Lévy-Steinitz for countable sets of series

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Abstract

The Lévy-Steinitz theorem characterizes the values that a conditionally convergent sequence in $\mathbb{R}^n$ can attain under permutations. We use material from [3] to extend this analysis to sequences in $\mathbb{R}^\omega$, under pointwise convergence, reproving a theorem of Stanimir Troyanski [6].

It is shown in [3] that there exists a c.c.c. partial order adding a permutation of ω making every conditionally convergent real series in the ground model converge to a value not in the ground model. Applying this forcing fact to a countable elementary submodel of a sufficiently large fragment of the universe, one gets the following fact : for any countable set S of conditionally convergent real series, and every countable $X \subseteq \mathbb{R}$, there is a permutation of ω making each member of S converge to a real number not in X. In this note we give a more direct proof of this fact, using the same machinery. The resulting theorem (due to Stanimir Troyanski [6]) is an extension of the Lévy-Steinitz theorem (which characterizes the values that a finite set of series can take under permutations) to countable sets of series. The proof uses the original Lévy-Steinitz theorem, as well as the Polygonal Refinement Theorem, which is used in the original proof of Lévy-Steinitz theorem. It is a simplified version of the proof of the theorem from [3] mentioned above.

We emphasize that our extension of the Lévy-Steinitz theorem applies to $\mathbb{R}^\omega$ under pointwise converge. Corollary 7.2.2 of [4] says that in each infinite-dimensional Banach space there is a series attaining exactly two values under rearrangements.

1 Preliminaries

We start with some material taken from [5], as rewritten in [3].

Given a sequence $\bar{a} = \langle a^i : i < d\rangle$ consisting of real-values series (for some $d \in \omega$), we let $K(\bar{a})$ be the set of $\langle s_i : i < d\rangle \in \mathbb{R}^d$ for which the series $\sum_{i \in d} s_i a^i$ is absolutely convergent. We let $R(\bar{a})$ be the orthogonal complement of $K(\bar{a})$

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(i.e., the set of vectors in $\mathbb{R}^d$ orthogonal to every element of $K(\bar{a})$). The sets $K(\bar{a})$ and $R(\bar{a})$ are each linear subspaces of $\mathbb{R}^d$, and their dimensions sum to d.

We say that a set I consisting of conditionally convergent series is *independent* if $K(\bar{a}) = \{\mathbf{0}\}$ for each finite sequence $\bar{a}$ from I. We let $S(\bar{a})$ be the set of values in $\mathbb{R}^d$ of the form $\sum_{n\in\omega}\langle a^i_{p(n)} : i < d\rangle$ for p a permutation of ω.

The following theorem from 1913 is due to Lévy and Steinitz (see [1, 4, 5]).

Theorem 1.1 (Lévy-Steinitz)**.** *If $\bar{a} = \langle a^i : i < d\rangle$ is a finite sequence of conditionally convergent real-valued series, then*

$$S(\bar{a}) = \{\langle \sum a^i : i < d\rangle + \bar{x} : \bar{x} \in R(\bar{a})\}.$$

One way to interpret the Lévy-Steinitz Theorem is to note that in the case where $\langle a^i : i < d\rangle$ is independent, it says that every value in $\mathbb{R}^n$ is attainable under some rearrangement. If $\langle a^i : i < d\rangle$ is an arbitrary sequence of conditionally convergent real series, then there exists a set $s \subseteq d$ such that $\{a^i : i \in s\}$ is independent, and such that, for each $j \in d \setminus s$ there exist scalars k_i ($i \in s$), not all 0 such that $\sum_{i\in s} k_i a^i + a^j$ is absolutely convergent. Given any permutation of ω, then, the value of such an a^j is determined by the values of a^i ($i \in s$). We carry out the version of this analysis for countable sets in the final section of this paper.

The following is the key lemma in the proof of the Lévy-Steinitz theorem (see [2, 5]).

Theorem 1.2 (The Polygonal Confinement Theorem; Steinitz)**.** *For each positive integer d there exists a constant C_d such that for each positive $n \in \omega$ and all vectors v_m ($m \in n$) from $\mathbb{R}^d$, if*

$$\sum_{m\in n} v_m = 0$$

and $\|v_m\| \leq 1$ for all $m \in n$, then there is a permutation p of $n \setminus \{0\}$ such that

$$\left\| v_0 + \sum_{m\in k\setminus\{0\}} v_{p(m)} \right\| \leq C_d$$

for every $m \in n + 1$.

The following immediate (and standard) consequence of the Polygonal Confinement Theorem is proved in [3].

