These theorems are proved the following way: Suppose $f : X \to Y$ is not a union of Baire class $m$ functions with $\mathbf{\Delta}^0_n$ domains. Then construct a $\mathbf{\Sigma}^0_{n-m+1}$ set $A$ whose preimage is not $\mathbf{\Sigma}^0_n$ (i.e. $f^{-1}(A)$ is $\mathbf{\Pi}^0_n$ hard).

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**Proposition**

*To prove the decomposability conjecture, it's enough to prove the case where $m = n - 1$.*

# Changing topology

Suppose we change our topology $(X, \tau)$ on $X$ to a new Polish topology $(X, \eta)$ where we make countably many $\mathbf{\Pi}^0_n$ sets in $(X, \tau)$ the new basic open sets of $(X, \eta)$.

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The converses of these statements are **very** false.

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# This idea actually works eventually if we just keep trying

Each time our idea fails, we get a canonical new $\mathbf{\Pi}^0_{n-2}$ set to add to our change of topology. (This requires a careful analysis of the $n = 3$, $m = 2$ decomposability proof).

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If we add this set to our change of topology and try again, we'll eventually succeed at some countable ordinal stage. If not, we would contradict the following:

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*Fix $\alpha < \omega_1$. There is no $\omega_1$ length sequence of distinct $\mathbf{\Pi}^0_\alpha$ sets.*

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We also need a new characterization of when a set is $\boldsymbol{\Sigma}^0_n$ hard for $n \geq 3$.

# Characterizing $\mathbf{\Sigma}^0_{n+2}$ hardness

Let $\mathcal{A}$ be a countable collection of subsets of $X$. Let $\tau(\mathcal{A})$ denote the topology generated by the subbasis $\mathcal{A}$, where the open sets are unions of finite intersections of elements of $\mathcal{A}$.

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Given a Polish space $X$, say that $\vec{\mathcal{A}} = \mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n$ is a **suitable sequence of length** $n+1$ of subsets of $X$ iff $\mathcal{A}_0$ is a countable basis of open sets for $X$, $\mathcal{A}_m$ is a countable set of $\mathbf{\Pi}^0_m$ subsets of $X$ for $m \geq 1$, every $\mathcal{A}_m$ is closed under finite intersections, and for all $m < n$,

1. If $B \in \mathcal{A}_0$, then $\overline{B} \in \mathcal{A}_1$, and $\mathcal{A}_m \subseteq \mathcal{A}_{m+1}$ for $m > 0$.
2. If $B \in \mathcal{A}_m$, then $X \setminus B \in \mathcal{A}_{m+1}$.
3. If $B \in \mathcal{A}_{m+1}$, then $B$ is closed in $\tau(\mathcal{A}_m)$.
4. If $B \in \mathcal{A}_{m+1}$ and $m > 0$, then $\overline{B}^{\mathcal{A}_{m-1}} \in \mathcal{A}_m$.

Properties (1)-(3) are simple properties which ensure that the topology $\tau(\mathcal{A}_m)$ is Polish. Property (4) here is the difficult property to satisfy and is key to the following theorem:

# Characterizing $\mathbf{\Sigma}^0_{n+2}$ hardness

**Theorem (Day-M.)**

*Suppose $X$ is Polish, $Y \subseteq X$, and $n \geq 1$. Then $Y$ is $\mathbf{\Sigma}^0_{n+2}$-hard (i.e. there exists a continuous reduction of a complete $\mathbf{\Sigma}^0_{n+2}$ set to $Y$) if and only if there exists a closed set $F \subseteq X$ and a suitable sequence of sets $\mathcal{A}_0, \ldots, \mathcal{A}_n$ on $F$ such that*

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2. *$Y$ is $\tau(\mathcal{A}_{n-1})$-comeager in $A$ for all $A \in \mathcal{A}_n$*

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The proof uses a priority argument. We make heavy use of the true stages machinery of Antonio Montalbán.

Thanks!