Strongly increasing sequences

Chris Lambie-Hanson[*]   Paul B. Larson[†]

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Abstract

Using a variation of Woodin's $\mathbb{P}_{\max}$ forcing, we force over a model of the Axiom of Determinacy to produce a model of ZFC containing a very strongly increasing sequence of length ω_2 consisting of functions from ω to ω. We also show that there can be no such sequence of length ω_4.

1 Introduction

Given functions f, g from ω to the ordinals, a relation R in $\{<, >, \leq, \geq, =\}$ and $n \in \omega$, we write

- $fR_n g$ to mean that $f(m)Rg(m)$ for all $m \in \omega \setminus n$;
- fR^*g to mean that $\{n \in \omega : \neg(f(n)Rg(n))\}$ is finite.

This paper concerns wellordered sequences of functions from the integers to the ordinals, which are increasing in the following senses.

1 Definition. Given a ordinals η and γ, we say that a sequence $\langle f_\alpha : \alpha < \gamma \rangle$ of functions from ω to η is *strongly increasing* if

1. for all $\alpha < \beta < \gamma$, $f_\alpha <^* f_\beta$
2. for each limit ordinal $\beta < \gamma$, there exist a club $C_\beta \subseteq \beta$ and an $n_\beta < \omega$ such that, for all α in C_β, $f_\alpha <_{n_\beta} f_\beta$.

2 Definition. We say that a strongly increasing sequence $\langle f_\alpha : \alpha < \gamma \rangle$ is *very strongly increasing* if

1. n_β in part (2) of Definition 1 can be chosen to be 0 for each limit ordinal β of uncountable cofinality;

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2. for each ordinal of the form $\beta + \omega < \gamma$, and each $n \in \omega$, there is an α in the interval $[\beta + 1, \beta + \omega)$ such that $f_\alpha \leq_n f_{\beta+\omega}$ and $f_\alpha(m) = 0$ for all $m < n$.

3 Remark. Condition (2) of Definition 2 is designed to make condition (2) of Definition 1 hold automatically when β has countable cofinality. In particular, it ensures that whenever $\bar{f} = \langle f_\alpha : \alpha < \gamma \rangle$ is very strongly increasing and γ has countable cofinality, there exists a function f_γ such that $\langle f_\alpha : \alpha \leq \gamma \rangle$ is very strongly increasing; moreover, if γ is not of the form $\beta + \omega$, then f_γ can be chosen to have any desired initial segment. Note that if $\langle f_\alpha : \alpha < \gamma \rangle$ is a strongly increasing sequence, then there is a sequence $\langle f'_\alpha : \alpha < \gamma \rangle$ satisfying condition (2) of Definition 2 such that $f_\alpha =^* f'_\alpha$ for all $\alpha < \gamma$.

Our interest in strongly increasing sequences is partially motivated by two longstanding open problems:

1. Is the Chang's Conjecture variant $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ consistent?

2. Is there consistently a pair of inner models $V \subseteq W$ of ZFC such that $(\aleph_{\omega+1})^V = (\aleph_2)^W$.

In Section 2, we review some of what is known about these questions and show that a positive answer to either would entail the consistency of the existence of an ordinal $\eta < \omega_2$ and a strongly increasing sequence of length ω_2 consisting of functions from ω to η. We also show that there can exist no such sequence of length ω_n for any $4 \leq n < \omega$.

In Section 3, we prove the main result of our paper, using a variant of Woodin's $\mathbb{P}_{\max}$ forcing to establish the consistency of the existence of a strongly increasing sequence of length ω_2 consisting of functions from ω to ω:

Theorem 4. *Suppose that $V = L(A, \mathbb{R})$, for some $A \subseteq \omega^\omega$, and that AD^+ holds. Then there is a forcing extension in which ZFC holds and there exists a very strongly increasing ω_2-sequence of functions from ω to ω.*

The axiom AD^+ is a strengthening of the Axiom of Determinacy due to Woodin (see [9]). The theorem implies the weaker version where $A = \emptyset$ and AD^+ is replaced with AD. The partial order used in the proof is a variation of Woodin's $\mathbb{P}_{\max}$ forcing [14]. The use of the hypothesis AD + $V = L(\mathbb{R})$ and the fact that ZFC holds in the corresponding extension are both part of the standard $\mathbb{P}_{\max}$ machinery.

2 Chang's Conjecture and strongly increasing sequences

In this section we motivate our result by linking it to some well-known questions about variants of Chang's Conjecture and collapsing successors of singular cardinals. None of the results in this section is essentially new,[1] but we include them for completeness. Some of the results have not been stated before in the precise forms we need.

We begin by noting the following immediate consequence of Remark 3.

Lemma 5. *If F is a very strongly increasing sequence in ω^ω, γ is an ordinal and $\gamma+1$ is the length of F, then there is a strongly increasing sequence in ω^ω of length $\gamma+\omega_1$ extending F.*

In particular, there is always a strongly increasing sequence in ω^ω of length ω_1. We next show that instances of Chang's Conjecture at singular cardinals of countable cofinality entail the existence of more interesting strongly increasing sequences. Recall that the Chang's Conjecture principle

$$(\kappa, \lambda) \twoheadrightarrow (\delta, \gamma)$$

says that whenever M is a structure of cardinality κ over some countable language, and B is a subset of M of cardinality λ, then M has an elementary substructure X of cardinality δ such that $|X \cap B| = \gamma$. We use equivalent formulations of these principles below.

