Splitting stationary sets from weak forms of Choice *

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Abstract

Working in the context of restricted forms of the Axiom of Choice, we consider the problem of splitting the ordinals below λ of cofinality θ into λ many stationary sets, where $\theta < \lambda$ are regular cardinals. This is a continuation of [5].

In this note we consider the issue of splitting stationary sets in the presence of weak forms of the Axiom of Choice plus the existence of certain types of ladder systems. Our primary interest is the theory ZF + DC plus the assertion that for some large enough cardinal λ, there is a ladder system for the members of λ of countable cofinality, that is, a function that assigns to every such $\alpha < \lambda$ a cofinal subset of ordertype ω. In this context, we show that for every $\gamma < \lambda$ of uncountable cofinality the set of $\alpha < \gamma$ of countable cofinality can be uniformly split into $cf(\gamma)$ many stationary sets. It follows from this and the results of [5] that there is no nontrivial elementary embedding from V into V, under the assumption of ZF + DC plus the assertion that the countable subsets of each ordinal can be wellordered. As a counterpoint to some of the results presented here, we give a symmetric forcing extension in which there are regressive functions on stationary sets not constant on stationary sets.

1 AC and DC

Given a nonempty set Z, the statement AC_Z says that whenever $\langle X_a : a \in Z\rangle$ is a collection of nonempty sets, there is a function f with domain Z such that $f(a) \in X_a$ for each $a \in Z$. If γ is an ordinal, the statement $\mathrm{AC}_{<\gamma}$ says that AC_η holds for all ordinals $\eta < \gamma$.

A *tree* T is a set of functions such that the domain of each function is an ordinal, and such that, whenever $f \in T$ and $\alpha \in dom(f)$, $f \restriction \alpha \in T$. Two

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elements f, g of a tree T are *compatible* if $f \subset g$ or $g \subset f$. A *branch* through a tree is a pairwise compatible collection of elements of T. A branch is *maximal* if it is not properly contained in any other branch.

Given an ordinal γ, the statement DC_γ says that for every tree $T \subset {}^{<\gamma}X$ (for some set X) there is a $b \subset T$ which is a maximal branch. The statement $\mathrm{DC}_{<\gamma}$ says that DC_η holds for all ordinals $\eta < \gamma$. It follows immediately from the definition of DC_γ that DC_γ implies DC_η for all $\eta < \gamma$. We write DC for DC_ω and AC for the statement that AC_Z holds for all sets Z.

Lemma 1.1 shows that DC_γ implies AC_γ for all ordinals γ.

Lemma 1.1. *Suppose that γ is a limit ordinal such that DC_γ holds, and T is a tree such that*

- *every $f \in T$ is a function with domain η, for some $\eta < \gamma$;*
- *for all limit ordinals $\eta < \gamma$, if f is a function with domain η such that $f \restriction \alpha \in T$ for all $\alpha \in \eta$, then $f \in T$;*
- *for every $f \in T$ there is a $g \in T$ properly containing f.*

Then there is a function f with domain γ such that $f \restriction \alpha \in T$ for all $\alpha < \gamma$.

Proof. Let b be a maximal branch of T, and let $f = \bigcup b$. Then f is a function whose domain is an ordinal $\eta \leq \gamma$. If $\eta < \gamma$, then $f \in T$ and f has a proper extension in T, contradicting its supposed maximality. □

2 Ladder systems

Notation. Given an ordinal δ, we let $cf(\delta)$ denote the cofinality of δ. Given an ordinal α and a set A, we let C^α_A denote the ordinals below α whose cofinality is in A. Given an ordinal λ and a function f, we let $\phi(\lambda, f)$ be the statement that there exists a sequence $\langle c_\delta : \delta \in C^\lambda_{dom(f)} \rangle$ such that each c_δ is a cofinal subset of δ of ordertype less than $f(cf(\delta))$.

Note that $\phi(\lambda, f)$ implies that $f(\gamma) \geq \gamma + 1$ for all regular cardinals $\gamma \in dom(f)$.

Notation. We let $\psi(\lambda, \theta)$ be the statement $\phi(\lambda, \{(\theta, \theta + 1)\})$. We say that a sequence $\langle c_\delta : \delta \in C^\lambda_{dom(f)} \rangle$ *witnesses* $\phi(\lambda, f)$ if each c_δ is a cofinal subset of δ of ordertype less than $f(cf(\delta))$, and similarly for $\psi(\lambda, \theta)$.

