Ground model definability in ZF

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Ground model definability

More than 3 decades after the introduction of forcing, the following fundamental question was asked and answered.

Question: Is the universe V definable in its forcing extensions $V[G]$?

Theorem: (Laver (2007), Woodin (2004)) A universe V is uniformly definable in its forcing extensions[1] with a parameter from V.

There is a single formula $\varphi(x, y)$ such that whenever $V[G]$ is a forcing extension of V, there is $a \in V$ such that

$$V = \{x \in V[G] \mid V[G] \models \varphi(x, a)\}.$$

[1] All forcing extensions in this talk are set-forcing extensions.

Cover and approximation properties

Definition (Hamkins, 2003) Suppose that $V \subseteq W$ are transitive models of (some fragment of) ZFC and δ is a cardinal in W.

- The pair $V \subseteq W$ satisfies the δ-cover property if for every $A \in W$ with $A \subseteq V$ and $|A|^W < \delta$, there is $B \in V$ with $A \subseteq B$ and $|B|^V < \delta$.

  "Every set of size less than δ in W that is contained in V can be covered by a set of size less than δ in V."

- The pair $V \subseteq W$ satisfies the δ-approximation property if whenever $A \in W$ with $A \subseteq V$ and $A \cap a \in V$ for every a of size less than δ in V, then $A \in V$.

  "If a set in W is contained in V and V has all pieces of it of size less than δ, then V has the set."

Theorem: (Hamkins) Suppose M, M', and N are transitive models of (a fragment of) ZFC such that:

- δ is regular in N,
- the pairs $M \subseteq N$ and $M' \subseteq N$ have the δ-cover and δ-approximation properties,
- $P^M(\delta) = P^{M'}(\delta)$,
- $(\delta^+)^M = (\delta^+)^N$.

Then $M = M'$.

The formula defining V

Suppose $V[G]$ is a forcing extension by a poset $\mathbb{P}$.

- Let $|\mathbb{P}| = \gamma$.
- Let $\delta = \gamma^+$.
- Let Z^* be a certain finite fragment of ZFC.
- Say that a transitive model $M \in V[G]$ is good if
  - $M \models \mathrm{Z}^*$,
  - $(\delta^+)^M = (\delta^+)^{V[G]}$,
  - $P^M(\delta) = P^V(\delta)$.

Lemma

- The pair $V_\gamma \subseteq V[G]_\gamma$ has the δ-cover and δ-approximation properties whenever they satisfy Z^*.
- There are unboundedly many ordinals γ such that V_γ and $V[G]_\gamma$ satisfy Z^*.

$$\varphi(x, P^V(\delta)) :=$$

x is an element of a good model M of height $\gamma \gg \delta$ such that $V[G]_\gamma \models \mathrm{Z}^*$ and the pair $M \subseteq V[G]_\gamma$ has the δ-cover and δ-approximation properties.

Set Theoretic Geology

Ground model definability makes it possible to study the structure of all ground models of a ZFC-universe in a first-order context.

Theorem: (Reitz, 2006) There is a formula $\psi(x, y)$ such that

- for every set r, the class $W_r = \{x \mid \psi(x, r)\}$ is a ground of V,
- for every ground W of V, there is a set r such that $W = \{x \mid \psi(x, r)\}$.

Several new and interesting inner models were introduced and studied using the definability of grounds.

Definition: The mantle is the intersection of all grounds of V.

Theorem: (Usuba, 2017) The mantle is a model of ZFC.

Theorem: (Usuba, 2018) If there exists an extendible cardinal, then the mantle is a (smallest) ground of V.

Ground model definability in ZF

Cover and approximation properties arguments do not easily generalize to choiceless models because they are fundamentally tied up with existence of cardinalities.

Open Question: Does (uniform) ground model definability hold for ZF-universes?

Partial results have used (combinations of) two incremental approaches.

- Isolate a class of partial orders for which ground model definability holds.
- Isolate axioms A such that ground model definability holds for $\mathrm{ZF} + A$.

Ground model definability in $\mathrm{ZF} + \mathrm{DC}_\delta$

Definition: (Lévy, 1964) The choice principle DC_δ asserts that for any non-empty set S and any binary relation R, if for each sequence $s \in S^{<\delta}$ there is a $y \in S$ such that $s\,R\,y$, then there is a function $f : \delta \to S$ such that $f \restriction \alpha\, R\, f(\alpha)$ for all $\alpha < \delta$.