Lemma 6. *Suppose that μ is a singular cardinal of countable cofinality, $\kappa < \mu$ is a cardinal, and $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$ holds. Then there exist $\eta < \kappa^+$ and a strongly increasing sequence $\langle f_\alpha : \alpha < \kappa^+ \rangle$ of functions from ω to η.*

Proof. Let $\langle \mu_i : i < \omega \rangle$ be an increasing sequence of regular cardinals, cofinal in μ. It is straightforward to recursively construct an increasing sequence $\vec{g} = \langle g_\beta : \beta < \mu^+ \rangle$ of functions in $\prod_{i<\omega} \mu_i$, letting n_β be the least i such that $\mu_i > \mathrm{cof}(\beta)$, for each limit ordinal β. Let θ be a sufficiently large regular cardinal, and use $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$ to find an elementary substructure $N \prec (H(\theta), \in, \mu^+, \mu, \vec{g})$ such that $|N \cap \mu^+| = \kappa^+$ and $|N \cap \mu| = \kappa$. Since N contains a surjection from μ to each element of $\mu^+ \cap N$, $\mathrm{otp}(N \cap \mu^+) = \kappa^+$.

By elementarity and the fact that $\vec{g}$ is strongly increasing, we know that, for every limit ordinal $\delta \in N \cap \mu^+$, there exist a club $C_\delta \subseteq \delta$ and an $n_\delta < \omega$ such that $C_\delta \in N$ and, for all $\beta \in C_\delta$, we have $g_\beta <_{n_\delta} g_\delta$. Let M be the transitive collapse of N, and let $\pi : N \to M$ be the collapse map.

Claim 7. *Suppose that $\delta \in N \cap \mu^+$ is a limit ordinal. Then $\pi(C_\delta)$ is closed in its supremum.*

Proof. Let $\bar{\beta} < \pi(\delta)$ be such that $\sup(\pi(C_\delta) \cap \bar{\beta}) = \bar{\beta}$. Let $\beta = \pi^{-1}(\bar{\beta})$, and note that β is a limit ordinal. If $\alpha < \beta$, then $\pi(C_\delta) \cap (\pi(\alpha), \bar{\beta}) \neq \emptyset$, so $C_\delta \cap (\alpha, \beta) \neq \emptyset$. It follows that $N \models$ "$C_\delta \cap \beta$ is unbounded in β," so, by elementarity, C_δ is in fact unbounded in β and, since C_δ is closed in δ, we have $\beta \in C_\delta$. But then $\bar{\beta} \in \pi(C_\delta)$, as desired. □

[1]See, e.g., [12], [1], or [11] for related results in slightly different contexts.

Since $\omega \subseteq N$, it follows that for all $\beta \in N \cap \mu^+$, we have $\mathrm{range}(g_\beta) \subseteq N$. Let $\langle \beta_\alpha : \alpha < \kappa^+ \rangle$ enumerate $M \cap \mu^+$ in increasing order. Define a sequence $\vec{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle$ of functions from ω to $\pi(\mu)$ by letting $f_\alpha(n) = \pi(g_{\beta_\alpha}(n))$ for all $\alpha < \kappa^+$ and all $n < \omega$. Claim 7 and the fact that $\pi(\mu) < \kappa^+$ imply that $\vec{f}$ is strongly increasing, as witnessed by n_{β_α} and $\pi(C_{\beta_\alpha})$ for each limit ordinal $\alpha < \kappa^+$. □

In [10], Levinski, Magidor, and Shelah prove, assuming the consistency of a certain large cardinal hypothesis, the consistency of a number of Chang's Conjecture variants involving singular cardinals, most notably $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$. However, in any instance $(\mu^+, \mu) \twoheadrightarrow (\kappa^+, \kappa)$ known to be consistent in which μ is a singular cardinal, we have $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$. This leads to the following natural folklore question.

8 Question. Is $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ consistent?

A closely related question, raised by Bukovský and Copláková-Hartová in [3], is the following.

9 Question. Is there consistently a pair of inner models $V \subseteq W$ of ZFC such that $(\aleph_{\omega+1})^V = (\aleph_2)^W$?

To see one connection between these two questions, observe that, if

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$$

and there is a Woodin cardinal, then there is a V-generic filter G for Woodin's stationary tower forcing such that $(\aleph_{\omega+1})^V = (\aleph_2)^{V[G]}$ (see [7]).

Essentially the same argument as that given in the proof of Lemma 6 yields the following result, which states that a positive answer to Question 9 would entail the existence of interesting strongly increasing sequences.

Lemma 10. *Suppose that $V \subseteq W$ are inner models of ZFC, $n < \omega$, and $(\aleph_{\omega+1})^V = (\aleph_{n+1})^W$. Then, in W, there is $\eta < \omega_{n+1}$ and a strongly increasing sequence $\langle f_\alpha : \alpha < \omega_{n+1} \rangle$ of functions from ω to η.*

For more information and partial progress on Questions 8 and 9, we refer the reader to [11] and [4].