The statement $\psi(\lambda, \omega)$ follows from the statement Ax^2_λ of [5] (in the case $\partial = \omega$), which says that there exists a well-orderable $\mathcal{A} \subset [\lambda]^{\aleph_0}$ such that every element of $[\lambda]^{\aleph_0}$ has infinite intersection with a member of $\mathcal{A}$. We will be primarily interested in statements $\phi(\lambda, f)$ where f is either the ordinal successor function or the cardinal successor function on some set of regular cardinals. The two following lemmas show that when the domain of f is a single regular cardinal, there is in some sense no statement strictly in between these two.

Lemma 2.1 (ZF)**.** *For each ordinal ordinal γ there exists a sequence*

$$\langle e_\delta : \delta < \gamma \rangle$$

such that each e_δ is a cofinal subset of δ of ordertype less than or equal to $|\gamma|$.

Proof. Let $\pi\colon |\gamma| \to \gamma$ be a bijection. For each $\delta < \gamma$, let e_δ be the set of ordinals of the form $\pi(\alpha)$, where $\alpha < |\gamma|$, $\pi(\alpha) < \delta$ and $\pi(\alpha) > \pi(\beta)$ for all $\beta < \alpha$ with $\pi(\beta) < \delta$. □

Notation. Given a set x of ordinals, we let $o.t.(x)$ denote the ordertype of x. Given an ordinal $\eta < o.t.(x)$, we let $x(\eta)$ be the η-th member of x, i.e., the unique $\alpha \in x$ such that $o.t.(x \cap \alpha) = \eta$.

Lemma 2.2 (ZF)**.** *Let λ be an ordinal, let θ be a regular cardinal, and let η be an ordinal less than θ^+. Then $\phi(\lambda, \{(\theta, \eta)\})$ implies $\psi(\lambda, \theta)$.*

Proof. Let $\langle c_\delta : \delta \in C^\lambda_{\{\theta\}} \rangle$ witness $\phi(\lambda, \{(\theta, \eta)\})$, and let $\langle e_\delta : \delta < \eta \rangle$ be such that each e_δ is a cofinal subset of δ of ordertype less than or equal to θ. For each $\delta \in C^\lambda_{\{\theta\}}$, letting α_δ be the ordertype of c_δ, let $d_\delta = \{c_\delta(\beta) \mid \beta \in e_{\alpha_\delta}\}$. Then each d_δ is a cofinal subset of δ of ordertype θ. □

3 Splitting C^λ_θ from DC_θ and AC_γ

Notation. Given ordinals α, β, η and a sequence of sets of ordinals $\bar{C} = \langle c_\delta : \delta \in S \rangle$ (for some set S), we let $S^\eta_{\alpha,\beta}(\bar{C})$ be the set of $\delta \in S$ such that $o.t.(c_\delta) > \eta$ and $c_\delta(\eta) \in [\alpha, \beta)$.

We are primarily interested in the following theorem in the case where θ and γ are both ω, in which case $\psi(\lambda, \omega)$ implies the existence of a sequence $\bar{C}$ satisfying the stated hypotheses.

Theorem 3.1 (ZF)**.** *Suppose that the following hold.*

- *$\theta \geq \aleph_0$ is a regular cardinal such that DC_θ holds;*
- *$\gamma \geq \theta$ is an ordinal such that AC_γ holds;*
- *λ is an ordinal of cofinality greater than γ;*
- *E is a club subset of λ;*
- *$\bar{C} = \langle c_\delta : \delta \in C^\lambda_{\{\theta\}} \cap E \rangle$ is a sequence such that each c_δ is a cofinal subset of δ of ordertype less than or equal to γ.*

Then

1. *there exists an $\eta^* < \gamma$ such that for each $\alpha < \lambda$ there exists a $\beta \in (\alpha, \lambda)$ such that $S^{\eta^*}_{\alpha,\beta}(\bar{C})$ is a stationary subset of λ;*

2. *there exist functions* $g\colon C^{\lambda}_{(\gamma,\lambda)} \to \gamma$, $h\colon C^{\lambda}_{(\gamma,\lambda)} \to \lambda$ *and a collection of ordinals* $\langle \alpha^{\xi}_{\beta} : \xi \in C^{\lambda}_{(\gamma,\lambda)}, \beta < h(\xi)\rangle$ *such that*

    - *for each* $\xi \in C^{\lambda}_{(\gamma,\lambda)}$, $h(\xi) < cf(\xi)^{+}$*;*
    - *for each* $\xi \in C^{\lambda}_{(\gamma,\lambda)}$, $\langle \alpha^{\xi}_{\beta} : \beta < h(\xi)\rangle$ *is a continuous increasing sequence cofinal in* ξ*,*
    - *for each* $\xi \in C^{\lambda}_{(\gamma,\lambda)}$ *and each* $\beta < h(\xi)$, $S^{g(\xi)}_{\alpha^{\xi}_{\beta},\alpha^{\xi}_{\beta+1}}(\bar{C}\restriction\xi)$ *is stationary.*

