Borel equivalence relations and symmetric models

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Definition

Let E be an equivalence relation on X. A **complete classification** of E is a map $c\colon X \longrightarrow I$ such that for any $x, y \in X$, xEy iff $c(x) = c(y)$. The elements of I are called **complete invariants**.

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$$\langle x_{i,j} \mid i, j < \omega \rangle \mapsto \{\{x_{i,j};\ j \in \omega\};\ i \in \omega\}.$$

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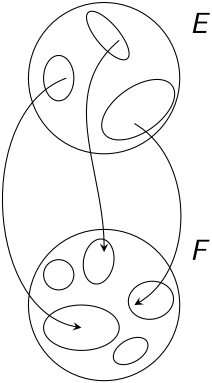

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Let E and F be Borel equivalence relations on Polish spaces X and Y respectively.

- A Borel map $f\colon X \to Y$ is a **homomorphism** from E to F, ($f\colon E \to_B F$), if for $x, x' \in X$, $x \mathrel{E} x' \implies f(x) \mathrel{F} f(x')$.

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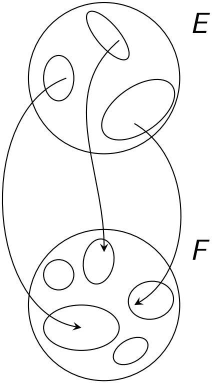

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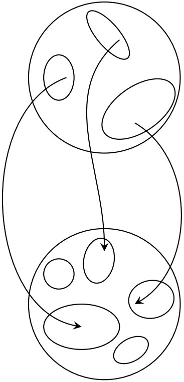

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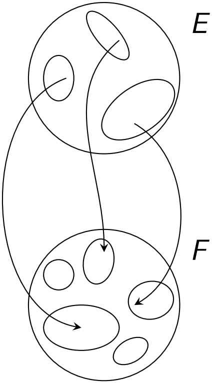

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- E and F are **Borel bireducible**, $(E \sim_B F)$, if $E \leq_B F$ and $F \leq_B E$.

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Theorem (Kanovei-Sabok-Zapletal 2013)

1. If $C \subseteq \mathbb{R}^\omega$ is comeager then $=^+ \restriction C$ is Borel bireducible to $=^+$.

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2. Let E be an analytic equivalence relation. Then either
   - $=^+$ is Borel reducible to E, or
   - any Borel homomorphism from $=^+$ to E maps a comeager subset of $\mathbb{R}^\omega$ into a single E-class.

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Then $F \sim_B =^{++}$.
Define $u \colon \mathbb{R}^\omega \times (2^\omega)^\omega \to \mathbb{R}^\omega$ by $u(x, z) = x$, $u \colon F \to_B =^+$.
We work in the comeager subset of $\mathbb{R}^\omega \times (2^\omega)^\omega$ where $\forall j \exists i (z(i)(j) = 1)$. So u maps $A_{(x,z)}$ to its union $\bigcup A_{(x,z)}$.

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Theorem (S.)

1. $F \restriction C \sim_B =^{++}$ for comeager $C \subseteq \mathbb{R}^\omega \times (2^\omega)^\omega$.

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\begin{tikzcd}
F \arrow[d, "u"] \arrow[dr, "\forall f"] & \sim_B =^{++} \\
=^+ \arrow[r, dashed, "\exists h"] & E
\end{tikzcd}
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1. $F \restriction C \sim_B =^{++}$ for comeager $C \subseteq \mathbb{R}^\omega \times (2^\omega)^\omega$.
2. for any analytic equivalence relation E either
   - F is Borel reducible to E, or
   - every homomorphism f from F to E factors through u on a comeager set.
     ($\exists h\colon =^+ \to_B E$ s.t. $(h \circ u)\ E\ f$, on a comeager set.)

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Let $x \in X$ be a Cohen generic real and let $A = A_x$.
There is a one-to-one correspondence between

- (partial) Borel homomorphisms $f \colon X \to Y$ from F to E (defined on a comeager set);
- sets $B \in V(A)$ such that B is an invariant for E and B is definable in $V(A)$ from A and parameters in V alone.

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Remark

The proof uses tools from Zapletal "Idealized Forcing" (2008) and Kanovei-Sabok-Zapletal "Canonical Ramsey theory on Polish Spaces" (2013).

A model of Monro (1973)

Let $(x, z) \in \mathbb{R}^\omega \times (2^\omega)^\omega$ be Cohen generic.
Let $A^1 = \{x(i);\, i \in \omega\}$, the $=^+$-invariant of x, and
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Suppose $B \in V(A^2)$ is a set of reals which is definable from A^2.
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- A Borel homomorphism f from F to $=^+$ corresponds to a set of reals B definable from A^2.
- Since $B \in V(A^1)$ is definable from A^1, it corresponds to a homomorphism h from $=^+$ to $=^+$.

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- Also $A^1 \in V(A^2)$ is the set of reals corresponding to the union homomorphism u.
- We conclude that f factors as $h \circ u$.