For the next result, we need some notions from PCF theory [12]. The following is a special case of a more general definition.

11 Definition. Suppose that $\vec{f} = \langle f_\alpha : \alpha < \gamma \rangle$ is a $<^*$-increasing sequence of functions from ω to the ordinals. A function g from ω to the ordinals is an *exact upper bound* (eub) for $\vec{f}$ if:

- g is an *upper bound*, i.e., for every $\alpha < \gamma$, we have $f_\alpha <^* g$;
- for every function h such that $h <^* g$, there is $\alpha < \gamma$ such that $h <^* f_\alpha$.

An ordinal $\beta < \gamma$ is called *good* for $\vec{f}$ if $\mathrm{cf}(\beta) > \omega$ and there is an eub h for $\vec{f} \restriction \beta = \langle f_\alpha : \alpha < \beta \rangle$ such that $\mathrm{cf}(h(i)) = \mathrm{cf}(\beta)$ for all but finitely many $i < \omega$.

The following theorem, due in a much more general form to Shelah [12], is a basic result in PCF theory. Recall that S^κ_λ denotes the set of ordinals below κ of cofinality λ. For a proof, we refer the reader to [5, Theorem 10.1].

Theorem 12. *Suppose that $\kappa < \lambda$ are uncountable regular cardinals and that $\vec{f} = \langle f_\alpha : \alpha < \lambda\rangle$ is a $<^*$-increasing sequence of functions from ω to the ordinals. If there are stationarily many $\beta \in S^\lambda_\kappa$ such that β is good for $\vec{f}$, then $\vec{f}$ has an eub h such that $\mathrm{cf}(h(i)) > \kappa$ for all $i < \omega$.*

We can now prove the following result bounding the lengths of strongly increasing sequences of functions.

Theorem 13. *Suppose that λ is an uncountable regular cardinal and $\epsilon < \lambda^{+3}$. Then there is no strongly increasing sequence $\vec{f} = \langle f_\alpha : \alpha < \lambda^{+3}\rangle$ of functions from ω to ϵ.*

Proof. Suppose for the sake of contradiction that $\vec{f} = \langle f_\alpha : \alpha < \lambda^{+3}\rangle$ is a strongly increasing sequence of functions from ω to ϵ. Fix a club-guessing sequence $\langle C_\xi \mid \xi \in S^{\lambda^{++}}_\lambda\rangle$ (see [12], for instance). In particular, each C_ξ is a club in ξ of order type λ and, for every club $D \subseteq \lambda^{++}$, there is a $\xi \in S^{\lambda^{++}}_\lambda$ such that $C_\xi \subseteq D$.

Our goal is to show that there are stationarily many elements of $S^{\lambda^{+3}}_\lambda$ that are good for $\vec{f}$. To this end, fix a club $C \subseteq \lambda^{+3}$. We now construct an increasing, continuous sequence $\langle \alpha_\eta : \eta < \lambda^{++}\rangle$ of elements of C. Begin by letting $\alpha_0 = \min(C)$. If $\eta < \lambda^{++}$ is a limit ordinal, then $\alpha_\eta = \sup\{\alpha_\zeta : \zeta < \eta\}$. Suppose now that $\eta < \lambda^{++}$ and $\langle \alpha_\zeta : \zeta \leq \eta\rangle$ has been constructed. We show how to obtain $\alpha_{\eta+1}$.

For each $\xi \in S^{\lambda^{++}}_\lambda$, ask whether there exist $\beta < \lambda^{+3}$ and $n < \omega$ such that, for all $\zeta \in C_\xi \cap (\eta+1)$, we have $f_{\alpha_\zeta} <_n f_\beta$. Note that, if β has this property, then so does every γ with $\beta \leq \gamma < \lambda^{+3}$. If the answer is yes, then choose an ordinal $\alpha^\xi_\eta \in C \setminus (\alpha_\eta + 1)$ witnessing this. If the answer is no, then let $\alpha^\xi_\eta = \min(C \setminus (\alpha_\eta + 1))$. Let $\alpha_{\eta+1} = \sup\{\alpha^\xi_\eta : \xi \in S^{\lambda^{++}}_\lambda\}$.

Let $\beta = \sup\{\alpha_\eta : \eta < \lambda^{++}\}$. By the fact that $\vec{f}$ is strongly increasing, we can find a club $D \subseteq \beta$ and a natural number n such that, for all $\alpha \in D$, we have $f_\alpha <_n f_\beta$. Let $E = \{\eta < \lambda^{++} : \alpha_\eta \in D\}$, and note that E is club in λ^{++}. Fix $\xi \in S^{\lambda^{++}}_\lambda$ such that $C_\xi \subseteq E$. For $\eta \in C_\xi$, let $\eta^\dagger$ denote $\min(C_\xi \setminus (\eta+1))$.

Suppose $\eta \in C_\xi$. When defining $\alpha_{\eta+1}$, the answer to the question asked about C_ξ was "yes," as witnessed by β. Therefore, $\alpha_{\eta+1}$ was chosen to be large enough so that, for some $n < \omega$ and all $\zeta \in C_\xi \cap (\eta+1)$, we have $f_{\alpha_\zeta} <_n f_{\alpha_{\eta+1}}$. The same obviously holds for all η' in the interval (η, λ^{++}). In particular, there is a natural number n_η such that, for all $\zeta \in C_\xi \cap (\eta+1)$, we have $f_{\alpha_\zeta} <_{n_\eta} f_{\alpha_{\eta^\dagger}}$.