Proof. We prove the first part first. Supposing that there is no such η^*, for each $\eta < \gamma$ let $\alpha^*_{\eta} < \lambda$ be the least $\alpha < \lambda$ such that $S^{\eta}_{\alpha,\beta}(\bar{C})$ is nonstationary for all $\beta \in (\alpha, \lambda)$. Using the fact that $cf(\lambda) > \gamma$, let α^* be the least element of $C^{\lambda}_{\{\theta\}} \cap E$ greater than or equal to the supremum of $\{\alpha^*_{\eta} : \eta < \gamma\}$. Now, applying DC_{θ} and AC_{γ}, we choose a continuous increasing sequence of ordinals $\langle \alpha_{\xi} : \xi < \theta\rangle$ and sets $D_{\xi,\eta}$ $(\xi < \theta, \eta < \gamma)$ by recursion on $\xi < \theta$ such that

1. $\alpha_0 = \alpha^*$
2. each $D_{\xi,\eta}$ is a club subset of E disjoint from $S^{\eta}_{\alpha^*,\alpha_{\xi}}(\bar{C})$
3. if $\xi < \theta$ a limit ordinal then $\alpha_{\xi} = \bigcup\{\alpha_{\zeta} : \zeta < \xi\}$
4. if $\xi = \zeta + 1$, then $\alpha_{\xi} = \min(\bigcap_{\rho \leq \zeta, \eta < \gamma} D_{\rho,\eta} \setminus (\alpha_{\zeta} + 1))$

Let $\alpha_{\theta} = \bigcup\{\alpha_{\xi} : \xi < \theta\}$. Then $\alpha_{\theta} < \lambda$ as $cf(\lambda) > \theta$, so $\alpha_{\theta} \in C^{\lambda}_{\{\theta\}} \cap E$. For some $\eta < \gamma$, $c_{\alpha_{\theta}}(\eta) > \alpha^*$, hence for some $\xi < \theta$, $c_{\alpha_{\theta}}(\eta) \in [\alpha_*, \alpha_{\xi})$. Then $\alpha_{\theta} \in S^{\eta}_{\alpha_0,\alpha_{\xi}}(\bar{C})$, contradicting the assumption that $\alpha_{\theta} \in D_{\xi,\eta}$.

To prove the second part of the lemma, fix $\xi \in C^{\lambda}_{(\gamma,\lambda)}$. Applying the first part of the lemma with ξ as λ, let $g(\xi)$ be the least $\eta \in \gamma$ such that for each $\alpha < \xi$ there exists a $\beta \in (\alpha, \xi)$ such that $S^{\eta}_{\alpha,\beta}(\bar{C}\restriction\xi)$ is a stationary subset of ξ. Then by recursion on $\beta < \xi$ we can choose an increasing continuous sequence of ordinals $\alpha^{\xi}_{\beta} < \lambda$ $(\beta < \xi)$ such that $\alpha^{\xi}_{0} = 0$, $\alpha^{\xi}_{\beta} = \cup\{\alpha^{\xi}_{\zeta} : \zeta < \beta\}$ for limit β, and, if $\beta = \zeta + 1$, then, if $\alpha^{\xi}_{\zeta} = \xi$ then $\alpha^{\xi}_{\beta} = \xi$, otherwise α^{ξ}_{β} is the minimal ordinal $\delta \in (\alpha^{\xi}_{\zeta}, \xi)$ such that $S^{g(\xi)}_{\alpha^{\xi}_{\zeta},\delta}(\bar{C}\restriction\xi)$ is stationary. Let $h(\xi)$ be the least β such that $\alpha^{\xi}_{\beta} = \xi$ if some such β exists, and let it be ξ otherwise. Since there is a club subset of ξ of cardinality $cf(\xi)$, and the sets $S^{g(\xi)}_{\alpha^{\xi}_{\beta},\alpha^{\xi}_{\beta+1}}(\bar{C}\restriction\xi)$ $(\beta < h(\xi))$ are disjoint stationary subsets of ξ, $h(\xi) < cf(\xi)^{+}$. This completes the definitions of g, h and $\langle \alpha^{\xi}_{\beta} : \xi \in C^{\lambda}_{(\gamma,\lambda)}, \beta < h(\xi)\rangle$. □

Corollary 3.2. *Suppose that the following hold.*

- $\theta \geq \aleph_0$ *is a regular cardinal such that* DC_{θ} *holds;*
- λ *is an ordinal of cofinality greater than* θ*;*
- A *is the set of regular cardinals in the interval* $[\theta, \lambda)$*.*

Then $\psi(\lambda,\theta)$ implies $\phi(\lambda,f)$, where f is the cardinal successor function on A.

