

Polish groupoids and $\mathcal{L}_{\omega_1\omega}$ -theories

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Philosophy

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syntax \longrightarrow semantics

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$$\mathcal{L}, \mathcal{M} \xrightarrow{\text{Aut}} G$$

Continuous logic

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Metric \mathcal{L} -structure \mathcal{M} (\mathcal{L} relational):

- ▶ underlying complete metric space M of diameter ≤ 1
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Up to isomorphism, Polish groups are precisely $\text{Aut}(\mathcal{M})$ for separable first-order metric structures \mathcal{M} (with $\mathcal{L} = \emptyset$, or on \mathbb{U}).

Urysohn sphere \mathbb{U} : universal ultrahomogeneous Polish metric space of diameter ≤ 1

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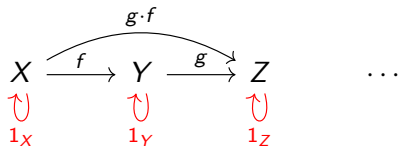
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The diagram illustrates the composition of two morphisms f and g . It shows a sequence of objects X , Y , and Z connected by arrows f and g . A red curved arrow labeled $g \cdot f$ connects X directly to Z , representing the composition of f and g . Ellipses \dots follow the Z object.

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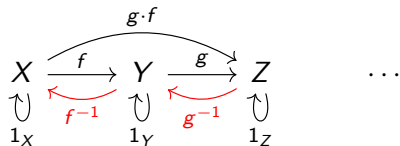
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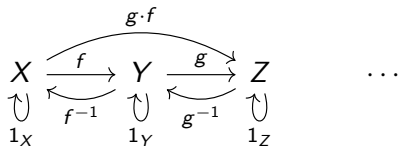
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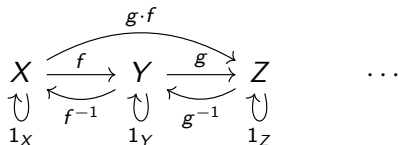


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obeying the usual axioms. Formally:

- ▶ sets G^0 , G^1 of **objects** and **morphisms**
- ▶ **source** and **target** $\sigma, \tau : G^1 \rightarrow G^0$
- ▶ **identity** $1_{(-)} : G^0 \rightarrow G^1$
- ▶ **multiplication** $\cdot : G^1 \times_{G^0} G^1 := \{(g, h) \mid \sigma(g) = \tau(h)\} \rightarrow G^1$
- ▶ **inverse** $(-)^{-1} : G^1 \rightarrow G^1$

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- ▶ formally, $(G \ltimes X)^1 := G \times X$ where $(g, x) = g : x \rightarrow g \cdot x$
- ▶ $\sigma = \text{projection} : G \times X \rightarrow X$ open

Logic actions

\mathcal{L} : countable relational language

Space of metric \mathcal{L} -structures $\text{Mod}_{\mathbb{U}}(\mathcal{L}) := \prod_{n\text{-ary } R \in \mathcal{L}} \overbrace{\text{Lip}(\mathbb{U}^n, [0, 1])}^{\text{1-Lipschitz maps w/ ptwise conv}}$

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For an \mathcal{L} -theory \mathcal{T} , $\text{Mod}_{\mathbb{U}}(\mathcal{L}, \mathcal{T}) := \{\mathcal{M} \in \text{Mod}_{\mathbb{U}}(\mathcal{L}) \mid \mathcal{M} \models \mathcal{T}\}$
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Groupoid of models $\text{Iso}(\mathbb{U}) \ltimes \text{Mod}_{\mathbb{U}}(\mathcal{L}, \mathcal{T})$ has

- ▶ objects: models of \mathcal{T} on \mathbb{U}
- ▶ morphisms: isomorphisms between models

Representation of Polish groupoids

Theorem (C.)

For any open Polish groupoid G , there is a language \mathcal{L} and an $\mathcal{L}_{\omega_1\omega}$ -sentence ϕ in continuous logic s.t.

$G \simeq_B \text{Iso}(\mathbb{U}) \times \text{Mod}_{\mathbb{U}}(\mathcal{L}, \phi)$ (*Borel equivalence of groupoids*).

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In particular, $\mathbb{E}_G \sim_B \mathbb{E}_{\text{Iso}(\mathbb{U})}^{\text{Mod}_{\mathbb{U}}(\mathcal{L}, \phi)}$, a Polish group action.

$X \mathbb{E}_G Y \iff \exists g : X \rightarrow Y \in G.$

This answers a question of [Lupini \(2017\)](#).

Representation of non-Archimedean groupoids

A topological groupoid G is **non-Archimedean** if every $1_X \in G^1$ has a neighborhood basis of open subgroupoids.

E.g., $G \ltimes X$, for non-Archimedean G .

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Theorem (C.)

For any non-Archimedean open Polish groupoid G , there is a language \mathcal{L} and an $\mathcal{L}_{\omega_1\omega}$ -sentence ϕ in discrete logic s.t.

$$G \simeq_B S_\infty \times \text{Mod}_{\mathbb{N}}(\mathcal{L}, \phi).$$

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Strong conceptual completeness

The preceding results say that (up to \simeq_B)

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ctbl cts (disc) $\mathcal{L}_{\omega_1\omega}$ -theories \longrightarrow (non-Arch) open Polish gpds

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Proof uses DST (Becker-Kechris) + topos theory (Joyal-Tierney).

Thank you