Since λ is regular and uncountable, we can fix a natural number n and an unbounded set $A \subseteq C_\xi$ such that, for all $\eta \in A$, we have $n_\eta = n$. Let $B = \{\alpha_{\eta^\dagger} : \eta \in A\}$. Then B is unbounded in α_ξ and, for all $\zeta < \eta$, both in B, we have $f_{\alpha_{\zeta^\dagger}} <_n f_{\alpha_{\eta^\dagger}}$. But now, if g is a function from ω to the ordinals such that, for all $n \leq m < \omega$, we have $g(m) = \sup\{f_{\eta^\dagger}(m) : \eta \in B\}$, it is easily verified that g is an eub for $\vec{f} \restriction \alpha_\xi$. But then g witnesses that α_ξ is good for $\vec{f}$. Moreover, by construction, $\alpha_\xi \in C$. Since C was arbitrary, we have shown that there are stationarily many elements of $S^{\lambda^{+3}}_\lambda$ that are good for $\vec{f}$.

By Theorem 12, it follows that there is an eub h for $\vec{f}$ such that $\mathrm{cf}(h(i)) > \lambda$ for all $i < \omega$.

Claim 14. $\mathrm{cf}(h(i)) \geq \lambda^{+3}$ *for all but finitely many* $i < \omega$.

Proof. If not, then there exist $k \in \{1, 2\}$ and an unbounded $A \subseteq \omega$ such that, for all $i \in A$, we have $\mathrm{cf}(h(i)) = \lambda^{+k}$. For each $i \in A$, let $\{\delta^i_\eta : \eta < \lambda^{+k}\}$ enumerate, in increasing fashion, a set of ordinals cofinal in $h(i)$. For each $\eta < \lambda^{+k}$, define a function h_η from ω to the ordinals by letting $h_\eta(i) = \delta^i_\eta$ if $i \in A$ and $h_\eta(i) = 0$ otherwise. For each $\eta < \lambda^{+k}$, we have $h_\eta <^* h$, so, since h is an eub for $\vec{f}$, there is $\beta_\eta < \lambda^{+3}$ such that $h_\eta <^* f_{\beta_\eta}$. Let $\gamma = \sup\{\beta_\eta : \eta < \lambda^{+k}\}$. Since $k < 3$, we have $\gamma < \lambda^{+3}$. Therefore, for all $\eta < \lambda^{+k}$, we have $h_\eta <^* f_\gamma$. Fix an unbounded $B \subseteq \lambda^{+k}$ and an $n < \omega$ such that, for all $\eta \in B$, we have $h_\eta <_n f_\gamma$. But then, for all $i \in A \setminus n$, we must have $f_\gamma(i) \geq \sup\{\delta^i_\eta : \eta \in B\} = h(i)$, contradicting the fact that h is an upper bound for $\vec{f}$. □

But this claim immediately contradicts the fact that $\vec{f}$ is a sequence of functions from ω to ϵ and $\epsilon < \lambda^{+3}$. This is because, by the claim, we must have $h(i) > \epsilon$ for all but finitely many $i < \omega$. But then the constant function, taking value ϵ, witnesses that h fails to be an eub. □

The results in this section lead to the following corollary.

Corollary 15. *Suppose that* $3 \leq n < \omega$.

1. *If* $\eta < \omega_{n+1}$*, then there is no strongly increasing sequence* $\langle f_\alpha : \alpha < \omega_{n+1}\rangle$ *of functions from* ω *to* η.
2. $(\aleph_{\omega+1}, \aleph_\omega) \not\twoheadrightarrow (\aleph_{n+1}, \aleph_n)$.
3. *There are no inner models* $V \subseteq W$ *of* ZFC *such that* $(\aleph_{\omega+1})^V = (\aleph_{n+1})^W$.

It also follows that the only regular cardinals that can possibly be lengths of strongly increasing sequences from ω^ω are $\aleph_n$ for $0 \leq n \leq 3$. We have seen that there are always such sequences of length $\aleph_0$ and $\aleph_1$. We will prove, in Section 3, the consistency of the existence of a strongly increasing sequence of length $\aleph_2$. The question about the consistency of the existence of a strongly increasing sequence of length $\aleph_3$ remains open.

3 Consistency via a $\mathbb{P}_{\max}$ variation

In this section we use a natural variation of Woodin's partial order $\mathbb{P}_{\max}$ to produce a very strongly increasing sequence in ω^ω of length ω_2.

We refer the reader to [14] for background on $\mathbb{P}_{\max}$, especially Chapter 4 and Section 9.2. The article [8] may also be helpful. Conditions in our partial order $\mathbb{P}$ are triples (M, F, a) such that

- M is a countable transitive model of $\mathsf{ZFC} + \mathsf{MA}_{\aleph_1}$ which is iterable with respect to $\mathrm{NS}_{\omega_1}^M$;
- F is in M a very strongly increasing sequence in ω^ω whose length is a successor ordinal less than ω_2^M (or 0);
- a is an element of $\mathcal{P}(\omega_1)^M$ such that, for some $x \in \mathcal{P}(\omega)^M$, $\omega_1^M = \omega_1^{L[a,x]}$.