The following is a consequence of the results of [5], Woodin's proof of Kunen's Theorem (see [2]) and the arguments in this section.

Corollary 3.3 (ZF + DC)**.** *Assume that for every ordinal λ there exists a wellorderable set $\mathcal{A} \subset [\lambda]^{\aleph_0}$ such that every element of $[\lambda]^{\aleph_0}$ has infinite intersection with a member of $\mathcal{A}$. Then there is no nontrivial elementary embedding from V into V.*

Proof. Suppose towards a contradiction that $j\colon V \to V$ is an elementary embedding. Let κ_0 be the critical point of j, and for each nonzero $n < \omega$, let $\kappa_{n+1} = j(\kappa_n)$. Let $\kappa_\omega = \cup\{\kappa_n : n < \omega\}$. Then $j(\kappa_\omega) = \kappa_\omega$ and $j(\kappa_\omega^+) = \kappa_\omega^+$. For no $\alpha < \kappa_0$ is there is a surjection from V_α onto κ_0 (to see this, consider $j(\pi)$, where π is such a surjection, in light of the fact that $j{\restriction}V_{\kappa_0}$ is the identity function). By elementarity, then, the same is true for each κ_n, and so the same is true for κ_ω. Then by the results of [5] (specifically, Lemma 2.13), κ_ω^+ is regular.

Let $\bar{C} = \langle c_\delta : \delta \in S^{\kappa_\omega^+}_\omega\rangle$ witness $\psi(\kappa_\omega^+,\omega)$. Applying Theorem 3.1, let $n_* \in \omega$ and $\bar{\alpha} = \langle \alpha_\xi : \xi < \kappa_\omega^+\rangle$ be such that $\bar{\alpha}$ is a continuous increasing sequence of elements of κ_ω^+ and such that $S^{n_*}_{\alpha_\xi,\alpha_{\xi+1}}(\bar{C})$ is a stationary subset of κ_ω^+ for each $\xi < \kappa_\omega^+$.

Let F be the set of limit ordinals $\delta < \kappa_\omega^+$ such that $j(\alpha) < \delta$ for every $\alpha < \delta$. Then F is a club. Let E be the set of members of F of cofinality less than κ_0. Then $j{\restriction}E$ is the identity function, and no stationary set of ordinals of countable cofinality is disjoint from E.

Let $\langle S'_\xi : \xi < \kappa_\omega^+\rangle = j(\langle S^{n_*}_{\alpha_\xi,\alpha_{\xi+1}} : \xi < \kappa_\omega^+\rangle)$. As j is an elementary embedding, $V \models$ "S'_{κ_0} is a stationary subset of λ disjoint from S'_ξ for $\xi \in \kappa_\omega^+ \setminus \{\kappa_0\}$." Hence, S'_{κ_0} is disjoint from $S'_{j(\xi)}$, for all $\xi < \kappa_\omega^+$. But

$$\bigcup_{\xi<\kappa_\omega^+} S'_{j(\xi)} \supset \bigcup_{\xi<\kappa_\omega^+} (S'_{j(\xi)} \cap E) = \bigcup_{\xi<\kappa_\omega^+} (S^{n_*}_{\alpha_\xi,\alpha_{\xi+1}} \cap E) = E.$$

□

4 Club guessing

In this section we show that the standard club-guessing arguments go through under weak forms of Choice plus the existence of ladder systems. Theorem 4.1 uses forms of DC, and Theorem 4.3 uses AC.

Theorem 4.1 (ZF)**.** *Let $\theta < \lambda$ be regular cardinals, with $\theta^+ < \lambda$, and suppose that DC_{θ^+} holds. Suppose that $\langle c_\delta : \delta \in C^\lambda_{\{\theta\}}\rangle$ is a sequence such that each c_δ is a closed cofinal subset of δ of ordertype less than θ^+. Then the following hold.*

- *There exists a sequence $\langle d_\delta : \delta \in C^\lambda_{\{\aleph_0\}}\rangle$ such that each d_δ is a cofinal subset of δ, and such that for every club subset $D \subset \lambda$ there is a $\delta \in C^\lambda_{\{\aleph_0\}}$ with $d_\delta \subset D$.*

- *If θ is uncountable, then there exists a sequence $\langle d_\delta : \delta \in C^\lambda_{\{\theta\}}\rangle$ such that each d_δ is a closed cofinal subset of c_δ, and such that for every club subset $D \subset \lambda$ there is a $\delta \in C^\lambda_{\{\theta\}}$ with $d_\delta \subset D$.*

Proof. We argue as in [4], Chapter III.

