Groundedness of infinitary sentences

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Joint work with
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Descriptive set theory of countable models

L countable. Let $X_L = \{$all L-structures with universe $\omega\}$.

- X_L Polish via $U_{\varphi(\overline{e})} := \{M \in X_L : M \models \varphi(\overline{e})\}$ clopen for all $\varphi, \overline{e} \in \omega^n$.
- $S_\infty = Sym(\omega)$ acts on X_L via homeomorphisms: $\sigma \cdot M \models \varphi(\sigma^{-1}(\overline{e})) \Leftrightarrow M \models \varphi(\overline{e})$.
- For a theory T or $\Phi \in L_{\omega_1,\omega}$, $\mathrm{Mod}_\omega(\Phi)$ is a Borel subset of X_L, invariant under the action of S_∞.
- The only Borel, invariant subspaces of X_L are $\mathrm{Mod}_\omega(\Phi)$ for some $\Phi \in L_{\omega_1,\omega}$.

Borel complexity

Study the complexity of $\mathrm{Mod}_\omega(T)/\cong_T$ (or $\mathrm{Mod}_\omega(\Phi)/\cong_\Phi$)

Friedman-Stanley: $(\mathrm{Mod}_\omega(\Phi), \cong_{L_1})$ is Borel reducible to $(\mathrm{Mod}_\omega(\Psi), \cong_{L_2})$, $\Phi \leq_B \Psi$, if there is a Borel $f : \mathrm{Mod}_\omega(\Phi) \to \mathrm{Mod}_\omega(\Psi)$ with $M \cong_{L_1} N$ iff $f(M) \cong_{L_2} f(N)$.

- $\cong_\Phi$ is always $\mathbf{\Sigma}^1_1$ (analytic) $[M \cong N$ iff $\exists f(\cdots)]$ but sometimes is Borel.

Maximal complexity

Φ is Borel complete if $\Psi \leq_B \Phi$ for all $\Psi \in L_{\omega_1,\omega}$.
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Until recently, only known example of Φ with $\cong_\Phi$ non-Borel, but not Borel complete was 'Abelian p-groups' where countable models are characterized by the Ulm invariants.

A first-order example

Ulrich-Rast-L: $\cong$ is not Borel for the first-order, weakly minimal REF(bin)=Th$(2^\omega, E_n)_{n\in\omega}$ (where $E_n(\eta,\nu)$ iff $\eta|n = \nu|n$).

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Show: REF(bin) is not Borel complete.

Strategy for showing non-Borel completeness

- Every countable M has a canonical Scott sentence $css(M) \in L_{\omega_1,\omega}$ characterizing M up to isomorphism among countable L-structures. Let $CSS(\Phi) := \{css(M) : M \in \mathrm{Mod}_\omega(\Phi)\}$.

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for every $\mathbb{V}[G] \supseteq \mathbb{V}$.

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Barwise: Every M (of any cardinality) has a canonical Scott sentence $\varphi_M \in L_{\infty,\omega}$ such that for any M, N

$$M \equiv_{\infty,\omega} N \quad \text{if and only if} \quad \varphi_M = \varphi_N$$

Groundedness

Credo #2: Every $\varphi \in L_{\infty,\omega}$ is potentially in $L_{\omega_1,\omega}$, i.e., $(\varphi \in L_{\omega_1,\omega})^{\mathbb{V}[G]}$ for some $\mathbb{V}[G] \supseteq \mathbb{V}$.

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$\Phi \in L_{\omega_1,\omega}$ is grounded if, for every $\varphi \in L_{\infty,\omega} \cap \mathbb{V}$ and $\varphi \models \Phi$, if φ has a model in some $\mathbb{V}[G] \supseteq \mathbb{V}$, then φ has a model in $\mathbb{V}$.

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Theorem (Ulrich-Rast-L)

If Φ, Ψ are both grounded and $\Phi \leq_B \Psi$, then $||\Phi|| \leq ||\Psi||$.