The order is : $(M, F, a) < (N, H, b)$ if there exists in M an iteration

$$j\colon (N, \mathrm{NS}_{\omega_1}^N) \to (N^*, \mathrm{NS}_{\omega_1}^{N^*})$$

such that

- $j(b) = a$;
- $\mathrm{NS}_{\omega_1}^{N^*} = \mathrm{NS}_{\omega_1}^M \cap N^*$;
- $j(H)$ is a proper initial segment of F.

The requirement above that F properly extends $j(H)$ simplifies the arguments below (by removing trivial cases), and adds no new complications, by Lemma 5.

16 Remark. If (M, F, a) is a condition in our partial order $\mathbb{P}$, then $\langle (M, \mathrm{NS}_{\omega_1}^M), a\rangle$ is a condition in $\mathbb{P}_{\max}$. Conversely, if $\langle (M, \mathrm{NS}_{\omega_1}^M), a\rangle$ is a $\mathbb{P}_{\max}$ condition, then $(M, \emptyset, a)$ is a condition in $\mathbb{P}$. If $(M, \emptyset, a)$ and (N, G, b) are conditions in $\mathbb{P}$ such that $\langle (M, \mathrm{NS}_{\omega_1}^M), a\rangle < \langle (N, \mathrm{NS}_{\omega_1}^N), b\rangle$ (as $\mathbb{P}_{\max}$ conditions), then there is an F in M such that $(M, F, a) < (N, G, b)$ (as conditions in $\mathbb{P}$). Finally, the order on each comparable pair of $\mathbb{P}$-conditions is witnessed by a unique iteration (this follows, for instance, from the corresponding fact for $\mathbb{P}_{\max}$).

17 Remark. Much of the standard $\mathbb{P}_{\max}$ machinery can be applied directly to the partial order $\mathbb{P}$. In particular, AD^+ implies the following facts about $\mathbb{P}$, each of which can be derived from Theorem 9.31 of [14] and Remark 16 (the first follows from the second).

- For each set $x \in H(\aleph_1)$, there exists a $\mathbb{P}$-condition (M, F, a) with x in $H(\aleph_1)^M$.
- For each $\mathbb{P}$ condition (M, F, a) and each $A \subseteq \omega^\omega$ there is a $\mathbb{P}$-condition $(N, F', b) < (M, F, a)$ such that
  - $(N, \mathrm{NS}_{\omega_1}^N)$ is A-iterable;
  - $\langle H(\aleph_1)^N, \in, A \cap N\rangle \prec \langle H(\aleph_1), \in, A\rangle$;

In conjunction with the previous two remarks, the proof of Theorem 4.43 of Woodin gives the ω-clousure of $\mathbb{P}$. We sketch the proof since the same details appear again in the proof of Theorem 19.

Lemma 18. *Every descending ω-sequence in $\mathbb{P}$ has a lower bound.*

Proof. (Sketch) Suppose that $p_i = (M_i, F_i, a_i)$ $(i \in \omega)$ is a descending ω-sequence in $\mathbb{P}$. By the first part of Remark 17 we can work inside a countable transitive model N such that that $\langle p_i : i \in \omega \rangle \in H(\aleph_1)^N$ and $(N, \mathrm{NS}^N_{\omega_1})$ is iterable. As in the proof of Theorem 4.43 of [14] we get by combining the iterations witnessing the order on the p_i's a sequence of $\mathbb{P}$-conditions $(\hat{M}_i, \hat{F}_i, \hat{a}_i)$ $(i \in \omega)$ such that, for each $i \in \omega$,

- $(\hat{M}_i, \mathrm{NS}^{\hat{M}_i}_{\omega_1})$ is an iterate of $(M_i, \mathrm{NS}^{M_i}_{\omega_1})$ and $\hat{F}_i$ and $\hat{a}_i$ are the corresponding images of F_i and a_i respectively,
- $\omega_1^{\hat{M}_i} = \omega_1^{\hat{M}_0}$,
- $\mathrm{NS}^{\hat{M}_i}_{\omega_1} = \mathrm{NS}^{\hat{M}_{i+1}}_{\omega_1} \cap \hat{M}_i$,
- $\hat{F}_i$ is a proper initial segment of $\hat{F}_{i+1}$ and
- $\hat{a}_i = \hat{a}_0$.

One can then iterate the sequence $\langle \hat{M}_i : i \in \omega \rangle$ to produce a sequence of iterations

$$j_i \colon (\hat{M}_i, \mathrm{NS}^{\hat{M}_i}_{\omega_1}) \to (M^*_i, \mathrm{NS}^{M^*_i}_{\omega_1}) \quad (i \in \omega)$$

such that

- each M^*_i is correct (in N) about stationary subsets of ω_1;
- $\bigcup_{i \in \omega} j_i(\hat{F}_i)$ (which we will call F) is a very strongly increasing sequence in ω^ω whose length is a limit ordinal of countable cofinality;
- each $j_i(\hat{a}_i)$ is the same set (which we will call a).