For the first part, for any two sets A, B, let $gl(A, B)$ denote the set

$$\{\sup(\alpha \cap B) \mid \alpha \in A \setminus (\min(B) + 1)\}.$$

Note that if A and $B \cap \gamma$ are club subsets of an ordinal γ, then $gl(A, B)$ is a club subset of $B \cap \gamma$ as well.

Supposing that the first conclusion of the theorem is false, choose for each $\zeta \leq \theta^+$ a club subset $D_\zeta \subset \lambda$ such that the following conditions are satisfied.

- D_0 does not contain c_δ for any $\delta \in C^\lambda_{\{\theta\}}$;
- for each $\zeta < \theta^+$, $D_{\zeta+1}$ is contained in the limit points of D_ζ, and $D_{\zeta+1}$ does not contain $gl(c_\delta, D_\zeta)$ for any $\delta \in C^\lambda_{\{\theta\}}$ which is a limit point of D_ζ.
- for each limit ordinal $\zeta \leq \theta^+$, $D_\zeta = \bigcap_{\xi<\zeta} D_\xi$.

Now fix a $\delta \in C^\lambda_{\{\theta\}}$ which is a limit point of D_{θ^+}. For each $\alpha \in c_\delta$, either there is a $\zeta < \theta^+$ such that $\alpha \leq \min(D_\zeta)$, or $\langle \sup(\alpha \cap D_\zeta) : \zeta < \theta^+\rangle$ is a nonincreasing sequence which reaches an eventually constant value. Since $|c_\delta| < \theta^+$, there is a $\zeta < \theta^+$ such that for each $\alpha \in c_\delta$, $\alpha > \min(D_\zeta)$ implies $\alpha > \min(D_{\zeta+1})$, and, if $\alpha > \min(D_\zeta)$, then $\sup(\alpha \cap D_\zeta) = \sup(\alpha \cap D_{\zeta+1})$. Then $gl(c_\delta, D_\zeta) = gl(c_\delta, D_{\zeta+1})$. However, $gl(c_\delta, D_{\zeta+1}) \subset D_{\zeta+1}$ and $D_{\zeta+1}$ was chosen not to contain $gl(c_\delta, D_\zeta)$, giving a contradiction.

For the second part, note that we can just take the intersection of c_δ and d_δ for each $\delta \in C^\lambda_{\{\theta\}}$, where d_δ is given by the first part. □

4.2 Question. Does DC_θ suffice for Theorem 4.1?

Theorem 4.3 (ZF)**.** *Suppose that*

- *$\theta < \lambda$ are regular uncountable cardinals;*
- *there is no surjection from $\mathcal{P}(\theta)$ to λ;*
- *AC_X holds, where X is the union of θ^+ and the set of club subsets of θ;*
- *$\langle c_\delta : \delta \in C^\lambda_{\{\theta\}}\rangle$ is a sequence such that each c_δ is a closed cofinal subset of δ of ordertype less than θ^+.*

Then there exists a sequence $\langle e_\delta : \delta \in C^\lambda_{\{\theta\}}\rangle$ such that each e_δ is a closed cofinal subset of c_δ of ordertype θ, and such that for every club subset $D \subset \lambda$ there is a $\delta \in C^\lambda_{\{\theta\}}$ with $e_\delta \subset D$.

Proof. Applying AC_X, let $\bar{D} = \langle d_\delta : \delta \in C^{\theta^+}_{\{\theta\}} \rangle$ be such that each d_δ is a club subset of δ of ordertype θ. For each $\delta \in C^\lambda_{\{\theta\}}$, let $c'_\delta = c_\delta \cap d_\delta$. Then each c'_δ is a closed, cofinal subset of δ of ordertype θ.