An application

Theorem (Ulrich-Rast-L)

1. *REF(bin) is grounded.*
2. *If T is weakly minimal, then* $|Mod(\Phi)/\equiv_{\infty,\omega}| \leq \beth_2$, *hence* $||REF(bin)|| = \beth_2$.

Corollary (Ulrich-Rast-L)

REF(bin) is not Borel complete. In fact, 'countable sets of countable sets of reals' $\not\leq_B REF(bin)$.

Understanding groundedness

Recall: Φ is grounded if, for any $\varphi \in L_{\infty,\omega} \cap \mathbb{V}$ with $\varphi \models \Phi$, if φ has a model in some $\mathbb{V}[G] \supseteq \mathbb{V}$, then φ has a model in $\mathbb{V}$.

For $\varphi \in L_{\omega_1,\omega}$, life is good.

Theorem (Karp's Completeness Theorem)

The following are equivalent for $\varphi \in L_{\omega_1,\omega}$:

1. *φ has a model;*
2. *φ has countable model in X_L;*
3. *φ does not have any 'formal contradictions';*
4. *$\mathbb{V} \models (\exists M \models \varphi)$ if and only if $\mathbb{V}[G] \models (\exists M \models \varphi)$ for some/every forcing extension $\mathbb{V}[G]$.*

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Thus, witnesses φ to non-groundedness of Φ must be in $L_{\infty,\omega}$ but not $L_{\omega_1,\omega}$.

A non-example

Let $L = \{U_n : n \in \omega\}$ and T:='Independent unary predicates'. Say $2^{\aleph_0} = \kappa$ and let $\{s_i : i \in \kappa\}$ enumerate $\mathscr{P}(\omega)$.

$$\delta := \bigwedge T \wedge \bigwedge_{i\in\kappa} \exists! x \Big(\bigwedge_{n\in s_i} U_n(x) \wedge \bigwedge_{n\notin s_i} \neg U_n(x) \Big)$$

$\delta \in L_{\kappa^+,\omega}$ and has a unique model (of size κ).

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Let $\theta_k := \bigwedge T_k \wedge \bigwedge_{n \in \omega} \forall \overline{x}(\overline{x}$ realizes a complete formula).

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Thus, if $\kappa > \aleph_k$, then $\delta \wedge \theta_k$ has no models in $\mathbb{V}$, but in the Levy collapse $Coll(\kappa, \aleph_0)$, $\mathbb{V}[G] \models (\delta \wedge \theta_k) \in L_{\omega_1,\omega}$ and has a model.

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So θ_k is not grounded in the vocabulary $\tau_k \cup \{U_n : n \in \omega\}$.

Some positive results

- (Larson-Zapletal) $L = \{E\}$, $T =$ acyclic graphs is grounded.
- (URL) $L = \{E_n : n \in \omega\}$, $REF =$ 'refining equivalence relations' (with arbitrary splitting) is grounded.
- (Kaplan-Shelah) Some classes of linear orders are grounded, but general question remains Open.

Good news:

1. If T is $\aleph_1$-categorical, then T is grounded.
2. If Φ is complete (i.e., Φ itself is a Scott sentence) then Φ is grounded.
3. If Φ is grounded, then:
   1. The Friedman-Stanley jump $J(\Phi)$ is grounded; and
   2. If $\Psi \vdash \Phi$, then Ψ is grounded.
4. Among $\{\Phi :\cong_\Phi \text{ Borel}\}$, the grounded ones and the non-grounded ones are $\leq_B$-cofinal.

Bad news:

1. There are first order, non-grounded T with $\cong$ Borel.
2. There exist first order, weakly minimal T that are non-grounded.
3. There exist first order, ω-stable T that are non-grounded.
4. There are first-order, Borel complete T that are grounded (e.g., REF(inf)).
5. The sentence θ_1 (in the vocabulary $\tau_1 \cup \{U_n : n \in \omega\}$) is grounded iff CH holds.

Thanks for listening!

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