There exists by Remark 3 a very strongly increasing sequence F^* in N which extends F by the addition of one function. Then (M, F^*, a) is a lower bound for $\langle p_i : i \in \omega \rangle$. $\square$

The remaining argument concerns descending ω_1-sequences in $\mathbb{P}$, from the point of view of some countable transitive model. As in Lemma 5.2 of [8], we phrase the construction of such a sequence in terms of a game, where player I takes care of certain steps (typically, meeting each member of some $\aleph_1$-sized collection of dense sets) which are repeated without change in our context.

Given a condition (M, F, a), the ω_1-*sequence game* for $p = (M, F, a)$ is the game of length ω_1 where players I and II pick the members of a descending ω_1-sequence of conditions $p_\alpha = (M_\alpha, F_\alpha, a_\alpha)$ $(\alpha < \omega_1)$ from $\mathbb{P}$, where $p_0 = p$, player I picks p_α for all successor ordinals α and player II picks p_α for all limit ordinals α. We have in addition that for each $\alpha < \omega_1$, letting $j_{\alpha,\alpha+1}$ be the embedding of M_α into $M_{\alpha+1}$ witnessing that $p_{\alpha+1} < p_\alpha$, $F_{\alpha+1}$ properly extends $j_{\alpha,\alpha+1}(F_\alpha)$. Such a sequence induces an elementary embedding of each model M_α into a structure M^*_α of cardinality $\aleph_1$ and sequence $F^* = \langle f^*_\alpha : \alpha < \gamma \rangle$ which is the union of the images of the sequences F_α under these embeddings. Player II wins a run of the game if and only if

- each M^*_α is correct about stationary subsets of ω_1, and
- there is a very strongly increasing sequence in ω^ω of successor length extending F^*.

The last condition above is equivalent to the existence of a function $f \in \omega^\omega$ such that $\{\alpha < \gamma : f_\alpha <_0 f\}$ contains a club subset of γ (which will have cofinality $\aleph_1$).

Lemma 19 below is the only new ingredient in adapting the standard $\mathbb{P}_{\max}$ machinery to give the desired consistency result. The lemma is an adaptation of Lemma 4.46 of [14]; the steps involving the sequences σ and the use of Hechler and club-shooting forcing are the only new elements. For those unfamiliar with $\mathbb{P}_{\max}$, the most mysterious part of the proof may be the iteration of the sequence $\langle \hat{M}_{\beta,i} : i < \omega \rangle$. For details on this sort of construction (which was already sketched in the proof of Lemma 18) we refer the reader to Corollary 4.20 of [14]. According to the standard arguments, the lemma is applied inside a countable transitive model which is correct about $\mathbb{P}$ and which contains a Woodin cardinal, as given for instance by Theorem 9.30 of [14]. The forcing in the proof of Lemma 19 (Hechler forcing followed by adding a club subset of ω_1) is then followed by a two-stage forcing to produce a condition in $\mathbb{P}$, first collapsing the Woodin cardinal to be ω_2 to make the nonstationary ideal on ω_1 precipitous, and then forcing with a c.c.c. forcing (which preserves precipitousness) to make $\mathsf{MA}_{\aleph_1}$ hold.

We will use the following objects from the theory of cardinal characteristics of the continuum. The cardinal characteristic $\mathfrak{d}$ is defined to be the least cardinal κ such that there is a set $A \subseteq \omega^\omega$ of cardinality κ such that every element of ω^ω has a $<^*$-upper bound in A (see [2]). We let $(\mathbb{H}, \leq_{\mathbb{H}})$ denote Hechler forcing [6], where

- $\mathbb{H}$ is the set of pairs (s, P) such that $s \in \omega^{<\omega}$ and $P \in [\omega^\omega]^{<\aleph_0}$ and
- $(s, P) \leq_{\mathbb{H}} (s', P')$ if s extends s', P contains P' and $s(n) > f(n)$ for all $f \in P'$ and $n \in |s| \setminus |s'|$.

The partial order $\mathbb{H}$ adds an element of ω^ω which is $<^*$ above each element of ω^ω in the ground model. Moreover, $\mathbb{H}$ is c.c.c. (in fact, σ-centered) as conditions with the same first coordinate are compatible, so it preserves stationary subsets of ω_1. It follows that, for any cardinal κ, MA_κ implies that $\mathfrak{d} > \kappa$.

Lemma 19. *For each condition $p = (M, F, a)$ in $\mathbb{P}$, there is a strategy for player II in the ω_1-sequence game for p such that each run of the game in V according to this strategy is winning for player II in some forcing extension which preserves the stationarity of each stationary subset of ω_1 in each model M^*_α arising from the run of the game.*

Proof. A run of the game builds a descending ω_1 sequence of $\mathbb{P}$ conditions $p_\alpha = (M_\alpha, F_\alpha, a_\alpha)$, with (stationary set preserving, but not elementary) embeddings $j_{\alpha,\beta} \colon M_\alpha \to M_\beta$. Let γ_α be such that F_α has length $\gamma_\alpha + 1$ in M_α. We have that $\gamma_\beta > j_{\alpha,\beta}(\gamma_\alpha)$ for all $\alpha < \beta$.

