Choice from Finite Sets: A Topos View

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Introduction

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Mostowski-Gauntt Permutation Group Criteria

Theorem (Local)

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

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Any model M of ZF. (Can allow atoms, can omit replacement, can omit regularity. Also M/I for any object I of M)
Fix a group G.
G-**Set**: topos of sets with (left) action of G and functions that commute with the actions.

More Examples

M-sets for any monoid M

Kripke structures for intuitionistic logic

Presheaves on any small category

Sheaves on any topological space (or any site)

Realizability topoi

Facts about $\mathbf{Set}^I$

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Notation: Abbreviate "In topos $\mathcal{E}$, every n-element set has a point" as $EP(n, \mathcal{E})$. And $EP(Z, \mathcal{E})$ means $EP(z, \mathcal{E})$ for all z in the finite set Z.

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So $EP(n, G\text{-}\mathbf{Set})$ says that every action of G on n has a fixed point. And $EP(Z, G\text{-}\mathbf{Set})$ says the same for actions of G on any $z \in Z$.

Local Mostowski-Gauntt in Topos Language

Theorem

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *For every topos $\mathcal{E}$ of the form* **Model of ZF**$/I$, *$EP(Z, \mathcal{E})$ implies $EP(n, \mathcal{E})$.*
- *For every topos $\mathcal{E}$ of the form G-**Set**, $EP(Z, \mathcal{E})$ implies $EP(n, \mathcal{E})$.*

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What other sorts of topoi does this equivalence apply to?

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- *For every topos $\mathcal{E}$, $EP(Z, \mathcal{E})$ implies $EP(n, \mathcal{E})$.*

General Global Theorem

Theorem

For any natural number n and any finite set Z of natural numbers, the following are equivalent.

- *ZF (or ZFA) proves that, if $C(Z,I)$ for all I then $C(n,I)$ for all I.*
- *For every topos $\mathcal{E}$ of the form* **Model of ZF**$/I$, *if $EP(Z,\mathcal{E}/F)$ for all finite decidable objects F of $\mathcal{E}$, then $EP(n,\mathcal{E})$.*
- *Every group that acts without fixed points on n has a subgroup that acts without fixed points on some $z \in Z$.*
- *For every topos $\mathcal{E}$ of the form G-**Set**, if $EP(Z,\mathcal{E}/F)$ for all finite decidable objects F of $\mathcal{E}$, then $EP(Z,\mathcal{E})$.*
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