For each $\delta \in C^\lambda_{\{\theta\}}$ and for each club $C \subset \theta$, let $c(C)_\delta = \{c'_\delta(\beta) \mid \beta \in C\}$. Supposing that the conclusion fails, choose $\langle E_C : C \subset \theta \text{ club}\rangle$ such that each E_C is a club subset of λ not containing $c(C)_\delta$ for any $\delta \in C^\lambda_{\{\theta\}}$. As there is no surjection from $\mathcal{P}(\theta)$ to λ,

$$E = \bigcap \{E_C : C \subset \theta \text{ club}\}$$

is a club subset of λ. Let δ be any limit member of E in $C^\lambda_{\{\theta\}}$, and let

$$C = \{\alpha < \gamma \mid c'_\delta(\alpha) \in E\}.$$

Then $c(C)_\delta = c'_\delta \cap E \subset E_C$, contradicting the choice of E_C. □

5 Splitting at higher cofinalities

In this section we consider the problem of using a ladder system to split $C^\lambda_{\{\theta\}}$ into stationary sets without the help of AC and DC. So the difference is that we try to split at cofinality θ without DC_θ.

Theorem 5.1 (ZF)**.** *Suppose that the following hold.*

- *$\theta < \lambda$ are regular uncountable cardinals;*
- *$\gamma \in [\theta, \lambda)$ is an ordinal;*
- *$\bar{C} = \langle c_\delta : \delta \in C^\lambda_{\{\theta\}} \rangle$ is a sequence such that each c_δ is a cofinal subset of δ of ordertype less than or equal to γ.*

Then either

1. *there exist $\eta < \gamma$ and a continuous increasing sequence $\langle \alpha_\xi : \xi < \lambda \rangle$ such that each $\alpha_\xi \in \lambda$ and each $S^\eta_{\alpha_\xi, \alpha_{\xi+1}}(\bar{C})$ is stationary, or*
2. *the following two statements hold:*
   - *(a) for some club $E \subset \lambda$ there exists a regressive function F on $E \cap C^\lambda_{\{\theta\}}$ such that $F^{-1}\{\beta\}$ not stationary for any $\beta < \lambda$, and*
   - *(b) if AC_γ holds, then for some $\alpha_* < \lambda$ there is a regressive function G on $C^\lambda_{\{\theta\}} \setminus (\alpha_* + 1)$ such that for each $\beta < \lambda$ the set of $\gamma \in C^\lambda_{\{\theta\}}$ such that $G(\gamma) < \beta$ is not stationary.*

Proof. Suppose first that there exists a $\eta < \gamma$ such that for each $\alpha < \lambda$ there exists a $\beta \in (\alpha, \lambda)$ such that $S^\eta_{\alpha,\beta}(\bar{C})$ is a stationary subset of λ. Then we can choose $\alpha_\xi < \lambda$ ($\xi < \lambda$) by induction on ξ, increasing continuously with ξ, such that $\alpha_0 = 0$ and

$$\alpha_{\xi+1} = min\{\alpha : \alpha_\xi < \alpha < \lambda \wedge S^\eta_{\alpha_\xi,\alpha}(\bar{C}) \text{ is stationary}\}.$$

Then the first conclusion of the lemma holds.

Suppose instead that there is no such η. Then for each $\eta < \gamma$, let $\alpha^*_\eta < \lambda$ be minimal such that for all $\beta \in (\alpha^*_\eta, \lambda)$, $S^\eta_{\alpha^*_\eta,\beta}(\bar{C})$ not a stationary subset of λ. Let $\alpha_* = sup\{\alpha^*_\eta : \eta < \gamma\}$. Then $\alpha_* < \lambda$, as $\lambda = cf(\lambda) > \gamma$.

Define $F\colon C^\lambda_{\{\theta\}} \setminus (\alpha_* + 1) \to \lambda \times \gamma$ by letting $F(\delta) = (\alpha, \eta)$ if α is the least element of c_δ greater than α_* and $\alpha = c_\delta(\eta)$. Then for no $(\alpha, \eta) \in \lambda \times \gamma$ is $F^{-1}\{(\alpha, \eta)\}$ stationary.

Let $H\colon \lambda \times \gamma \to \lambda$ be the function $H(\alpha, \eta) = \gamma \cdot \alpha + \eta$, and let E be the set of $\alpha \in (\alpha_*, \lambda)$ such that $H(\beta, \eta) < \alpha$ for all $\beta < \alpha$ and $\eta < \gamma$. Then E is a club set. Furthermore, the function $H \circ F$ is regressive on $E \cap C^\lambda_{\{\theta\}}$ and not constant on a stationary set, as desired.

Finally, suppose that AC_γ holds. For each $\beta \in (\alpha_*, \lambda)$ and each $\eta < \gamma$, $S^\eta_{\alpha_*,\beta}(\bar{C})$ is nonstationary. It follows (from AC_γ) that for each $\beta \in (\alpha_0, \lambda)$, $S_\beta = \bigcup_{\eta<\gamma} S^\eta_{\alpha_*,\beta}(\bar{C})$ is nonstationary. Now define $G\colon C^\lambda_{\{\theta\}} \setminus (\alpha_* + 1) \to \lambda$ by letting $G(\delta)$ be the least element of c_δ greater than α_*. Then for every $\beta \in \lambda$, the set of $\delta \in C^\lambda_\theta \setminus (\alpha_* + 1)$ with $G(\delta) < \beta$ is nonstationary. □

6 A model of ZF and a regressive function

In this section we give a proof of the following theorem, which is complementary to Theorem 5.1.