We show how to play for player II, i.e., how to choose M_β, F_β and a_β for each limit ordinal β, assuming that p_α ($\alpha < \beta$) have already been chosen. Let us say that a countable limit ordinal β is *relevant* if β is the supremum of $\{\omega_1^{M_\alpha} : \alpha < \beta\}$. At the end of the game, the set of relevant ordinals will be a club subset of ω_1. For limit ordinals β which are not relevant, player II can let p_β be any lower bound for $\{p_\alpha : \alpha < \beta\}$ (lower bounds exist by Lemma 18).

As we carry out our construction, by some bookkeeping we associate to each pair (σ, A), where $\sigma \in \omega^{<\omega}$ and A is, in some M_α a stationary subset of $\omega_1^{M_\alpha}$, a stationary set $B_{\sigma,A} \subseteq \omega_1^V$, in such a way that the associated sets $B_{\sigma,A}$ are disjoint for distinct pairs (σ, A). We show how to play for II in such a way that, for all relevant limit ordinals β, if β is in $B_{\sigma,A}$, for some A which is a stationary subset of $\omega_1^{M_{\alpha'}}$ for some $\alpha' < \beta$, then β is in the induced image of A (i.e., $j_{\alpha',\alpha}(A)$ for all $\alpha \in [\beta, \omega_1]$ (note that $\omega_1^{M_\beta}$ will be greater than β) and σ is an initial segment of the first member of F_β which is not in the images of the preceding F_α's (i.e., the member of F_β indexed by the supremum of $\{j_{\alpha,\beta}(\gamma_\alpha) : \alpha < \beta\}$).

For each relevant limit ordinal β, in round β the triples $(M_\alpha, F_\alpha, a_\alpha)$ will have been chosen, along with the maps $j_{\alpha,\delta}$ ($\alpha \leq \delta < \beta$) witnessing that this is a descending sequence of conditions. Choosing a cofinal ω-sequence $\langle \alpha_i : i \in \omega\rangle$ in β, and composing the maps $j_{\alpha_i,\alpha_{i+1}}$ ($i \in \omega$), we get sequence $\langle \hat{M}_{\beta,i} : i < \omega\rangle$, where each $\hat{M}_{\beta,i}$ is the corresponding image of M_{α_i}. Let $\hat{\jmath}_{\alpha_i,\beta}$ be the induced embedding of M_{α_i} into $\hat{M}_{\beta,i}$, for each $i < \omega$. Note then that $\omega_1^{\hat{M}_{\beta,i}} = \beta$ for all $i < \omega$, and that each $\hat{M}_{\beta,i}$ is a subset of the corresponding $\hat{M}_{\beta,i+1}$, and correct in $\hat{M}_{\beta,i+1}$ about stationary subsets of its ω_1. Fix a countable transitive model M_β of ZFC with $\langle \hat{M}_{\beta,i} : i < \omega\rangle$, $\langle \hat{f}^\beta_\delta : \delta < \hat{\gamma}_\beta\rangle$, $\langle p_\alpha : \alpha < \beta\rangle$ and $\{\alpha_i : i \in \omega\}$ in $H(\aleph_1)^{M_\beta}$, and such that $(M_\beta, \mathrm{NS}^{M_\beta}_{\omega_1})$ is iterable. Working in M, iterate $\langle \hat{M}_{\beta,i} : i < \omega\rangle$ in such a way that the image of each model $\hat{M}_{\beta,i}$ is correct in M_β about stationary subsets of ω_1, and let j^*_β be the corresponding embedding. If β is in $B_{\sigma,A}$, for some A which is a stationary subset of $\omega_1^{M_{\alpha'}}$ for some $\alpha' < \beta$, then put the corresponding image of A ($\hat{\jmath}_{\alpha_i,\beta}(j_{\alpha',\alpha_i}(A))$ for any i such that $\alpha_i \geq \alpha'$) into the first filter in the iteration, so that $\beta \in j^*_\beta(\hat{\jmath}_{\alpha_i,\beta}(j_{\alpha',\alpha_i}(A)))$. Having constructed the iteration j^*_β, extend the union of the sets $j^*_\beta(\hat{\jmath}_{\alpha_i,\beta}(F_{\alpha_i}))$ (the length of which has countable cofinality, since each $\gamma_{\alpha_{i+1}}$ is greater than $j_{\alpha_i,\alpha_{i+1}}(\gamma_{\alpha_i})$) with one element having σ as an initial segment (which we can do by the "moreover" part of Remark 3), and let F_β be this extension. Let a_β be the common value of $j^*_\beta(\hat{\jmath}_{\alpha_i,\beta}(a_{\alpha_i}))$ ($i < \omega$). This completes the choice of M_β, F_β and a_β.

Having constructed the entire run of the game, letting a be the union of the sets a_α ($\alpha < \omega_1$), there is for each $\alpha < \omega_1$ a unique iteration j_{α,ω_1} of $(M_\alpha, \mathrm{NS}^{M_\alpha}_{\omega_1})$ sending a_α to a. Let γ^*_α denote the image of γ_α under this iteration, and let F^*_α be the corresponding image of F_α. Let $F^* = \bigcup_{\alpha<\omega_1} F^*_\alpha$ and let γ^* be the length of F^*. For each countable limit ordinal β, let $\eta_\beta = \sup\{\gamma^*_\alpha : \alpha < \beta\}$. The η_β's are closed below their supremum γ, which has cofinality ω_1.

