Polish groupoids and $\mathcal{L}_{\omega_1\omega}$-theories

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$$\mathcal{L}, \mathcal{M} \xmapsto{\mathrm{Aut}} G$$

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Metric $\mathcal{L}$-structure $\mathcal{M}$ ($\mathcal{L}$ relational):

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Urysohn sphere $\mathbb{U}$: universal ultrahomogeneous Polish metric space of diameter ≤ 1

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obeying the usual axioms. Formally:

- sets G^0, G^1 of objects and morphisms
- source and target $\sigma, \tau : G^1 \to G^0$
- identity $1_{(-)} : G^0 \to G^1$
- multiplication $\cdot : G^1 \times_{G^0} G^1 := \{(g, h) \mid \sigma(g) = \tau(h)\} \to G^1$
- inverse $(-)^{-1} : G^1 \to G^1$

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- σ = projection : $G \times X \to X$ open

Logic actions

$\mathcal{L}$: countable relational language

$$\text{Space of metric } \mathcal{L}\text{-structures } \mathrm{Mod}_{\mathbb{U}}(\mathcal{L}) := \prod_{n\text{-ary } R \in \mathcal{L}} \overbrace{\mathrm{Lip}(\mathbb{U}^n, [0,1])}^{\text{1-Lipschitz maps w/ ptwise conv}}$$

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For an $\mathcal{L}$-theory $\mathcal{T}$, $\mathrm{Mod}_{\mathbb{U}}(\mathcal{L}, \mathcal{T}) := \{\mathcal{M} \in \mathrm{Mod}_{\mathbb{U}}(\mathcal{L}) \mid \mathcal{M} \models \mathcal{T}\}$ ($\mathrm{Iso}(\mathbb{U})$-invariant)

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Groupoid of models $\mathrm{Iso}(\mathbb{U}) \ltimes \mathrm{Mod}_{\mathbb{U}}(\mathcal{L}, \mathcal{T})$ has

- objects: models of $\mathcal{T}$ on $\mathbb{U}$
- morphisms: isomorphisms between models

Representation of Polish groupoids

Theorem (C.)

For any open Polish groupoid G, there is a language $\mathcal{L}$ and an $\mathcal{L}_{\omega_1\omega}$-sentence ϕ in continuous logic s.t. $G \simeq_B \mathrm{Iso}(\mathbb{U}) \ltimes \mathrm{Mod}_{\mathbb{U}}(\mathcal{L}, \phi)$ (Borel equivalence of groupoids).

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$G \simeq_B \mathsf{Iso}(\mathbb{U}) \ltimes \mathsf{Mod}_{\mathbb{U}}(\mathcal{L},\phi)$ *(Borel equivalence of groupoids).*
In particular, $\mathbb{E}_G \sim_B \mathbb{E}_{\mathsf{Iso}(\mathbb{U})}^{\mathsf{Mod}_{\mathbb{U}}(\mathcal{L},\phi)}$, a Polish group action.

$$X \ \mathbb{E}_G \ Y \ :\Longleftrightarrow \ \exists g : X \to Y \in G.$$

This answers a question of Lupini (2017).

Representation of non-Archimedean groupoids

A topological groupoid G is non-Archimedean if every $1_X \in G^1$ has a neighborhood basis of open subgroupoids.

E.g., $G \ltimes X$, for non-Archimedean G.

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Theorem (C.)

For any non-Archimedean open Polish groupoid G, there is a language $\mathcal{L}$ and an $\mathcal{L}_{\omega_1\omega}$-sentence ϕ in discrete logic s.t. $G \simeq_B S_\infty \ltimes \mathrm{Mod}_{\mathbb{N}}(\mathcal{L}, \phi)$.
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Strong conceptual completeness

The preceding results say that (up to $\simeq_B$)

$$\begin{aligned}\text{syntax} &\twoheadrightarrow \text{semantics}\\ \text{ctbl cts (disc) } \mathcal{L}_{\omega_1\omega}\text{-theories} &\twoheadrightarrow \text{(non-Arch) open Polish gpds}\\ (\mathcal{L}, \mathcal{T}) &\longmapsto \mathrm{Mod}_{\mathbb{U}(\mathbb{N})}(\mathcal{L}, \mathcal{T}).\end{aligned}$$

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Proof uses DST (Becker–Kechris) + topos theory (Joyal–Tierney).

Thank you