Lemma 1.3. *Let m and d be positive integers, let ρ be a positive real number and let b and v_i ($i \in m$) be elements of $\mathbb{R}^d$. Suppose that*

$$\sum_{i\in m} v_i = b,$$

$\|b\| \leq \rho$ *and* $\|v_i\| \leq \rho$ *for all* $i \in m$*. Then there is a permutation* p *of* $m \setminus \{0\}$ *such that*

$$\left\| v_0 + \sum_{i \in j \setminus \{0\}} v_{p(i)} \right\| \leq \rho C_d + \|b\|$$

for every $j \in m+1$*.*

2 Countable independent sets

We prove in this section the version of the Lévy-Steinitz theorem for countable independent sets. The proof is an adaptation of arguments from [3]. The general version is proved in the next section.

Theorem 2.1. *Let* $\langle a^i : i < \omega \rangle$ *be an independent sequence of conditionally convergent real series and let* $\langle x_i : i < \omega \rangle$ *be a sequence of real numbers. Then there is a permutation* p *of* ω *such that, for each* $i \in \omega$*,* $\sum_{j \in \omega} a^i_{p(j)} = x_i$*.*

For the rest of this section, fix $\langle a^i : i < \omega \rangle$ and $\langle x_i : i < \omega \rangle$ as in the statement of Theorem 2.1, and a nondecreasing sequence of constants C_d as given by the Polygonal Confinement Theorem. We define a partial order P from which our desired permutation will be induced by a suitable descending sequence. Conditions in P are triples (f, d, ϵ) such that

- f is an injection from some $n \in \omega$ to ω;
- d is a positive integer;
- ϵ is a positive rational number;
- $\left\| \sum_{k<n} \langle a^i_{f(k)} : i < d \rangle - \langle x_i : i < d \rangle \right\| < \epsilon$;
- for all $m \in \omega \setminus \mathrm{Range}(f)$, $\left\| \langle a^i_m : i < d \rangle \right\| < \epsilon / C_d$.

The order on P_I is defined by : $(g, e, \delta) \leq (f, d, \epsilon)$ if

- g extends f;
- $e \geq d$;
- for all $m \in \mathrm{Dom}(g) + 1$, $\left\| \sum_{k \in m \setminus \mathrm{Dom}(f)} \langle a^i_{g(k)} : i < d \rangle \right\| < 2\epsilon$;
- $2\delta + \left\| \sum_{k \in \mathrm{Dom}(g) \setminus \mathrm{Dom}(f)} \langle a^i_{g(k)} : i < d \rangle \right\| \leq 2\epsilon$.

Observe that if ϵ is greater than both $|x_0|$ and $|\sup\{C_1 a^0_m : m \in \omega\}|$, then $(\emptyset, 1, \epsilon)$ is a condition in P. If $\langle (f_n, d_n, \epsilon_n) : n \in \omega \rangle$ is a descending sequence in P such that

- $\bigcup_{n \in \omega} f_n$ is a permutation of ω,

- $\omega = \bigcup_{n\in\omega} d_n$ and
- $\lim_{n\to\infty} \epsilon_n = 0$

then $\bigcup_{n\in\omega} f_n$ is as desired. Theorem 2.1 follows then from Lemma 2.2.

Lemma 2.2. *For each $(f, d, \epsilon) \in P$ and each $n \in \omega$, there exists a condition $(g, d+1, \delta) \leq (f, d, \epsilon)$ with $n \subseteq \mathrm{Dom}(g) \cap \mathrm{Range}(g)$ and $\delta < 1/n$.*

Proof. Let (f, d, ϵ) and n be given. By the Lévy-Steinitz theorem, there is a permutation p of ω extending f such that

$$\sum_{n\in\omega} \langle a^i_{p(n)} : i < d+1 \rangle = \langle x_i : i < d+1 \rangle.$$

Let $\eta < \epsilon$ be such that $\left\| \langle a^i_m : i < d \rangle \right\| < \eta / C_d$ for all $m \in \omega \setminus \mathrm{Dom}(f)$, and let $\delta \in \mathbb{Q}^+$ be smaller than both $1/n$ and $(\epsilon - \eta)/2$. Fix $n_* \geq n$ such that

- $n \subseteq \mathrm{Range}(p \restriction n_*)$,
- $\left\| \sum_{m\in n_* \setminus \mathrm{Dom}(f)} \langle a^i_{p(m)} : i < d \rangle \right\| < \epsilon$,
- $\left\| \sum_{m\in n_*} \langle a^i_{p(m)} : i < d+1 \rangle - \langle x_i : i < d+1 \rangle \right\| < \delta$ and
- $\left\| \langle a^i_m : i < d+1 \rangle \right\| < \delta / C_{d+1}$ for each $m \in \omega \setminus n_*$.