Theorem 6.1 (ZFC)**.** *Let $\theta < \lambda$ be regular cardinals. There is a partial order P such that in the P-extension of V there is an inner model M with the following properties.*

- *M and V have the same ordinals of cofinality θ;*
- *λ is a regular cardinal in M;*
- *M satisfies ZF + $DC_{<\theta}$ + $\phi(\lambda, f)$, where f is the ordinal successor function on the regular cardinals below θ;*
- *there exists in M a regressive function on $(C^\lambda_{[\theta,\lambda)})^M$ which is not constant on a stationary set.*

The strategy for the proof is a direct modification of Cohen's original proof of the independence of AC (see [1]).

Assume that ZFC holds and that $\theta < \lambda$ are uncountable regular cardinals. Given a set $X \subset \lambda \times \lambda$, let P_X be the partial order whose conditions consist of pairs (f, d) such that

- f is a regressive function on $C^\lambda_{[\theta,\lambda)}$ whose domain is $\alpha \cap C^\lambda_{[\theta,\lambda)}$ for some successor ordinal $\alpha < \lambda$;
- d is a partial function whose domain is a subset of X of cardinality less than λ such that for each (α, β) in the domain of d, $d(\alpha, \beta)$ is a closed, bounded subset of $max(dom(f)) + 1$ disjoint from $f^{-1}\{\alpha\}$.

The order on P_X is given by: $(f, d) \leq (g, e)$ if $g \subset f$, $dom(e) \subset dom(d)$ and $d(\alpha, \beta) \cap (max(dom(g)) + 1) = e(\alpha, \beta)$ for all $(\alpha, \beta) \in dom(e)$.

The partial order P_X is closed under decreasing sequences of length less than θ and therefore does not add sets of ordinals of cardinality less than θ. Furthermore, if $|X|^+ < \lambda$, then below densely many conditions (conditions (f, d) with $|dom(f)| > |X|$) every descending sequence in P_X of length less than λ has a lower bound, so P_X does not add sequences from V of length less than λ. We will see below that P_X is in some sense homogeneous.

Given $X \subset \lambda \times \lambda$ and a regressive function F on $C^\lambda_{[\theta,\lambda)}$, let $Q_{F,X}$ denote the partial order whose conditions are partial functions d with domain a subset of X of cardinality less than λ, such that for each (α, β) in the domain of d, $d(\alpha, \beta)$ is a closed, bounded subset of λ disjoint from $F^{-1}\{\alpha\}$. If X is a subset of $\lambda \times \lambda$ such that $|X|^+ < \lambda$, and $Y \subset \lambda \times \lambda$ is disjoint from X, then, since P_X does not add bounded subsets of λ, $P_{X\cup Y}$ is forcing-isomorphic to $P_X * Q_{\dot{F},Y}$, where $\dot{F}$ represents the generic regressive function added by P_X.

Let $\bar{D} = \langle d_\delta : \delta \in C^\lambda_{\{\theta\}} \rangle$ be a sequence in V such that each d_δ is a cofinal subset of δ of ordertype $cf(\delta)$. Let ${}^{<\theta}Ord$ be the class of functions whose domain is an ordinal less than θ and whose range is contained in the ordinals.

A V-generic filter for P_X is naturally represented by a pair $(F, \bar{C})$, where F is a regressive function on $(C^\lambda_{[\theta,\lambda)})^V$, $\bar{C}$ has the form $\langle C_{\alpha,\beta} : (\alpha, \beta) \in X \rangle$, and each $C_{\alpha,\beta}$ is a club subset of λ disjoint from $F^{-1}\{\alpha\}$. Fixing such a pair, let M be the smallest transitive inner model of ZF containing $\bar{D}$, F, ${}^{<\theta}Ord$ and every function from γ to $\{C_{\alpha,\beta} : (\alpha, \beta) \in X\}$, for any $\gamma < \theta$. Every set in M is definable in M from $\bar{D}$, F, a member of ${}^{<\theta}Ord$ and a function from a $\gamma < \theta$ to $\{C_{\alpha\beta} : (\alpha, \beta) \in X\}$. It follows that M is closed under sequences of length less than θ in $V[F, \bar{C}]$, and therefore that M satisfies $\mathrm{DC}_{<\theta}$. Since $\bar{D}$ is in M, and since V and $V[F, \bar{C}]$ have the same ordinals of cofinality less than θ, M satisfies $\phi(\lambda, f)$, where f is the ordinal successor function on the regular cardinals below θ. Since $V[F, \bar{C}]$ and V have the same sequences of ordinals of length less than θ, M is definable in $V[F, \bar{C}]$ from $\bar{D}$, F and the (unordered) set $\{C_{\alpha,\beta} : (\alpha, \beta) \in X\}$.