"We can make δ-many dependent choices along any relation without terminal nodes."

Definition: A poset $\mathbb{P}$ admits a gap at a cardinal δ if it has the form $\mathbb{R} * \dot{\mathbb{Q}}$ where

- $|\mathbb{R}| < \delta$ (in particular, $\mathbb{R}$ is well-orderable) is a non-trivial forcing,
- $\Vdash_{\mathbb{R}}$ "$\dot{\mathbb{Q}}$ is $\leq \delta$-strategically closed".

Posets admitting a gap at δ preserve DC_δ to the forcing extension.

Theorem: (G., Johnstone, 2014) Suppose $V \models \mathrm{ZF} + \mathrm{DC}_\delta$ and $\mathbb{P}$ is a poset admitting a gap at δ. Then every forcing extension $V[G]$ by $\mathbb{P}$ satisfies DC_δ.

Theorem: (G., Johnstone, 2014) A universe $V \models \mathrm{ZF} + \mathrm{DC}_\delta$ is uniformly definable in all its forcing extensions by posets admitting a gap at δ.

There is a single formula $\varphi(x, y)$ such that whenever $V \models \mathrm{ZF} + \mathrm{DC}_\delta$ and $V[G]$ is a forcing extension of V by a poset $\mathbb{P}$ admitting a gap at δ, then

$$V = \{x \in V[G] \mid V[G] \models \varphi(x, P^V(\delta))\}.$$

Löwenheim-Skolem cardinals

Definition: An uncountable cardinal κ is a Lowenheim-Skolem cardinal (LS) if for every $\gamma < \kappa \leq \alpha$ and $x \in V_\alpha$, there is $\beta > \alpha$ and $X \prec V_\beta$ such that

- $V_\gamma \subseteq X$,
- $x \in X$,
- the transitive collapse of X belongs to V_κ,
- $(X \cap V_\alpha)^{V_\gamma} \subseteq X$.

Proposition: Suppose $V \models \mathrm{ZFC}$. Then the LS cardinals are precisely the $\beth$-fixed point cardinals.

Definition: (Woodin) Suppose $V \models \mathrm{ZF}$. A cardinal κ is supercompact if for every $\alpha \geq \kappa$, there is $\beta \geq \alpha$, a transitive set N and an elementary embedding $j : V_\beta \to N$ such that $\mathrm{crit}(j) = \kappa$, $\alpha < j(\kappa)$, and $N^{V_\alpha} \subseteq N$.

Proposition: Suppose $V \models \mathrm{ZF}$. Then every supercompact cardinal is an LS cardinal.

Theorem: (Usuba, 2019) The assertion that there is a proper class of LS cardinals is absolute between V and its forcing extensions.

Corollary: If AC is forceable over $V \models \mathrm{ZF}$, then V has a proper class of LS-cardinals.

Example: $L(\mathbb{R})$ has a proper class of LS cardinals.

Universes without LS cardinals

Theorem: (Usuba, 2019) Suppose $V \models \mathrm{ZF}$. If κ is an LS cardinal, then the club filter on κ^+ is κ-complete.

Theorem: (Karagila, 2018) Every universe $V \models \mathrm{ZFC} + \mathrm{GCH}$ has an extension to a universe $W \models \mathrm{ZF}$ with the same cofinalities in which the club filter is not σ-complete for any regular cardinal δ.

Corollary: It is consistent that there are universes $V \models \mathrm{ZF}$ without any LS cardinals.

Ground model definability in ZF + proper class of LS cardinals

Theorem: (Usuba, 2019) A universe $V \models \mathrm{ZF}$ + proper class of LS cardinals is uniformly definable in all its forcing extensions.

There is a single formula $\varphi(x, y)$ such that whenever

$$V \models \mathrm{ZF} + \text{proper class of } \mathrm{LS} \text{ cardinals}$$

and $V[G]$ is a forcing extension by a poset $\mathbb{P} \in V_\kappa$ for an LS cardinal κ, then

$$V = \{x \in V[G] \mid V[G] \models \varphi(x, V_\kappa)\}.$$

The proof is a modification of the cover and approximation properties arguments, where cardinality is replaced by a rough measure on sets:

- (Usuba) Define for a set x that the measure $\|x\|$ is the least ordinal α such that there is a surjection from V_α onto x.

Corollary: Uniform ground model definability holds for the following universes.

- Universes satisfying ZF + proper class of supercompact cardinals.
- Universes over which AC is forceable.