By Lemma 1.3, there is an injection g from n_* to ω extending f, with the same range as $p \restriction n_*$, such that

$$\begin{aligned} \left\| \langle a^i_{g(\mathrm{Dom}(f))} : i < d \rangle + \sum_{k\in m \setminus (\mathrm{Dom}(f)+1)} \langle a^i_{g(k)} : i < d \rangle \right\| &\leq (\eta/C_d)C_d + (\epsilon - \delta) \\ &< 2\epsilon - 2\delta \end{aligned}$$

for every $m \in n_* + 1$. Then (g, A, δ) is as desired. □

3 Arbitrary sequences

We adapt the notation introduced in Section 1 to countable sequences. Given a sequence $\bar{a} = \langle a^i : i < \omega \rangle$ consisting of real-values series, we let $K(\bar{a})$ be the set of $\langle s_i : i < \omega \rangle \in \mathbb{R}^\omega$ for which the following hold:

- the set $\{i \in \omega : d_i \neq 0\}$ is finite;
- the series $\sum_{i\in\omega} d_i a^i$ is absolutely convergent.

We let $R(\bar{a})$ be the orthogonal complement of $K(\bar{a})$ (i.e., the set of vectors in $\mathbb{R}^\omega$ orthogonal to every element of $K(\bar{a})$). The sets $K(\bar{a})$ and $R(\bar{a})$ are each linear subspaces of $\mathbb{R}^\omega$. We let $S(\bar{a})$ be the set of values in $\mathbb{R}^\omega$ of the form $\sum_{n\in\omega} \langle a^i_{p(n)} : i < d \rangle$ for p a permutation of ω.

The following natural generalization of the Lévy-Steinitz theorem to countable sequences was first proved by Troyasnki [6].

Theorem 3.1 (Lévy-Steinitz for countable sets)**.** *If $\bar{a} = \langle a^i : i < \omega\rangle$ is a sequence of conditionally convergent real-valued series, then*

$$S(\bar{a}) = \{\langle \sum a^i : i < \omega\rangle + \bar{x} : \bar{x} \in R(\bar{a})\}.$$

Proof. For each $i \in \omega$, let $s_i = \sum a^i$. Let $I \subseteq \omega$ be such that $\{a^i : i \in I\}$ is independent, and such that, for each $j \in \omega \setminus i$ there exist $c_j \in \mathbb{R}$ and $d^j_k \in \mathbb{R}$ $(k \in I \cap j)$ such that $\sum_{k\in I\cap j} d^j_k a^k + a^j$ is absolutely convergent, with sum c_j.

For one direction of the desired equality, let $\langle x_i : i < \omega\rangle$ be in $R(\bar{a})$. We want to find a permutation p of ω such that, for each $i \in \omega$, $\sum_{n\in\omega} a^i_{p(n)} = s_i + x_i$. By Theorem 2.1, there is a permutation p such that this equation holds for all $i \in I$. Suppose now that j is in $\omega \setminus I$. Since $\langle x_i : i \in \omega\rangle$ is in $R(\bar{a})$, $\sum_{k\in I\cap j} x_k d^j_k + x_j = 0$. Since $\sum_{k\in I\cap j} d^j_k a^k + a^j$ is absolutely convergent with sum c_j, $\sum_{k\in I\cap j} d^j_k s_k + s_j = c_j$ and

$$\begin{aligned}\sum_{n\in\omega} a^j_{p(n)} &= c_j - \sum_{k\in I\cap j} d^j_k \sum_{n\in\omega} a^k_{p(n)} \\ &= c_j - \sum_{k\in I\cap j} d^j_k (s_k + x_k) \\ &= (c_j - \sum_{k\in I\cap j} d^j_k s_k) - \sum_{k\in I\cap j} d^j_k x_k \\ &= s_j + x_j\end{aligned}$$

as desired.

For the other direction, let p be a permutation of ω such that $\sum_{n\in\omega} a^i_{p(n)}$ converges for all $n \in \omega$. For each $i \in \omega$, let $x_i = \sum_{n\in\omega} a^i_{p(n)} - s_i$. We want to see that $\bar{x} = \langle x_i : i \in \omega\rangle$ is in $R(\bar{a})$. To do this, fix $\langle d_i : i < \omega\rangle$ in $\mathbb{R}^\omega$ such that $\{i \in \omega : d_i \neq 0\}$ is a finite set D, and such that $\sum_{i\in D} d_i a^i$ is absolutely convergent with sum e. Then

$$\begin{aligned}\langle x_i : i < \omega\rangle \cdot \langle d_i : i < \omega\rangle &= \sum_{i\in D} x_i d_i \\ &= \sum_{i\in D} d_i (\sum_{n\in\omega} a^i_{p(n)} - \sum_{n\in\omega} a^i_n) \\ &= (\sum_{n\in\omega}\sum_{i\in D} d_i a^i_{p(n)}) - (\sum_{n\in\omega}\sum_{i\in D} d_i a^i_n) \\ &= e - e \\ &= 0.\end{aligned}$$

$\square$

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