Force (over V) with Hechler forcing to add a $g \in \omega^\omega$ dominating each real in V mod-finite. This forcing is c.c.c., and therefore preserves stationary sets.

We want to see that, for each A which is, for some $\alpha < \omega_1$, a stationary subset of $\omega_1^{M_\alpha}$ in M_α, the set of $\beta \in j_{\alpha,\omega_1}(A)$ such that g dominates $f^*_{\eta_\beta}$ everywhere is stationary. To see that this is the case, consider a Hechler condition (σ, P) and an A. In any club $C \subseteq \omega_1$, we can find a $\beta \in C \cap B_{\sigma,A}$ such that

$$\sup\{\omega_1^{M_\alpha} : \alpha < \beta\} = \beta.$$

Then $(\sigma, P \cup \{\beta\}) \leq (\sigma, P)$, and $(\sigma, P \cup \{\beta\})$ forces that the generic real g will dominate $f^*_{\eta_\beta}$ everywhere.

Finally, force to shoot a club E (via the standard forcing with countable conditions) through the set of $\beta < \omega_1$ such that g dominates $f^*_{\eta_\beta}$ everywhere. By the previous paragraph, this forcing preserves the stationarity of each set $j_{\alpha,\omega_1}(A)$. It follows that in $V[g][E]$, $F^* \cup \{(\gamma^*, g)\}$ witnesses that the run of the ω_1-sequence game just produced is winning for player II. $\square$

20 Remark. Lemma 19 and standard $\mathbb{P}_{\max}$ arguments (see Theorems 9.32 and 9.34 of [14]) give the following.

- If $G \subseteq \mathbb{P}$ is a filter, and $A_G = \bigcup\{a : (M, F, a) \in G\}$, then for each $p = (M, F, a) \in G$ there is a unique iteration $j_{p,G}$ of $(M, \mathrm{NS}^M_{\omega_1})$ sending a to A_G.
- If W is an inner model of AD^+ and $G \subseteq \mathbb{P}$ is W-generic, then every element of $\mathcal{P}(\omega_1)^{W[G]}$ is an element of $j_{p,G}[\mathcal{P}(\omega_1)^M]$ for some $p = (M, F, a)$ in G.
- If $A \subseteq \omega^\omega$ is such that $L(A, \omega^\omega) \models \mathsf{AD}^+$, and $G \subseteq \mathbb{P}$ is $L(A, \omega^\omega)$-generic, then $L(A, \omega^\omega)[G]$ satisfies AC + $2^{\aleph_0} = \aleph_2$, and

  $$\bigcup\{j_{p,G}(F) : p = (M, F, a) \in G\}$$

  is a very strongly increasing sequence of length ω_2.

21 Remark. One could naturally try to reproduce the result proved here using an iterated forcing consisting alternately of Hechler forcing and adding club subsets of ω_1. One issue with this approach is that if $\langle f_\alpha : \alpha < \omega_1 \rangle$ is a strongly increasing sequence in ω^ω, and $g \in \omega^\omega$ is Hechler-generic over V, there will be cofinally many $m \geq n$ such that the ground model set $\{\alpha < \omega_1 : f_\alpha(m) \geq g(m)\}$ is stationary, and the stationarity of some if these sets will have to be destroyed into order to use the Hechler real to continue the sequence. The argument from Lemma 19 avoids this issue by requiring preservation only of the sets $j_{\alpha,\omega_1}(A)$ coming from the models in the sequence being built. That is, it is not necessary to preserve the stationarity of every stationary subset of ω_1 in V. The Semi-Properness Iteration Lemma on pages 485-486 of [13] offers a parallel degree of freedom, as (using the notation there) for each successor j the quotient P_j/P_i is required to be semiproper only for arbitrarily large nonlimit $i < j$.

We conclude with a couple of remaining open questions. We first remark that the $\mathbb{P}_{\max}$ machinery is well-suited for constructing models with strongly increasing sequences of length ω_2 but does not seem to be readily adaptable to construct models with longer strongly increasing sequences. We know that there cannot exist strongly increasing sequences of length ω_4 of functions from ω to ω, but the corresponding question about ω_3 remains open:

22 Question. Is it consistent with ZFC that there exists a strongly increasing sequence of length ω_3 consisting of functions from ω to ω?

Lastly, we forced over a model satisfying $\mathrm{AD} + V = L(\mathbb{R})$ to obtain our main consistency result. We do not know if these hypotheses are optimal, or even if the existence of a strongly increasing ω_2-sequence of functions from ω to ω carries any large cardinal strength at all.

23 Question. What (if any) is the consistency strength of "$\mathsf{ZFC}+$ there exists a strongly increasing ω_2-sequence of functions from ω to ω"?

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Institute of Mathematics, CAS
Žitná 25
115 67 Praha 1
Czechia
lambiehanson@math.cas.cz

Department of Mathematics
Miami University
Oxford, Ohio
USA
larsonpb@miamioh.edu