Given $Y \subset X$, let N_Y denote $V[F, \langle C_{\alpha,\beta} : (\alpha, \beta) \in Y \rangle]$.

Lemma 6.2. *Suppose that $X = Z \times Z$, for some uncountable $Z \subset \lambda$, and that $(F, \bar{C})$ is V-generic for P_X. Then every subset of V in M exists in N_Y for some $Y \subset X$ of cardinality less than θ.*

Proof. Given such a set A, we can fix $Y \subset X$ of cardinality less than θ such that Y is of the form $W \times W$ for some $W \subset \lambda$ and such that A is definable in M

from $\bar{D}$, F, an element x of ${}^{<\theta}Ord$ and $\langle C_{\alpha,\beta} : (\alpha,\beta) \in Y\rangle$. Let ϕ be a formula such that

$$A = \{a \mid M \models \phi(a, \bar{D}, F, x, \langle C_{\alpha,\beta} : (\alpha,\beta) \in Y\rangle)\}.$$

We have that P_X is forcing-equivalent to $P_Y * Q_{\dot{F}, X\setminus Y}$. Suppose that there are two conditions d and e in $Q_{F,X\setminus Y}$ (in N_Y) and some $a \in V$ such that

$$d \Vdash M \models \phi(a, \bar{D}, F, x, \langle C_{\alpha,\beta} : (\alpha,\beta) \in Y\rangle)$$

and

$$e \Vdash M \models \neg\phi(a, \bar{D}, F, x, \langle C_{\alpha,\beta} : (\alpha,\beta) \in Y\rangle).$$

There are conditions $d' \leq d$ and $e' \leq e$ in $Q_{F,X\setminus Y}$ such that

- for every $(\alpha,\beta) \in dom(d')$ there is a β' such that $(\alpha,\beta') \in dom(e')$ and $e'(\alpha,\beta') = d'(\alpha,\beta)$, and
- for every $(\alpha,\beta) \in dom(e')$ there is a β' such that $(\alpha,\beta') \in dom(d')$ and $d'(\alpha,\beta') = e'(\alpha,\beta)$.

There is then a natural isomorphism π between $Q_{F,X\setminus Y}$ below d' and $Q_{F,X\setminus Y}$ below e'. This isomorphism π has the property that, given two generic filters $G_{d'}$ and $G_{e'}$ for $Q_{F,X\setminus Y}$ with $\pi[G_{d'}] = G_{e'}$, the (unordered) generic set $\{C_{\alpha,\beta} : (\alpha,\beta) \in X \setminus Y\}$ is the same in the two extensions. Then M is the same model in the two extensions, contradicting the claim that

$$d \Vdash M \models \phi(a, \bar{D}, F, x, \langle C_{\alpha,\beta} : (\alpha,\beta) \in Y\rangle)$$

and

$$e \Vdash M \models \neg\phi(a, \bar{D}, F, x, \langle C_{\alpha,\beta} : (\alpha,\beta) \in Y\rangle).$$

□

It follows from Lemma 6.2 that M and V have the same sequences of ordinals of length less than λ, so λ is a regular cardinal in M. In the case that $X = \lambda \times \lambda$, then, M satisfies ZF + $\mathrm{DC}_{<\theta}$ + $\phi(\lambda, f)$, where f is the ordinal successor function on the regular cardinals below θ, and there exists in M a regressive function on $C^{\lambda}_{[\theta,\lambda)}$ which is not constant on a stationary set.

References

[1] T. Jech, **Set Theory**, Springer-Verlag, Berlin, 2003

[2] A. Kanamori, **The Higher Infinite. Large cardinals in set theory from their beginnings**, 2nd edition, Springer Monographs in Mathematics, 2003

[3] S. Shelah, **Proper and Improper Forcing**, Springer-Verlag, Berlin, 1998

[4] S. Shelah, **Cardinal Arithmetic**, Oxford University Press, 1994

[5] S. Shelah, *PCF without choice*, publication number 835, preprint